Binomial and normal distribution [77 marks]

1. [Maximum mark: 6] 19N.1.SL.TZ0.T_12

The Malthouse Charity Run is a 5 kilometre race. The time taken for each runner to complete the race was recorded. The data was found to be normally distributed with a mean time of 28 minutes and a standard deviation of 5 minutes.

A runner who completed the race is chosen at random.

(a) Write down the probability that the runner completed the race in more than 28 minutes. [1]

Markscheme

$0.5\ \left(\frac{1}{2},\ 50\%\right)$ ***(A1) (C1)***

[1 mark]

(b) Calculate the probability that the runner completed the race in less than 26 minutes. [2]

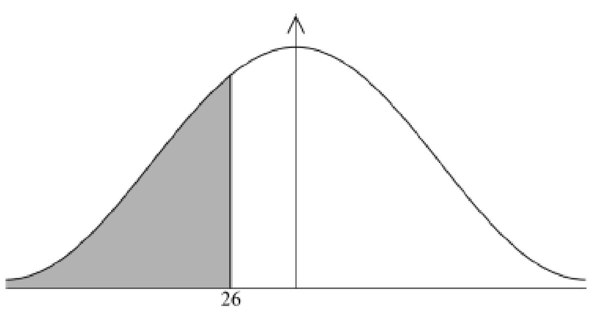
Markscheme

$\mathrm{P}(X \leq 26)$ ***(M1)***

Note: Award ***(M1)*** for a correct mathematical statement.

OR

Award ***(M1)*** for a diagram that shows the value 26 labelled to the left of the mean and the correct shaded region.



$3.45\ (0.344578\ldots,\ 34.5\%)$ ***(A1) (C2)***

[2 marks]

(c) It is known that 20% of the runners took more than 28 minutes and less than k minutes to complete the race.

Find the value of k. [3]

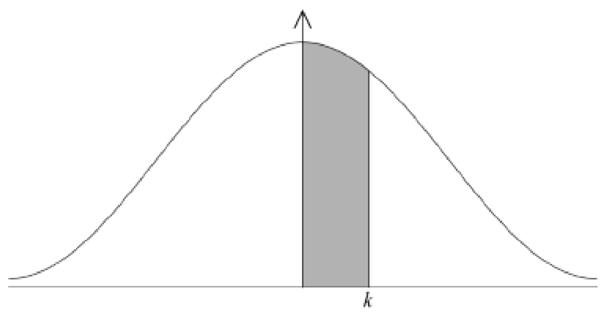
Markscheme

0.7 **OR** 0.3 (seen) ***(A1)***

Note: Award ***(A1)*** for 0.7 or 0.3 seen.

$\mathrm{P}(\text{time} < 7) = 0.7$ **OR** $\mathrm{P}(\text{time} > k) = 0.3$ ***(M1)***

Note: Award ***(M1)*** for a correct mathematical statement.
OR
Award ***(M1)*** for a diagram that shows k greater than the mean and shading in the region below k, above k, or between k and the mean.



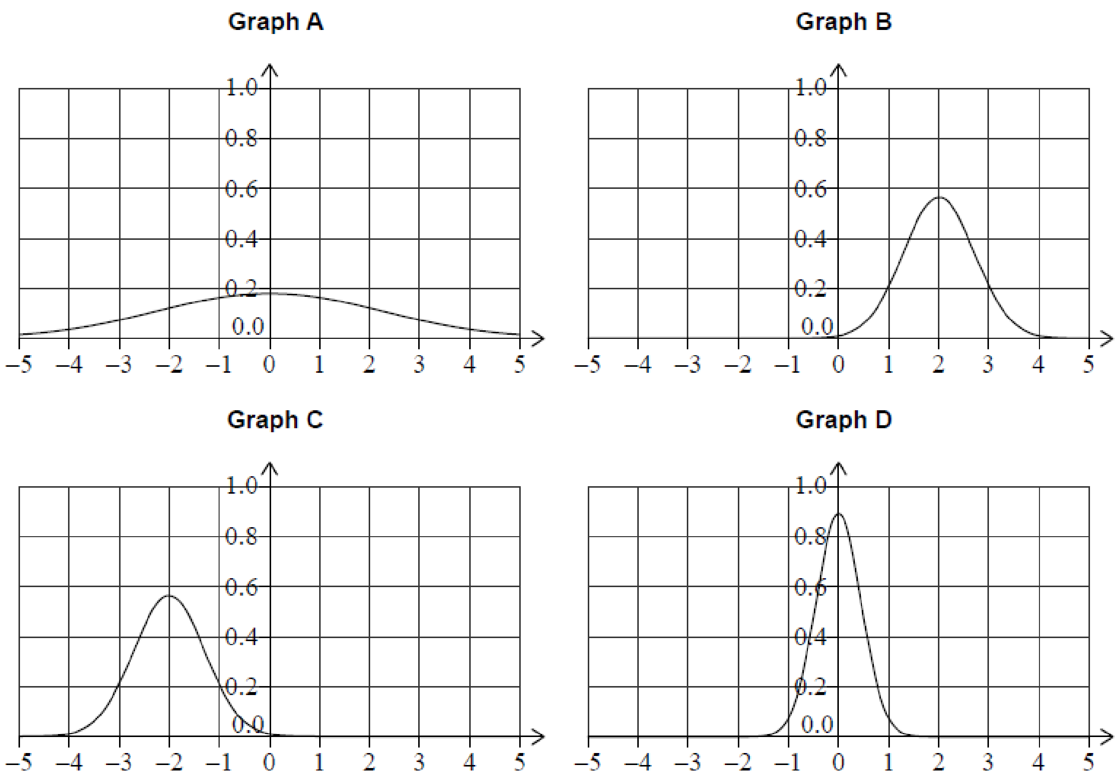
$(k =)\ 30.6\ (30.6220\ldots)$ (minutes) ***(A1)*** ***(C3)***

Note: Accept "30 minutes and 37 seconds" or (from 3 sf k value) "30 minutes and 36 seconds".

[3 marks]

2. [Maximum mark: 6] 19M.1.SL.TZ1.T_11

Consider the following graphs of normal distributions.



(a) In the following table, write down the letter of the corresponding graph next to the given mean and standard deviation.

Mean and standard deviation	Graph
Mean $= -2$; standard deviation $= 0.707$	
Mean $= 0$; standard deviation $= 0.447$	

[2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

Mean and standard deviation	Graph
Mean $= -2$; standard deviation $= 0.707$	C
Mean $= 0$; standard deviation $= 0.447$	D

(A1)

(A1) (C2)

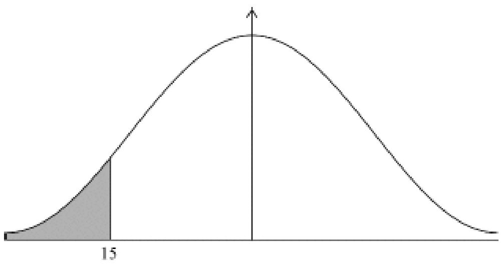
Note: Award *(A1)* for each correct entry.

[2 marks]

At an airport, the weights of suitcases (in kg) were measured. The weights are normally distributed with a mean of 20 kg and standard deviation of 3.5 kg.

(b) Find the probability that a suitcase weighs less than 15 kg. [2]

Markscheme



(M1)

Note: Award *(M1)* for sketch with 15 labelled and left tail shaded **OR** for a correct probability statement, P(*X* < 15).

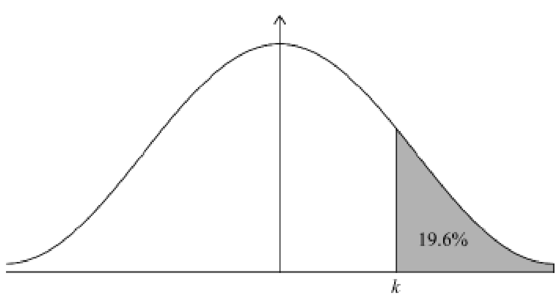
0.0766 (0.0765637…, 7.66%) ***(A1) (C2)***

[2 marks]

(c) Any suitcase that weighs more than k kg is identified as excess baggage.
19.6 % of the suitcases at this airport are identified as excess baggage.

Find the value of k. [2]

Markscheme



(M1)

Note: Award *(M1)* for a sketch showing correctly shaded region to the right of the mean with 19.6% labelled (accept shading of the complement with 80.4% labelled) **OR** for a correct probability statement, P(*X* > k) = 0.196 or P(*X* ≤ k) = 0.804.

23.0 (kg) (22.9959… (kg)) ***(A1) (C2)***

[2 marks]

3. [Maximum mark: 7] 22M.1.SL.TZ2.5

A polygraph test is used to determine whether people are telling the truth or not, but it is not completely accurate. When a person tells the truth, they have a 20% chance of failing the test. Each test outcome is independent of any previous test outcome.

10 people take a polygraph test and all 10 tell the truth.

(a) Calculate the expected number of people who will pass this polygraph test. [2]

Markscheme

$(\mathrm{E}(X) =)\ 10 \times 0.8$ ***(M1)***

8 (people) ***A1***

[2 marks]

(b) Calculate the probability that exactly 4 people will fail this polygraph test. [2]

Markscheme

recognition of binomial probability ***(M1)***

$0.0881 \quad (0.0880803\ldots)$ ***A1***

[2 marks]

(c) Determine the probability that fewer than 7 people will pass this polygraph test. [3]

Markscheme

0.8 and 6 seen **OR** 0.2 and 3 seen ***(A1)***

attempt to use binomial probability ***(M1)***

$0.121 \quad (0.120873\ldots)$ ***A1***

[3 marks]

4. [Maximum mark: 6] 23M.1.SL.TZ2.9

The lengths of the seeds from a particular mango tree are approximated by a normal distribution with a mean of $4\ \text{cm}$ and a standard deviation of $0.25\ \text{cm}$.

A seed from this mango tree is chosen at random.

(a) Calculate the probability that the length of the seed is less than $3.7\ \text{cm}$. [2]

Markscheme

$X \sim \text{N}(4,\ 0.25^2)$

EITHER

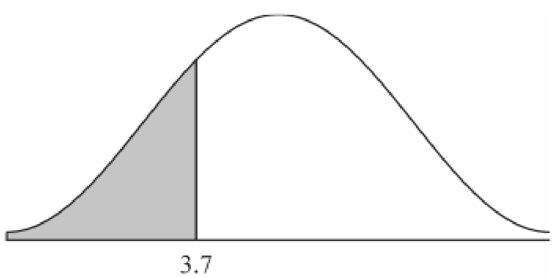
correct probability expression ***(M1)***

$\text{P}(X < 3.7)$

Note: Accept a weak or strict inequality, and any label instead of X, e.g. length or L.

OR

normal curve with vertical line, left of mean, labelled 3.7, and shaded region ***(M1)***



THEN

$0.115\ (0.115069\ldots,\ 11.5\%)$ ***A1***

Note: Award ***M1A0*** for 0.12 if no previous working.

[2 marks]

It is known that 30% of the seeds have a length greater than k cm.

(b) Find the value of k. [2]

Markscheme

EITHER

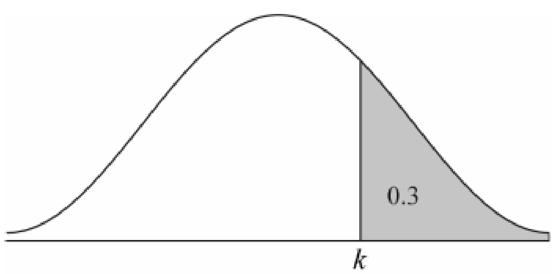
Correct probability expression ***(M1)***

$\mathrm{P}(X < k) = 0.7$ **OR** $\mathrm{P}(X > k) = 0.3$

Note: Accept a weak or strict inequality, and any label instead of X e.g., length or L.

OR

normal curve with vertical line to the right of the mean and shaded region, correctly labelled either 0.3 or 0.7 ***(M1)***



THEN

$(k =)\ 4.13\ (4.13110\ldots)$ ***A1***

Note: Award ***M1A0*** for 4.1 if no previous working.

[2 marks]

For a seed of length d cm, chosen at random, $\mathrm{P}(4 - m < d < 4 + m) = 0.6$.

(c) Find the value of m. [2]

Markscheme

EITHER

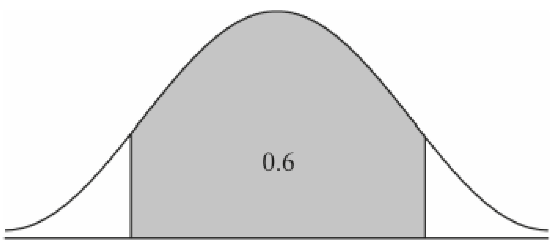
correct probability equation ***(M1)***

$\mathrm{P}(\text{length} < 4 + m) = 0.8$ **OR** $\mathrm{P}(\text{length} < 4 - m) = 0.2$

Note: Accept any letter instead of "length" e.g., X or L.

OR

normal curve with vertical lines symmetrical about the mean line with a correct indication of an area of 0.6 or 0.2 or 0.8 ***(M1)***



THEN

$0.210\ (0.210405\ldots)$ ***A1***

Note: Award ***(M1)A0*** for an answer of 3.7895 or 4.2105 seen without working. Condone 0.21 seen and award ***(M1)A1***.

[2 marks]

5. [Maximum mark: 7] 23M.1.SL.TZ1.12

On a specific day, the speed of cars as they pass a speed camera can be modelled by a normal distribution with a mean of $67.3\ \text{km h}^{-1}$.

A speed of $75.7\ \text{km h}^{-1}$ is two standard deviations from the mean.

(a) Find the standard deviation for the speed of the cars. [2]

Markscheme

attempt to find the difference between 75.7 and 67.3 ***(M1)***

$\frac{75.7-67.3}{2}$

$4.2\ (\text{km h}^{-1})$ ***A1***

[2 marks]

Speeding tickets are issued to all drivers travelling at a speed greater than $72\ \text{km h}^{-1}$.

(b) Find the probability that a randomly selected driver who passes the speed camera receives a speeding ticket. [2]

Markscheme

recognition of normal distribution that includes 72 ***(M1)***

e.g., sketch of normal distribution curve with 72 labelled to the right of the mean **OR** Normal CDF calculation using 72

$0.132\ \ (0.131559\ldots,\ 13.2\%,\ 13.1559\ldots\%)$ ***A1***

[2 marks]

It is found that 82% of cars on this road travel at speeds between $p\ \text{km h}^{-1}$ and $q\ \text{km h}^{-1}$, where $p < q$. This interval includes cars travelling at a speed of $74\ \text{km h}^{-1}$.

(c) Show that the region of the normal distribution between p and q is **not** symmetrical about the mean. [3]

Markscheme

METHOD 1 (Comparing areas above and below the mean)

$\mathrm{P}(67.3 <$speed$< 74)$ **OR** Normal CDF$(67.3,\ 74,\ 67.3,\ 4.2)$ **OR** sketch of normal distribution with 67.3 and 74 labelled and shaded between ***(M1)***

area of region between mean and q is at least $0.445\ (0.444670\ldots)$ ***A1***

Hence no more than $0.375\ (0.375329\ldots)$ between mean and p ***R1***

The region between p and q is not symmetrical ***AG***

METHOD 2 (Comparing areas in the tails)

attempt to calculate probability that speed $< p$ and speed$> q$ with $q = 74$ ***(M1)***

P(speed< 74)$= 0.944670\ldots$

P(speed$< p$)$= (0.944670\ldots - 0.82 =)\ 0.124670\ldots$

P(speed$> q$)$= (1 - 0.944670\ldots =)\ 0.0553295\ldots$ ***A1***

if $q \geq 74$, then P(speed$> q$)≤ 0.0553295 and P(speed$< p$)≥ 0.124670 so

P(speed$> q$) will never equal P(speed$< p$) ***R1***

the region between p and q is not symmetrical ***AG***

METHOD 3 (Assumption of symmetry comparing speeds)

attempt to calculate area below q assuming distribution is symmetrical ***(M1)***

e.g. P(speed$< q$)$= 0.82 + 1/2 \times 0.18\ \ (0.91)$

EITHER

$(q =)\ 72.9\ \ (72.9311\ldots)$ ***A1***

$72.9 < 74$ so 74 would not be in the region ***R1***

the region between p and q is not symmetrical ***AG***

OR

$\mathrm{P}(\text{speed} < 74) = 0.945\ \ (0.944670\ldots)$ ***A1***

$0.945 > 0.91$ so 74 would not be in the region ***R1***

the region between p and q is not symmetrical ***AG***

METHOD 4 (Assumption of symmetry comparing areas)

attempt to calculate symmetrical area with 74 as a boundary ***(M1)***

$\mathrm{P}(60.6 < \text{speed} < 74)$ **OR** Normal CDF$(60.6,\ 74,\ 67.3,\ 4.2)$ **OR**

$\mathrm{P}(67.3 < \text{speed} < 74)$ **OR** Normal CDF(67.3, 74, 67.3, 4.2)

EITHER

$0.889\ \ (0.889340\ldots)$ ***A1***

$0.889 > 0.82$ so 74 would not be in the region ***R1***

the region between p and q is not symmetrical ***AG***

OR

$0.445\ (0.444670\ldots)$ ***A1***

$0.4459 > 0.82 \div 2$ so 74 would not be in the region ***R1***

the region between p and q is not symmetrical ***AG***

[3 marks]

6. [Maximum mark: 5] 22N.1.SL.TZ0.8

Roy is a member of a motorsport club and regularly drives around the Port Campbell racetrack.

The times he takes to complete a lap are normally distributed with mean 59 seconds and standard deviation 3 seconds.

(a) Find the probability that Roy completes a lap in less than 55 seconds. [2]

Markscheme

$\mathrm{P}(T < 55)$ ***(M1)***

$0.0912 \ (0.0912112\ldots)$ ***A1***

Note: Award ***M1*** for a correct calculator notation such as $\text{normal cdf}(0, 55, 59, 3)$ or $\text{normal cdf}\left(-1^{99}, 55, 59, 3\right)$.

[2 marks]

Roy will complete a 20 lap race. It is expected that 8.6 of the laps will take more than t seconds.

(b) Find the value of t. [3]

Markscheme

correct use of expected value

$8.6 = 20 \times p$ **OR** $(p =)\ 0.43$ seen ***(M1)***

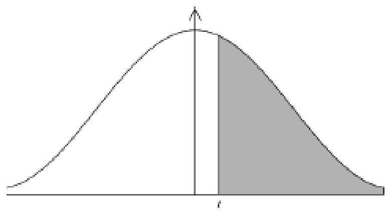
EITHER

correct probability statement

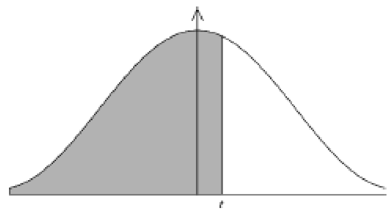
$\mathrm{P}(T > t) = 0.43$ **OR** $P(T < t) = 0.57$ ***(M1)***

OR

t indicated on sketch to communicate correct area ***(M1)***



OR



THEN

$(t =)\ 59.5$ (seconds) $(59.5291\ldots)$ *A1*

[3 marks]

7. [Maximum mark: 6] 22M.1.SL.TZ1.8

A factory produces bags of sugar with a labelled weight of 500 g. The weights of the bags are normally distributed with a mean of 500 g and a standard deviation of 3 g.

(a) Write down the percentage of bags that weigh more than 500 g. [1]

Markscheme

50% ***A1***

Note: Do not accept 0.5 or $\frac{1}{2}$.

[1 mark]

A bag that weighs less than 495 g is rejected by the factory for being underweight.

(b) Find the probability that a randomly chosen bag is rejected for being underweight. [2]

Markscheme

$0.0478 \quad (0.0477903\ldots,\ 4.78\%)$ ***A2***

[2 marks]

(c) A bag that weighs more than k grams is rejected by the factory for being overweight. The factory rejects 2% of bags for being overweight.

Find the value of k. [3]

Markscheme

$\mathrm{P}(X < k) = 0.98$ **OR** $\mathrm{P}(X > k) = 0.02$ ***(M1)***

Note: Award ***(M1)*** for a sketch with correct region identified.

$506\text{ g} \quad (506.161\ldots)$ ***A2***

[3 marks]

8. [Maximum mark: 5] 22M.1.SL.TZ2.10

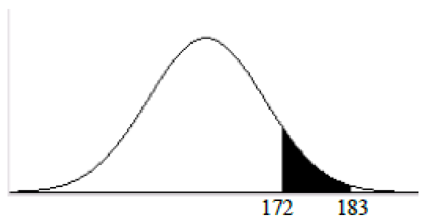
The masses of Fuji apples are normally distributed with a mean of $163\,\mathrm{g}$ and a standard deviation of $6.83\,\mathrm{g}$.

When Fuji apples are picked, they are classified as small, medium, large or extra large depending on their mass. Large apples have a mass of between $172\,\mathrm{g}$ and $183\,\mathrm{g}$.

(a) Determine the probability that a Fuji apple selected at random will be a large apple. [2]

Markscheme

sketch of normal curve with shaded region to the right of the mean and correct values ***(M1)***



$0.0921 \quad (0.0920950\ldots)$ ***A1***

[2 marks]

Approximately 68% of Fuji apples have a mass within the medium-sized category, which is between k and $172\,\mathrm{g}$.

(b) Find the value of k. [3]

Markscheme

EITHER

$(\mathrm{P}(x < 172))$

$0.906200\ldots$ ***(A1)***

$(0.906200\ldots - 0.68)$

$0.226200\ldots$ ***(A1)***

OR

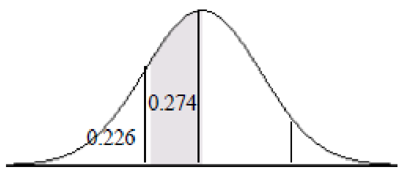
$(\mathrm{P}(163 < x < 172))$

$0.406200\ldots$ *(A1)*

$0.5-(0.68-0.406200\ldots)$ **OR** $0.5+(0.68-0.406200\ldots)$

$0.226200\ldots$ **OR** $0.773799\ldots$ *(A1)*

OR



(A1)(A1)

Note: Award ***A1*** for a normal distribution curve with a vertical line on each side of the mean and a correct probability of either 0.406 or 0.274 or 0.906 shown, ***A1*** for a probability of 0.226 seen.

THEN

$(k=)\ 158\,\text{g}\ \ (157.867\ldots\,\text{g})$ ***A1***

[3 marks]

9. [Maximum mark: 15] 22M.2.SL.TZ1.5

The aircraft for a particular flight has 72 seats. The airline's records show that historically for this flight only 90% of the people who purchase a ticket arrive to board the flight. They assume this trend will continue and decide to sell extra tickets and hope that no more than 72 passengers will arrive.

The number of passengers that arrive to board this flight is assumed to follow a binomial distribution with a probability of 0.9.

(a) The airline sells 74 tickets for this flight. Find the probability that more than 72 passengers arrive to board the flight. [3]

Markscheme

(let T be the number of passengers who arrive)

$(\mathrm{P}(T > 72) =)\ \mathrm{P}(T \geq 73)$ **OR** $1 - \mathrm{P}(T \leq 72)$ ***(A1)***

$T \sim \mathrm{B}(74,\ 0.9)$ **OR** $n = 74$ ***(M1)***

$= 0.00379 \quad (0.00379124\ldots)$ ***A1***

Note: Using the distribution $\mathrm{B}(74,\ 0.1)$, to work with the 10% that do not arrive for the flight, here and throughout this question, is a valid approach.

[3 marks]

(b.i) Write down the expected number of passengers who will arrive to board the flight if 72 tickets are sold. [2]

Markscheme

72×0.9 ***(M1)***

64.8 ***A1***

[2 marks]

(b.ii) Find the maximum number of tickets that could be sold if the expected number of passengers who arrive to board the flight must be less than or equal to 72. [2]

Markscheme

$n \times 0.9 = 72$ ***(M1)***

80 ***A1***

[2 marks]

Each passenger pays $\$150$ for a ticket. If too many passengers arrive, then the airline will give $\$300$ in compensation to each passenger that cannot board.

(c) Find, to the nearest integer, the expected increase or decrease in the money made by the airline if they decide to sell 74 tickets rather than 72. [8]

Markscheme

METHOD 1

EITHER

when selling 74 tickets

	$T \le 72$	$T = 73$	$T = 74$
Income minus compensation (I)	11100	10800	10500
Probability	0.9962…	0.003380…	0.0004110…

top row ***A1A1***

bottom row ***A1A1***

Note: Award ***A1A1*** for each row correct. Award ***A1*** for one correct entry and ***A1*** for the remaining entries correct.

$$\mathrm{E}(I) = 11100 \times 0.9962\ldots + 10800 \times 0.00338\ldots + 10500 \times 0.000411 \approx 11099$$

(M1)A1

OR

income is $74 \times 150 = 11100$ ***(A1)***

expected compensation is

$0.003380\ldots \times 300 + 0.0004110\ldots \times 600 \ \ (= 1.26070\ldots)$ ***(M1)A1A1***

expected income when selling 74 tickets is $11100 - 1.26070\ldots$ ***(M1)***

$= 11098.73\ldots \quad (= \$11099)$ ***A1***

THEN

income for 72 tickets $= 72 \times 150 = 10800$ ***(A1)***

so expected gain $\approx 11099 - 10800 = \$299$ ***A1***

METHOD 2

for 74 tickets sold, let C be the compensation paid out

$\mathrm{P}(T = 73) = 0.00338014\ldots, \quad \mathrm{P}(T = 74) = 0.000411098\ldots$ ***A1A1***

$\mathrm{E}(C) = 0.003380\ldots \times 300 + 0.0004110\ldots \times 600 \ \ (= 1.26070\ldots)$ ***(M1)A1A1***

extra expected revenue $= 300 - 1.01404\ldots - 0.246658\ldots \quad (300 - 1.26070\ldots)$
(A1)(M1)

Note: Award ***A1*** for the 300 and ***M1*** for the subtraction.

$= \$299$ (to the nearest dollar) ***A1***

METHOD 3

let D be the change in income when selling 74 tickets.

	$T \le 72$	$T = 73$	$T = 74$
Change in income	300	0	−300

(A1)(A1)

Note: Award ***A1*** for one error, however award ***A1A1*** if there is no explicit mention that $T = 73$ would result in $D = 0$ and the other two are correct.

$\mathrm{P}(T \le 73) = 0.9962\ldots, \quad \mathrm{P}(T = 74) = 0.000411098\ldots$ ***A1A1***

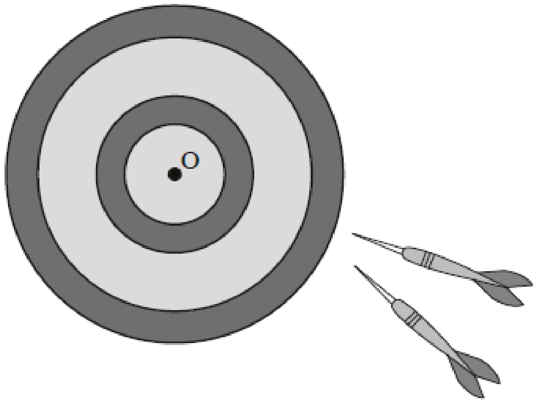
$\mathrm{E}(D) = 300 \times 0.9962\ldots + 0 \times 0.003380\ldots - 300 \times 0.0004110$ ***(M1)A1A1***

$= \$299$ ***A1***

[8 marks]

10. [Maximum mark: 14] 21N.2.SL.TZ0.5

Arianne plays a game of darts.



The distance that her darts land from the centre, O, of the board can be modelled by a normal distribution with mean $10\,\mathrm{cm}$ and standard deviation $3\,\mathrm{cm}$.

Find the probability that

(a.i) a dart lands less than $13\,\mathrm{cm}$ from O. [2]

Markscheme

Let X be the random variable "distance from O".

$X \sim \mathrm{N}\left(10,\ 3^2\right)$

$\mathrm{P}(X < 13) = 0.841\ \ (0.841344\ldots)$ ***(M1)(A1)***

[2 marks]

(a.ii) a dart lands more than $15\,\mathrm{cm}$ from O. [1]

Markscheme

$(\mathrm{P}(X > 15) =)\ \ 0.0478\ \ (0.0477903)$ ***A1***

[1 mark]

Each of Arianne's throws is independent of her previous throws.

(b) Find the probability that Arianne throws two consecutive darts that land more than $15\,\text{cm}$ from O. [2]

Markscheme

$\text{P}(X > 15) \times \text{P}(X > 15)$ ***(M1)***

$= 0.00228\ \ (0.00228391\ldots)$ ***A1***

[2 marks]

In a competition a player has three darts to throw on each turn. A point is scored if a player throws **all** three darts to land within a central area around O. When Arianne throws a dart the probability that it lands within this area is 0.8143.

(c) Find the probability that Arianne does **not** score a point on a turn of three darts. [2]

Markscheme

$1 - (0.8143)^3$ ***(M1)***

$0.460\ \ (0.460050\ldots)$ ***A1***

[2 marks]

In the competition Arianne has ten turns, each with three darts.

(d.i) Find the probability that Arianne scores at least 5 points in the competition. [3]

Markscheme

METHOD 1

let Y be the random variable "number of points scored"

evidence of use of binomial distribution ***(M1)***

$Y \sim \text{B}(10,\ 0.539949\ldots)$ ***(A1)***

$(\text{P}(Y \geq 5) =)\ 0.717\ \ (0.716650\ldots)$. ***A1***

METHOD 2

let Q be the random variable "number of times a point is not scored"

evidence of use of binomial distribution ***(M1)***

$Q \sim \mathrm{B}(10,\ 0.460050\ldots)$ ***(A1)***

$(\mathrm{P}(Q \leq 5) =)\ 0.717\ \ (0.716650\ldots)$ ***A1***

[3 marks]

(d.ii) Find the probability that Arianne scores at least 5 points and less than 8 points. [2]

Markscheme

$\mathrm{P}(5 \leq Y < 8)$ ***(M1)***

$0.628\ \ (0.627788\ldots)$ ***A1***

Note: Award ***M1*** for a correct probability statement or indication of correct lower and upper bounds, 5 and 7.

[2 marks]

(d.iii) Given that Arianne scores at least 5 points, find the probability that Arianne scores less than 8 points. [2]

Markscheme

$\frac{\mathrm{P}(5 \leq Y < 8)}{\mathrm{P}(Y \geq 5)}\ \left(= \frac{0.627788\ldots}{0.716650\ldots}\right)$ ***(M1)***

$0.876\ \ (0.876003\ldots)$ ***A1***

[2 marks]