Derivatives [191 marks]

1. [Maximum mark: 6] 19M.2.SL.TZ1.S_3

Consider the function $f\left(x\right)=x^{2}\mathrm{e}^{3x}$, $x\in\mathbb{R}$.

(a) Find $f'\left(x\right)$. [4]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

choosing product rule ***(M1)***

eg $uv'+vu'$, $\left(x^{2}\right)'\left(\mathrm{e}^{3x}\right)+\left(\mathrm{e}^{3x}\right)'x^{2}$

correct derivatives (must be seen in the rule) ***A1A1***

eg $2x$, $3\mathrm{e}^{3x}$

$f'\left(x\right)=2x\mathrm{e}^{3x}+3x^{2}\mathrm{e}^{3x}$ ***A1 N4***

[4 marks]

(b) The graph of f has a horizontal tangent line at $x=0$ and at $x=a$. Find a. [2]

Markscheme

valid method ***(M1)***

eg $f'\left(x\right)=0$, 

$a=-0.667\left(=-\frac{2}{3}\right)$ (accept $x=-0.667$) ***A1 N2***

[2 marks]

2. [Maximum mark: 7] 22M.1.SL.TZ1.9

The function f is defined by $f(x) = \frac{2}{x} + 3x^2 - 3,\ x \neq 0$.

(a) Find $f\prime(x)$. [3]

Markscheme

$f\prime(x) = -2x^{-2} + 6x$ **OR** $f\prime(x) = -\frac{2}{x^2} + 6x$ ***A1(M1)A1***

Note: Award ***A1*** for $6x$ seen, and ***(M1)*** for expressing $\frac{1}{x}$ as x^{-1} (this can be implied from either x^{-2} or $\frac{2}{x^2}$ seen in their final answer), ***A1*** for $-\frac{2}{x^2}$. Award at most ***A1(M1)A0*** if any additional terms are seen.

[3 marks]

(b) Find the equation of the normal to the curve $y = f(x)$ at $(1,\ 2)$ in the form $ax + by + d = 0$, where $a,\ b,\ d \in \mathbb{Z}$. [4]

Markscheme

finding gradient at $x = 1$

$\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{x=1} = 4$ ***A1***

finding the perpendicular gradient ***M1***

$m_{\perp} = -\frac{1}{4}$

$2 = -\frac{1}{4}(1) + c$ **OR** $y - 2 = -\frac{1}{4}(x - 1)$ ***M1***

Note: Award ***M1*** for correctly substituting $x = 1$ and $y = 2$ and their $m_{\perp}$.

$x + 4y - 9 = 0$ ***A1***

Note: Do not award the final ***A1*** if the answer is not in the required form. Accept integer multiples of the equation.

[4 marks]

3. [Maximum mark: 7] EXN.1.SL.TZ0.7

Consider the curve $y = x^2 - 4x + 2$.

(a) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. [1]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 4$ **A1**

[1 mark]

(b) Show that the normal to the curve at the point where $x = 1$ is $2y - x + 3 = 0$. [6]

Markscheme

Gradient at $x = 1$ is -2 **M1**

Gradient of normal is $\frac{1}{2}$ **A1**

When $x = 1$ $y = 1 - 4 + 2 = -1$ **(M1)A1**

EITHER

$y + 1 = \frac{1}{2}(x - 1)$ **M1**

$2y + 2 = x - 1$ or $y + 1 = \frac{1}{2}x - \frac{1}{2}$ **A1**

OR

$-1 = \frac{1}{2} \times 1 + c$ **M1**

$y = \frac{1}{2}x - \frac{3}{2}$ **A1**

THEN

$2y - x + 3 = 0$ **AG**

[6 marks]

4. [Maximum mark: 16] 18M.1.SL.TZ2.S_10

Consider a function f. The line L_1 with equation $y = 3x + 1$ is a tangent to the graph of f when $x = 2$

(a.i) Write down $f'(2)$. [2]

Markscheme

recognize that $f'(x)$ is the gradient of the tangent at x ***(M1)***

eg $f'(x) = m$

$f'(2) = 3$ (accept $m = 3$) ***A1 N2***

[2 marks]

(a.ii) Find $f(2)$. [2]

Markscheme

recognize that $f(2) = y(2)$ ***(M1)***

eg $f(2) = 3 \times 2 + 1$

$f(2) = 7$ ***A1 N2***

[2 marks]

Let $g(x) = f(x^2 + 1)$ and P be the point on the graph of g where $x = 1$.

(b) Show that the graph of g has a gradient of 6 at P. [5]

Markscheme

recognize that the gradient of the graph of g is $g'(x)$ ***(M1)***

choosing chain rule to find $g'(x)$ ***(M1)***

eg $\frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x},\ u = x^2 + 1,\ u' = 2x$

$g'(x) = f'(x^2 + 1) \times 2x$ ***A2***

$g'(1) = 3 \times 2$ ***A1***

$g'(1) = 6$ ***AG N0***

[5 marks]

(c) Let L_2 be the tangent to the graph of g at P. L_1 intersects L_2 at the point Q.

Find the y-coordinate of Q. [7]

Markscheme

at Q, $L_1 = L_2$ (seen anywhere) ***(M1)***

recognize that the gradient of L_2 is $g'(1)$ (seen anywhere) ***(M1)***
eg $m = 6$

finding $g(1)$ (seen anywhere) ***(A1)***
eg $g(1) = f(2),\ g(1) = 7$

attempt to substitute gradient and/or coordinates into equation of a straight line ***M1***
eg $y - g(1) = 6(x - 1),\ y - 1 = g'(1)(x - 7),\ 7 = 6(1) + \mathrm{b}$

correct equation for L_2

eg $y - 7 = 6(x - 1),\ y = 6x + 1$ ***A1***

correct working to find Q ***(A1)***
eg same *y*-intercept, $3x = 0$

$y = 1$ ***A1 N2***

[7 marks]

5. [Maximum mark: 8] EXN.1.AHL.TZ0.15

Consider the function $f(x) = \sqrt{-ax^2 + x + a},\ a \in \mathbb{R}^+$.

(a) Find $f\prime(x)$. [2]

Markscheme

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$f\prime(x) = (-2ax + 1) \times \frac{1}{2} \times \left(-ax^2 + x + a\right)^{-\frac{1}{2}}$

Note: M1 is for use of the chain rule.

$= \frac{-2ax+1}{2\sqrt{-ax^2+x+a}}$ **M1A1**

[2 marks]

For $a > 0$ the curve $y = f(x)$ has a single local maximum.

(b) Find in terms of a the value of x at which the maximum occurs. [2]

Markscheme

$-2ax + 1 = 0$ **(M1)**

$x = \frac{1}{2a}$ **A1**

[2 marks]

(c) Hence find the value of a for which y has the smallest possible maximum value. [4]

Markscheme

Value of local maximum $= \sqrt{-a \times \frac{1}{4a^2} + \frac{1}{2a} + a}$ **M1A1**

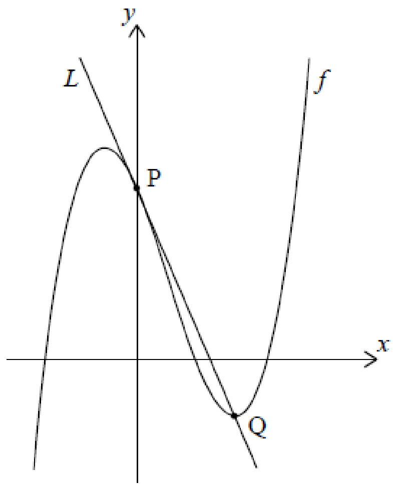
$= \sqrt{\frac{1}{4a} + a}$

This has a minimum value when $a = 0.5$ **(M1)A1**

[4 marks]

6. [Maximum mark: 16] 18N.1.SL.TZ0.S_10

Let $f\left(x\right) = x^3 - 2x^2 + ax + 6$. Part of the graph of f is shown in the following diagram.



The graph of f crosses the y-axis at the point P. The line *L* is tangent to the graph of f at P.

(a) Find the coordinates of P. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

valid approach ***(M1)***

eg $f\left(0\right)$, $0^3 - 2(0)^2 + a\left(0\right) + 6$, $f\left(0\right) = 6$, $\left(0,\ y\right)$

(0, 6) (accept x = 0 **and** y = 6) ***A1 N2***

[2 marks]

(b.i) Find $f'\left(x\right)$. [2]

Markscheme

$f' = 3x^2 - 4x + a$ ***A2 N2***

[2 marks]

(b.ii) Hence, find the equation of *L* in terms of a. [4]

Markscheme

valid approach ***(M1)***

eg $f'(0)$

correct working ***(A1)***

eg $3(0)^2 - 4(0) + a$, slope = a, $f'(0) = a$

attempt to substitute gradient and coordinates into linear equation ***(M1)***

eg $y - 6 = a(x - 0)$, $y - 0 = a(x - 6)$, $6 = a(0) + c$, $L = ax + 6$

correct equation ***A1 N3***

eg $y = ax + 6$, $y - 6 = ax$, $y - 6 = a(x - 0)$

[4 marks]

(c) The graph of f has a local minimum at the point Q. The line L passes through Q.

Find the value of a. [8]

Markscheme

valid approach to find intersection ***(M1)***

eg $f(x) = L$

correct equation ***(A1)***

eg $x^3 - 2x^2 + ax + 6 = ax + 6$

correct working ***(A1)***

eg $x^3 - 2x^2 = 0$, $x^2(x - 2) = 0$

$x = 2$ at Q ***(A1)***

valid approach to find minimum ***(M1)***

eg $f'(x) = 0$

correct equation ***(A1)***

eg $3x^2 - 4x + a = 0$

substitution of **their** value of x at Q into **their** $f'(x) = 0$ equation ***(M1)***

eg $3(2)^2 - 4(2) + a = 0$, $12 - 8 + a = 0$

$a = -4$ ***A1 N0***

[8 marks]

7. [Maximum mark: 7] 18M.1.SL.TZ1.S_7

Consider $f(x)$, $g(x)$ and $h(x)$, for $x \in \mathbb{R}$ where $h(x) = (f \circ g)(x)$.

Given that $g(3) = 7$, $g'(3) = 4$ and $f'(7) = -5$, find the gradient of the normal to the curve of h at $x = 3$. [7]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

recognizing the need to find h' ***(M1)***

recognizing the need to find $h'(3)$ (seen anywhere) ***(M1)***

evidence of choosing chain rule ***(M1)***

eg $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x},\ f'(g(3)) \times g'(3),\ f'(g) \times g'$

correct working ***(A1)***

eg $f'(7) \times 4,\ -5 \times 4$

$h'(3) = -20$ ***(A1)***

evidence of taking **their** negative reciprocal for normal ***(M1)***

eg $-\frac{1}{h'(3)},\ m_1 m_2 = -1$

gradient of normal is $\frac{1}{20}$ ***A1 N4***

[7 marks]

8. [Maximum mark: 6] 22M.1.AHL.TZ1.8

Consider the curve $y = 2x(4 - \mathrm{e}^x)$.

(a.i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

Markscheme

use of product rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2(4 - \mathrm{e}^x) + 2x(-\mathrm{e}^x)$ ***A1***

$= 8 - 2\mathrm{e}^x - 2x\mathrm{e}^x$

[2 marks]

(a.ii) Find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. [2]

Markscheme

use of product rule ***(M1)***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\mathrm{e}^x - 2\mathrm{e}^x - 2x\mathrm{e}^x$ ***A1***

$= -4\mathrm{e}^x - 2x\mathrm{e}^x$

$= -2(2 + x)\mathrm{e}^x$

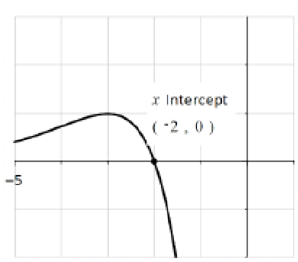
[2 marks]

(b) The curve has a point of inflexion at $(a,\ b)$.

Find the value of a. [2]

Markscheme

$-2(2 + a)\mathrm{e}^a = 0$ **OR** sketch of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ with x-intercept indicated **OR** finding the local maximum of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $(-2,\ 8.27)$ ***(M1)***



$(a =)\ -2$ ***A1***

[2 marks]

9. [Maximum mark: 5] 23M.1.SL.TZ1.5

Line L_1 is tangent to the graph of a function $f(x)$ at the point $\mathrm{P}(3,\ -1)$. Line L_2 is given by the equation $y=-\frac{1}{2}x-\frac{5}{2}$ and is perpendicular to L_1.

(a) Write down the gradient of L_1. [1]

Markscheme

2 ***A1***

[1 mark]

(b) Find the equation of L_1 in the form $y=mx+c$. [2]

Markscheme

attempt to substitute their part (a) and point $(3,\ -1)$ into the slope-intercept form or point-slope form of an equation ***(M1)***

$-1=2\times3+c$ **OR** $y+1=2(x-3)$

$y=2x-7$ ***A1***

Note: Equation must be in the form $y=mx+c$ for ***A1*** to be awarded.

[2 marks]

(c) Show that L_2 is not the line that is normal to $f(x)$ at point P. [2]

Markscheme

METHOD 1

attempt to show that P does not lie on L_2 ***(M1)***

e.g. $-\frac{1}{2}(3)-\frac{5}{2}$ **OR** graph showing L_2 and P in approximate correct locations

$-1\neq-\frac{1}{2}(3)-\frac{5}{2}$ $(-1\neq-4)$ **OR** $(3,\ -1)$ does not lie on the graph of L_2 ***R1***

hence L_2 is not the normal line to $f(x)$ at point P ***AG***

METHOD 2

attempt to find the equation of the normal line at $(3,\ -1)$ ***(M1)***

($-1=-\frac{1}{2}(3)+c$ **OR** $y+1=-\frac{1}{2}(x-3)$)

the normal line is $y=-\frac{1}{2}x+\frac{1}{2}$ ***R1***

hence L_2 is not the normal line to $f(x)$ at point P ***AG***

METHOD 3

attempt to find the intersection of L_1 and L_2 ***(M1)***

Intersection of $y=2x-7$ and $y=-\frac{1}{2}x-\frac{5}{2}$ is $(1.8,\ -3.4)$

$x=1.8\neq 3$ **OR** $y=-3.4\neq -1$ ***R1***

hence L_2 is not the normal line to $f(x)$ at point P ***AG***

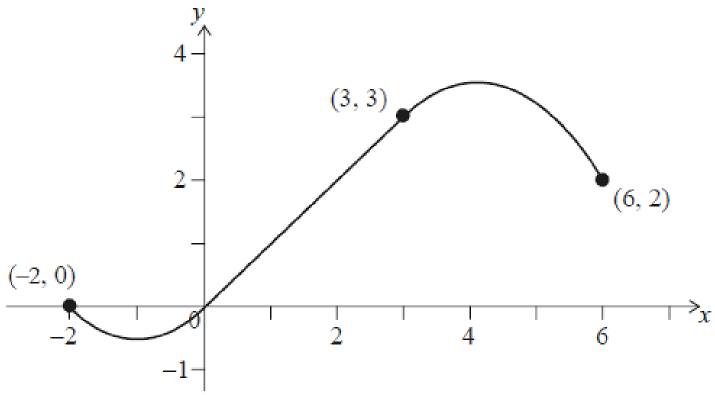
Note: Accept equivalent written arguments provided values are seen.

Methods 1 and 2 are independent of the answers in (a) and (b) but ***FT*** marks can be given for Method 3.

[2 marks]

10. [Maximum mark: 9] 23M.1.AHL.TZ1.10

A decorative hook can be modelled by the curve with equation $y = f(x)$. The graph of $y = f(x)$ is shown and consists of a line segment from $(0,\ 0)$ to $(3,\ 3)$ and two sections formed by quadratic curves.



(a) Write down the equation of the line segment for $0 \leq x \leq 3$. [1]

Markscheme

$y = x$ ***A1***

[1 mark]

The quadratic curve, with endpoints $(-2,\ 0)$ and $(0,\ 0)$, has the same gradient at $(0,\ 0)$ as the line segment.

(b) Find the equation of the curve between $(-2,\ 0)$ and $(0,\ 0)$. [3]

Markscheme

METHOD 1

equation has the form $y = ax^2 + bx + c$

when $x = 0,\ y = 0$ so $c = 0$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b$

attempt to find the value of b by setting *their* derivative equal to 1 when x is 0 ***(M1)***

$2a(0) + b = 1$

$b = 1$ ***(A1)***

when $x = -2,\ y = 0$

$a = \frac{1}{2}$ (and hence $y = \frac{1}{2}x^2 + x$) ***A1***

METHOD 2

equation has the form $y = ax(x+2)$ OR $y = ax^2 + 2ax$ ***A1***

$\frac{dy}{dx} = 2ax + 2a$

attempt to find the value of a by setting *their* derivative equal to 1 when x is 0 ***(M1)***

$a = \frac{1}{2}$ (and hence $y = \frac{1}{2}x^2 + x$) ***A1***

Note: Writing $y = x(x+2)$ is incorrect and gains no marks.

[3 marks]

The second quadratic curve, with endpoints $(3,\ 3)$ and $(6,\ 2)$, has the same gradient at $(3,\ 3)$ as the line segment.

(c) Find the equation of this curve. [4]

Markscheme

equation is $y = ax^2 + bx + c$

finding an expression for $\frac{dy}{dx}$ with unknown coefficients ***(M1)***

$\frac{dy}{dx} = 2ax + b$

setting up two equations using two points AND/OR one equation using the

gradient function ***(M1)***

three correct equations ***(A1)***

$9a + 3b + c = 3$

$36a + 6b + c = 2$

$6a + b = 1$

$a = -\frac{4}{9},\ b = \frac{11}{3},\ c = -4\ \left(a = -0.444444\ldots,\ b = 3.66666\ldots,\ c = -4\right)$ ***A1***

(and hence $y = -\frac{4}{9}x^2 + \frac{11}{3}x - 4$)

[4 marks]

(d) Write down f as a piecewise function. [1]

Markscheme

$$f\left(x\right)=\begin{cases}\frac{1}{2}x^2+x & ,-2\le x\le 0\\ x & ,\ 0\le x\le 3\\ -\frac{4}{9}x^2+\frac{11}{3}x-4, & 3<x\le 6\end{cases}$$ ***A1***

Note: Condone open or closed endpoints for all intervals.

Condone y in place of $f(x)$.

Allow ***FT*** from parts (a), (b) and (c).

[1 mark]

11. [Maximum mark: 7] 22M.1.SL.TZ2.11

Consider the function $f(x) = x^2 - \frac{3}{x},\ x \neq 0$.

(a) Find $f\prime(x)$. [2]

Markscheme

$(f\prime(x) =)\ 2x + \frac{3}{x^2}$ ***A1A1***

Note: Award ***A1*** for $2x$, ***A1*** for $+\frac{3}{x^2}$ **OR** $= 3x^{-2}$

[2 marks]

Line L is a tangent to $f(x)$ at the point $(1,\ -2)$.

(b) Use your answer to part (a) to find the gradient of L. [2]

Markscheme

attempt to substitute 1 into their part (a) ***(M1)***

$(f\prime(1) =)\ 2(1) + \frac{3}{1^2}$

5 ***A1***

[2 marks]

(c) Determine the number of lines parallel to L that are tangent to $f(x)$. Justify your answer. [3]

Markscheme

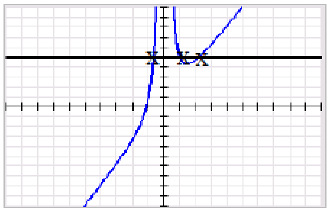
EITHER

$5 = 2x + \frac{3}{x^2}$ ***M1***

$x = -0.686,\ 1,\ 2.19\ \ (-0.686140\ldots,\ 1,\ 2.18614\ldots)$ ***A1***

OR

sketch of $y = f\prime(x)$ with line $y = 5$ ***M1***



three points of intersection marked on this graph ***A1***

(and it can be assumed no further intersections occur outside of this window)

THEN

there are two other tangent lines to $f(x)$ that are parallel to L ***A1***

Note: The final ***A1*** can be awarded provided two solutions other than $x = 1$ are shown **OR** three points of intersection are marked on the graph.

Award ***M1A1A1*** for an answer of "3 lines" where L is considered to be parallel with itself (given guide definition of parallel lines), but only if working is shown.

[3 marks]

12. [Maximum mark: 14] 19M.2.SL.TZ1.T_6

The function $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + kx + 5$ has a local maximum and a local minimum. The local maximum is at $x = -3$.

(a) Show that $k = -6$. [5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$x^2 + x + k$ ***(A1)(A1)(A1)***

Note: Award ***(A1)*** for each correct term. Award at most ***(A1)(A1)(A0)*** if additional terms are seen or for an answer $x^2 + x - 6$. If their derivative is seen in parts (b), (c) or (d) and not in part (a), award at most ***(A1)(A1)(A0)***.

$(-3)^2 + (-3) + k = 0$ ***(M1)(M1)***

Note: Award ***(M1)*** for substituting in $x = -3$ into their derivative and ***(M1)*** for setting it equal to zero. Substituting $k = -6$ invalidates the process, award at most (***A1)(A1)(A1)(M0)(M0)***.

$(k =) - 6$ ***(AG)***

Note: For the final ***(M1)*** to be awarded, no incorrect working must be seen, and must lead to the conclusion $k = -6$. The final ***(AG)*** must be seen.

[5 marks]

(b) Find the coordinates of the local **minimum**. [2]

Markscheme

(2, –2.33) **OR** $\left(2, -\frac{7}{3}\right)$ ***(A1)(A1)***

Note: Award ***(A1)*** for each correct coordinate. Award ***(A0)(A1)*** if parentheses are missing. Accept $x = 2, y = -2.33$. Award ***(M1)(A0)*** for their derivative, a quadratic expression with –6 substituted for k, equated to zero but leading to an incorrect answer.

[2 marks]

(c) Write down the interval where the gradient of the graph of $f(x)$ is negative. [2]

Markscheme

$-3 < x < 2$ ***(A1)*(ft)*(A1)***

Note: Award ***(A1)*** for $x > -3$, ***(A1)*(ft)** for $x < 2$. Follow through for their "2" in part (b). It is possible to award ***(A0)(A1)***. For $-3 < y < 2$ award ***(A1)(A0)***. Accept equivalent notation such as (–3, 2). Award ***(A0)(A1)*(ft)** for $-3 \leqslant x \leqslant 2$.

[2 marks]

(d) Determine the equation of the normal at $x = -2$ in the form $y = mx + c$. [5]

Markscheme

−4 ***(A1)*(ft)**

Note: Award ***(A1)*(ft)** for the gradient of the tangent seen. If an incorrect derivative was used in part (a), then working for their $f'(-2)$ must be seen. Follow through from their derivative in part (a).

gradient of normal is $\frac{1}{4}$ ***(A1)*(ft)**

Note: Award ***(A1)*(ft)** for the negative reciprocal of their gradient of tangent. Follow through within this part. Award ***(G2)*** for an unsupported gradient of the normal.

$\frac{49}{3}$ $\left(f(-2) = \frac{1}{3}(-2)^3 + \frac{1}{2}(-2)^2 - 6(-2) + 5 = \frac{49}{3}\right)$ ***(A1)***

Note: Award ***(A1)*** for $\frac{49}{3}$ (16.3333…) seen.

$\frac{49}{3} = \frac{1}{4}(-2) + c$ **OR** $y - \frac{49}{3} = \frac{1}{4}(x - -2)$ ***(M1)***

Note: Award ***(M1)*** for substituting their normal gradient into equation of line formula.

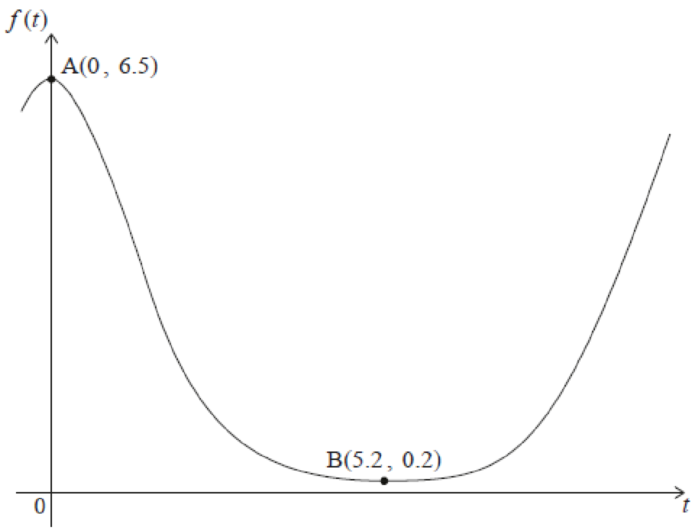
$y = \frac{1}{4}x + \frac{101}{6}$ **OR** $y = 0.25x + 16.8333\ldots$ ***(A1)*(ft)*(G4)***

Note: Award ***(G4)*** for the correct equation of line in correct form without any prior working. The final ***(A1)*(ft)** is contingent on $y = \frac{49}{3}$ and $x = -2$.

[5 marks]

13. [Maximum mark: 8] 22M.1.AHL.TZ1.17

A function f is of the form $f(t) = pe^{q\cos(rt)},\ p,\ q,\ r \in \mathbb{R}^{+}$. Part of the graph of f is shown.



The points A and B have coordinates $\mathrm{A}(0,\ 6.5)$ and $\mathrm{B}(5.2,\ 0.2)$, and lie on f.

The point A is a local maximum and the point B is a local minimum.

Find the value of p, of q and of r. [8]

Markscheme

substitute coordinates of A

$f(0) = pe^{q\cos(0)} = 6.5$

$6.5 = pe^{q}$ ***(A1)***

substitute coordinates of B

$f(5.2) = pe^{q\cos(5.2r)} = 0.2$

EITHER

$f\prime(t) = -pqr\sin(rt)e^{q\cos(rt)}$ ***(M1)***

minimum occurs when $-pqr\sin(5.2r)e^{q\cos(5.2r)} = 0$

$\sin(rt) = 0$

$r \times 5.2 = \pi$ ***(A1)***

OR

minimum value occurs when $\cos(rt) = -1$ ***(M1)***

$r \times 5.2 = π$ *(A1)*

OR

period $= 2 \times 5.2 = 10.4$ *(A1)*

$r = \frac{2π}{10.4}$ *(M1)*

THEN

$r = \frac{π}{5.2} = 0.604152\ldots\ (0.604)$ *A1*

$0.2 = p\mathrm{e}^{-q}$ *(A1)*

eliminate p or q *(M1)*

$\mathrm{e}^{2q} = \frac{6.5}{0.2}$ **OR** $0.2 = \frac{p^2}{6.5}$

$q = 1.74\ (1.74062\ldots)$ *A1*

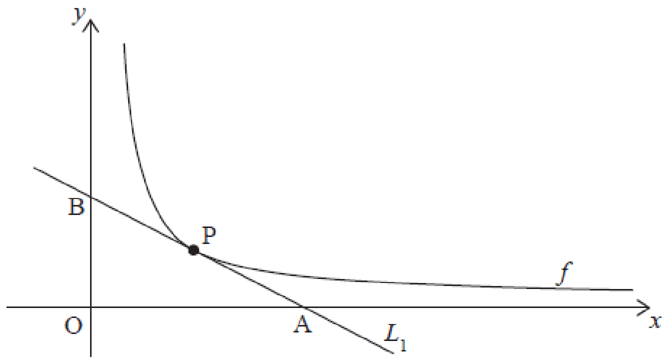
$p = 1.14017\ldots\ (1.14)$ *A1*

[8 marks]

14. [Maximum mark: 15] 20N.1.SL.TZ0.S_10

The following diagram shows part of the graph of $f(x) = \frac{k}{x}$, for $x > 0,\ k > 0$.

Let $\mathrm{P}\left(p,\ \frac{k}{p}\right)$ be any point on the graph of f. Line L_1 is the tangent to the graph of f at P.



(a.i) Find $f\prime(p)$ in terms of k and p. [2]

Markscheme

$f\prime(x) = -kx^{-2}$ *(A1)*

$f\prime(p) = -kp^{-2}\ \left(= -\frac{k}{p^2}\right)$ ***A1 N2***

[2 marks]

(a.ii) Show that the equation of L_1 is $kx + p^2y - 2pk = 0$. [2]

Markscheme

attempt to use point and gradient to find equation of L_1 ***M1***

eg $y - \frac{k}{p} = -kp^{-2}(x - p),\ \ \frac{k}{p} = -\frac{k}{p^2}(p) + b$

correct working leading to answer ***A1***

eg $p^2y - kp = -kx + kp,\ \ y - \frac{k}{p} = -\frac{k}{p^2}x + \frac{k}{p},\ \ y = -\frac{k}{p^2}x + \frac{2k}{p}$

$kx + p^2y - 2pk = 0$ ***AG N0***

[2 marks]

Line L_1 intersects the x-axis at point $\mathrm{A}(2p,\ 0)$ and the y-axis at point B.

(b) Find the area of triangle AOB in terms of k. [5]

Markscheme

METHOD 1 – area of a triangle

recognizing $x = 0$ at B ***(M1)***

correct working to find y-coordinate of null ***(A1)***

eg $p^2y - 2pk = 0$

y-coordinate of null at $y = \frac{2k}{p}$ (may be seen in area formula) ***A1***

correct substitution to find area of triangle ***(A1)***

eg $\frac{1}{2}(2p)\left(\frac{2k}{p}\right), \ p \times \left(\frac{2k}{p}\right)$

area of triangle $\mathrm{AOB} = 2k$ ***A1 N3***

METHOD 2 – integration

recognizing to integrate L_1 between 0 and $2p$ ***(M1)***

eg $\int_0^{2p} L_1 \ \mathrm{d}\ x\ , \ \int_0^{2p} -\frac{k}{p^2}x + \frac{2k}{p}$

correct integration of **both** terms ***A1***

eg $-\frac{kx^2}{2p^2} + \frac{2kx}{p}\ , \ -\frac{k}{2p^2}x^2 + \frac{2k}{p}x + c\ , \ \left[-\frac{k}{2p^2}x^2 + \frac{2k}{p}x\right]_0^{2p}$

substituting limits into **their** integrated function and subtracting (in either order) ***(M1)***

eg $-\frac{k(2p)^2}{2p^2} + \frac{2k(2p)}{p} - (0), \ -\frac{4kp^2}{2p^2} + \frac{4kp}{p}$

correct working ***(A1)***

eg $-2k + 4k$

area of triangle $\mathrm{AOB} = 2k$ ***A1 N3***

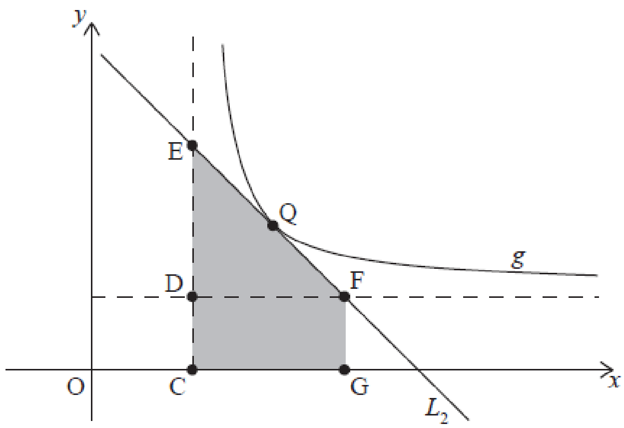
[5 marks]

(c) The graph of f is translated by $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ to give the graph of g.

In the following diagram:

- point Q lies on the graph of g
- points C, D and E lie on the vertical asymptote of g
- points D and F lie on the horizontal asymptote of g
- point G lies on the x-axis such that FG is parallel to DC.

Line L_2 is the tangent to the graph of g at Q, and passes through E and F.



Given that triangle EDF and rectangle CDFG have equal areas, find the gradient of L_2 in terms of p . [6]

Markscheme

Note: In this question, the second ***M*** mark may be awarded independently of the other marks, so it is possible to award ***(M0)(A0)M1(A0)(A0)A0***.

recognizing use of transformation ***(M1)***

eg area of triangle AOB = area of triangle DEF, $g(x) = \frac{k}{x-4} + 3$, gradient of $L_2 =$ gradient of L_1, $\mathrm{D}(4,\ 3),\ 2\mathrm{p}{+}4,$ one correct shift

correct working ***(A1)***

eg area of triangle $\mathrm{DEF} = 2k,\ \mathrm{CD} = 3,\ \mathrm{DF} = 2p,\ \mathrm{CG} = 2p,\ \mathrm{E}\left(4,\ \frac{2k}{p} + 3\right),\ \mathrm{F}(2p+4,\ 3),\ \mathrm{Q}\left(p+4,\ \frac{k}{p} + 3\right),$

gradient of $L_2 = -\frac{k}{p^2},\ g\prime(x) = -\frac{k}{(x-4)^2}$, area of rectangle $\mathrm{CDFG} = 2k$

valid approach ***(M1)***

eg $\frac{\mathrm{ED}\times\mathrm{DF}}{2} = \mathrm{CD}\times\mathrm{DF},\ 2p\cdot 3 = 2k\ ,\ \mathrm{ED} = 2\mathrm{CD}\ ,\ \int_4^{2p+4} L_2\,\mathrm{d}x = 4k$

correct working ***(A1)***

eg $\mathrm{ED} = 6,\ \mathrm{E}(4,\ 9),\ k = 3p,\ \text{gradient} = \frac{3-\left(\frac{2k}{p}+3\right)}{(2p+4)-4},\ \frac{-6}{\left(\frac{2k}{3}\right)},\ -\frac{9}{k}$

correct expression for gradient (in terms of p) ***(A1)***

eg $\frac{-6}{2p},\ \frac{9-3}{4-(2p+4)},\ -\frac{3p}{p^2},\ \frac{3-\left(\frac{2(3p)}{p}+3\right)}{(2p+4)-4},\ -\frac{9}{3p}$

gradient of L_2 is $-\frac{3}{p}\ \ \left(= -3p^{-1}\right)$ ***A1 N3***

[6 marks]

15. [Maximum mark: 6] 20N.1.SL.TZ0.T_13

Consider the graph of the function $f(x) = x^2 - \frac{k}{x}$.

(a) Write down $f\prime(x)$. [3]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$2x + \frac{k}{x^2}$ ***(A1)(A1)(A1) (C3)***

Note: Award ***(A1)*** for $2x$, ***(A1)*** for $+k$, and ***(A1)*** for x^{-2} or $\frac{1}{x^2}$.
Award at most ***(A1)(A1)(A0)*** if additional terms are seen.

[3 marks]

The equation of the tangent to the graph of $y = f(x)$ at $x = -2$ is $2y = 4 - 5x$.

(b) Write down the gradient of this tangent. [1]

Markscheme

$-2.5 \ \left(\frac{-5}{2}\right)$ ***(A1) (C1)***

[1 mark]

(c) Find the value of k. [2]

Markscheme

$-2.5 = 2 \times (-2) + \frac{k}{(-2)^2}$ ***(M1)***

Note: Award ***(M1)*** for equating their gradient from part (b) to their substituted derivative from part (a).

$(k =)\ 6$ ***(A1)*(ft)** ***(C2)***

Note: Follow through from parts (a) and (b).

[2 marks]

16. [Maximum mark: 15] 19M.1.SL.TZ2.S_9

Let θ be an **obtuse** angle such that $\sin\theta = \frac{3}{5}$.

(a) Find the value of $\tan\theta$. [4]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

evidence of valid approach ***(M1)***

eg sketch of triangle with sides 3 and 5, $\cos^2\theta = 1 - \sin^2\theta$

correct working ***(A1)***

eg missing side is 4 (may be seen in sketch), $\cos\theta = \frac{4}{5}$, $\cos\theta = -\frac{4}{5}$

$\tan\theta = -\frac{3}{4}$ ***A2 N4***

[4 marks]

(b) Line L passes through the origin and has a gradient of $\tan\theta$. Find the equation of L. [2]

Markscheme

correct substitution of either gradient **or** origin into equation of line ***(A1)***

(do not accept $y = mx + b$)

eg $y = x\tan\theta,\ y - 0 = m(x - 0),\ y = mx$

$y = -\frac{3}{4}x$ ***A2 N4***

Note: Award ***A1A0*** for $L = -\frac{3}{4}x$.

[2 marks]

Let $f(x) = \mathrm{e}^x \sin x - \frac{3x}{4}$.

(c) Find the derivative of f. [5]

Markscheme

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{-3x}{4}\right) = -\frac{3}{4}$ (seen anywhere, including answer) ***A1***

choosing product rule ***(M1)***

eg $uv' + vu'$

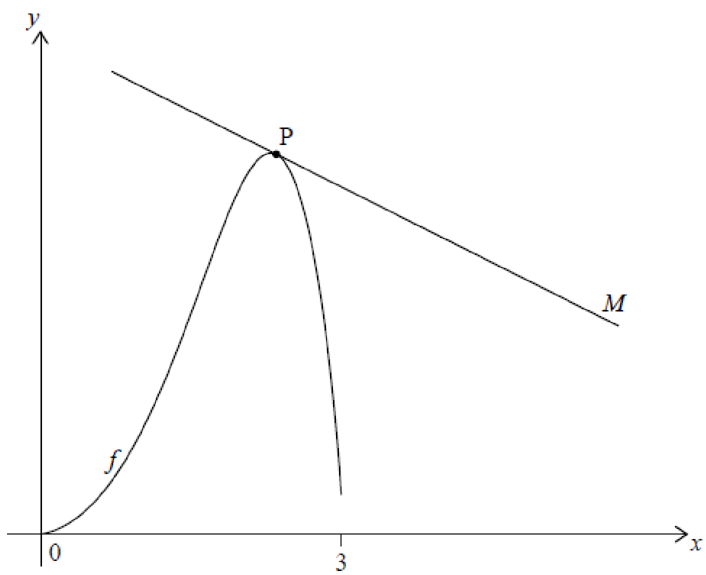
correct derivatives (must be seen in a correct product rule) ***A1A1***

eg $\cos x$, e^x

$f'(x) = e^x \cos x + e^x \sin x - \frac{3}{4} \ \left(= e^x \ (\cos x + \sin x) - \frac{3}{4}\right)$ ***A1 N5***

[5 marks]

(d) The following diagram shows the graph of f for $0 \le x \le 3$. Line M is a tangent to the graph of f at point P.



Given that M is parallel to L, find the x-coordinate of P. [4]

Markscheme

valid approach to equate **their** gradients ***(M1)***

eg $f' = \tan\theta,\ f' = -\frac{3}{4}$, $e^x \cos x + e^x \sin x - \frac{3}{4} = -\frac{3}{4},\ e^x \ (\cos x + \sin x) - \frac{3}{4} = -\frac{3}{4}$

correct equation without e^x ***(A1)***

eg $\sin x = -\cos x$, $\cos x + \sin x = 0$, $\frac{-\sin x}{\cos x} = 1$

correct working ***(A1)***

eg $\tan\theta = -1$, $x = 135^\circ$

$x = \frac{3\pi}{4}$ (do not accept 135°) ***A1 N1***

Note: Do not award the final ***A1*** if additional answers are given.

[4 marks]

17. [Maximum mark: 16] 19M.2.SL.TZ1.S_9

Let $f(x) = \frac{16}{x}$. The line L is tangent to the graph of f at $x = 8$.

(a) Find the gradient of L. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to find $f'(8)$ ***(M1)***

eg $f'(x)$, y', $-16x^{-2}$

−0.25 (exact) ***A1 N2***

[2 marks]

L can be expressed in the form $\boldsymbol{r} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} + t\boldsymbol{u}$.

(b) Find $\boldsymbol{u}$. [2]

Markscheme

$\boldsymbol{u} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ or any scalar multiple ***A2 N2***

[2 marks]

The direction vector of $y = x$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(c) Find the acute angle between $y = x$ and L. [5]

Markscheme

correct scalar product and magnitudes ***(A1)(A1)(A1)***

scalar product $= 1 \times 4 + 1 \times -1 \quad (= 3)$

magnitudes $= \sqrt{1^2 + 1^2}, \sqrt{4^2 + (-1)^2} \left(= \sqrt{2}, \sqrt{17}\right)$

substitution of their values into correct formula ***(M1)***

eg $\frac{4-1}{\sqrt{1^2+1^2}\sqrt{4^2+(-1)^2}}$, $\frac{-3}{\sqrt{2}\sqrt{17}}$, 2.1112, 120.96°

1.03037 , 59.0362°

angle = 1.03 , 59.0° ***A1 N4***

[5 marks]

(d.i) Find $(f \circ f)(x)$. [3]

Markscheme

attempt to form composite $(f \circ f)(x)$ ***(M1)***

eg $f(f(x))$, $f\left(\frac{16}{x}\right)$, $\frac{16}{f(x)}$

correct working ***(A1)***

eg $\frac{16}{\frac{16}{x}}$, $16 \times \frac{x}{16}$

$(f \circ f)(x) = x$ ***A1 N2***

[3 marks]

(d.ii) Hence, write down $f^{-1}(x)$. [1]

Markscheme

$f^{-1}(x) = \frac{16}{x}$ (accept $y = \frac{16}{x}$, $\frac{16}{x}$) ***A1 N1***

Note: Award ***A0*** in part (ii) if part (i) is incorrect.
Award ***A0*** in part (ii) if the candidate has found $f^{-1}(x) = \frac{16}{x}$ by interchanging x and y.

[1 mark]

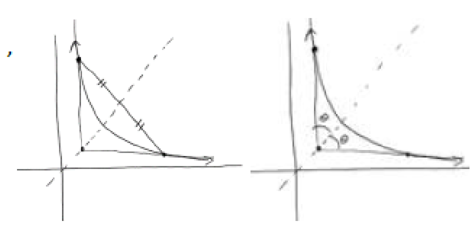
(d.iii) Hence or otherwise, find the obtuse angle formed by the tangent line to f at $x = 8$ and the tangent line to f at $x = 2$. [3]

Markscheme

METHOD 1

recognition of symmetry about $y = x$ ***(M1)***

eg (2, 8) ⇔ (8, 2)



evidence of doubling **their** angle ***(M1)***

eg 2×1.03, 2×59.0

2.06075, 118.072°

2.06 (radians) (118 degrees)   ***A1 N2***

METHOD 2

finding direction vector for tangent line at $x = 2$ ***(A1)***

eg $\begin{pmatrix}-1\\4\end{pmatrix}, \begin{pmatrix}1\\-4\end{pmatrix}$

substitution of **their** values into correct formula (must be from vectors)   ***(M1)***

eg $\frac{-4-4}{\sqrt{1^2+4^2}\sqrt{4^2+(-1)^2}}, \frac{8}{\sqrt{17}\sqrt{17}}$

2.06075, 118.072°

2.06 (radians) (118 degrees)   ***A1 N2***

METHOD 3

using trigonometry to find an angle with the horizontal   ***(M1)***

eg $\tan\theta = -\frac{1}{4},\ \tan\theta = -4$

finding both angles of rotation   ***(A1)***

eg $\theta_1 = 0.244978,\ 14.0362^\circ,\ \theta_1 = 1.81577,\ 104.036^\circ$

2.06075, 118.072°

2.06 (radians) (118 degrees)   ***A1 N2***

[3 marks]

18. [Maximum mark: 7] 19M.2.SL.TZ2.T_5

Consider the function $f\left(x\right)=\frac{1}{3}x^{3}+\frac{3}{4}x^{2}-x-1$.

(d) Find $f'\left(x\right)$. [3]

Markscheme

$x^{2}+\frac{3}{2}x-1$ ***(A1)(A1)(A1)***

Note: Award ***(A1)*** for each correct term. Award at most ***(A1)(A1)(A0)*** if there are extra terms.

[3 marks]

(e) Find the gradient of the graph of $y=f\left(x\right)$ at $x=2$. [2]

Markscheme

$2^{2}+\frac{3}{2}\times 2-1$ ***(M1)***

Note: Award ***(M1)*** for correct substitution of 2 in their derivative of the function.

6 ***(A1)*****(ft)*****(G2)***

Note: Follow through from part (d).

[2 marks]

(f) Find the equation of the tangent line to the graph of $y=f\left(x\right)$ at $x=2$. Give the equation in the form $ax+by+d=0$ where, $a,b,$ and $d\in\mathbb{Z}$. [2]

Markscheme

$\frac{8}{3}=6\left(2\right)+c$ ***(M1)***

Note: Award ***(M1)*** for 2, their part (a) and their part (e) substituted into equation of a straight line.

$c=-\frac{28}{3}$

OR

$\left(y-\frac{8}{3}\right)=6\left(x-2\right)$ ***(M1)***

Note: Award ***(M1)*** for 2, their part (a) and their part (e) substituted into equation of a straight line.

OR

$y=6x-\frac{28}{3}\ \left(y=6x-9.33333\ldots\right)$ ***(M1)***

Note: Award ***(M1)*** for their answer to (e) and intercept $-\frac{28}{3}$ substituted in the gradient-intercept line equation.

$-18x+3y+28=0$ (accept integer multiples) ***(A1)*****(ft)*****(G2)***

Note: Follow through from parts (a) and (e).

[2 marks]

19. [Maximum mark: 4] 19M.2.AHL.TZ1.H_1

Let l be the tangent to the curve $y = x\mathrm{e}^{2x}$ at the point (1, e^2).

Find the coordinates of the point where l meets the x-axis. [4]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

equation of tangent is $y = 22.167\ldots x - 14.778\ldots$ **OR** $y = -7.389\ldots = 22.167\ldots(x - 1)$ ***(M1)(A1)***

meets the x-axis when $y = 0$

$x = 0.667$

meets x-axis at (0.667, 0) $\left(= \left(\frac{2}{3},\ 0\right)\right)$ ***A1A1***

Note: Award ***A1*** for $x = \frac{2}{3}$ or $x = 0.667$ seen and ***A1*** for coordinates (x, 0) given.

METHOD 1

Attempt to differentiate ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{2x} + 2x\mathrm{e}^{2x}$

when $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\mathrm{e}^2$ ***(M1)***

equation of the tangent is $y - \mathrm{e}^2 = 3\mathrm{e}^2\,(x - 1)$

$y = 3\mathrm{e}^2x - 2\mathrm{e}^2$

meets x-axis at $x = \frac{2}{3}$

$\left(\frac{2}{3},\ 0\right)$ ***A1A1***

Note: Award ***A1*** for $x = \frac{2}{3}$ or $x = 0.667$ seen and ***A1*** for coordinates (x, 0) given.

[4 marks]

20. [Maximum mark: 6] 18N.1.SL.TZ0.T_11

Consider the curve $y = 5x^3 - 3x$.

(a) Find $\frac{dy}{dx}$. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$15x^2 - 3$ ***(A1)(A1) (C2)***

Note: Award ***(A1)*** for $15x^2$, ***(A1)*** for -3. Award at most ***(A1)(A0)*** if additional terms are seen.

[2 marks]

The curve has a tangent at the point P(−1, −2).

(b) Find the gradient of this tangent at point P. [2]

Markscheme

$15\,(-1)^2 - 3$ ***(M1)***

Note: Award ***(M1)*** for substituting −1 into their $\frac{dy}{dx}$.

$= 12$ ***(A1)*****(ft)** ***(C2)***

Note: Follow through from part (a).

[2 marks]

(c) Find the equation of this tangent. Give your answer in the form $y = mx + c$. [2]

Markscheme

$(y - (-2)) = 12\,(x - (-1))$ ***(M1)***

OR

$-2 = 12(-1) + c$ ***(M1)***

Note: Award ***(M1)*** for point **and** their gradient substituted into the equation of a line.

$y = 12x + 10$ ***(A1)*****(ft)** ***(C2)***

Note: Follow through from part (b).

[2 marks]

21. [Maximum mark: 6] 18M.1.SL.TZ2.T_14

Consider the function $f\left(x\right)=\frac{x^4}{4}$.

(a) Find $f'(x)$ [1]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

x^3 ***(A1) (C1)***

Note: Award ***(A0)*** for $\frac{4x^3}{4}$ and not simplified to x^3.

[1 mark]

(b) Find the gradient of the graph of f at $x=-\frac{1}{2}$. [2]

Markscheme

$\left(-\frac{1}{2}\right)^3$ ***(M1)***

Note: Award ***(M1)*** for correct substitution of $-\frac{1}{2}$ into their derivative.

$-\frac{1}{8}$ (-0.125) ***(A1)*(ft) *(C2)***

Note: Follow through from their part (a).

[2 marks]

(c) Find the x-coordinate of the point at which the **normal** to the graph of f has gradient $-\frac{1}{8}$. [3]

Markscheme

$x^3=8$ ***(A1)(M1)***

Note: Award ***(A1)*** for 8 seen maybe seen as part of an equation $y=8x+c$, ***(M1)*** for equating their derivative to 8.

$(x=)\ 2$ ***(A1) (C3)***

Note: Do not accept (2, 4).

[3 marks]