Models [71 marks]

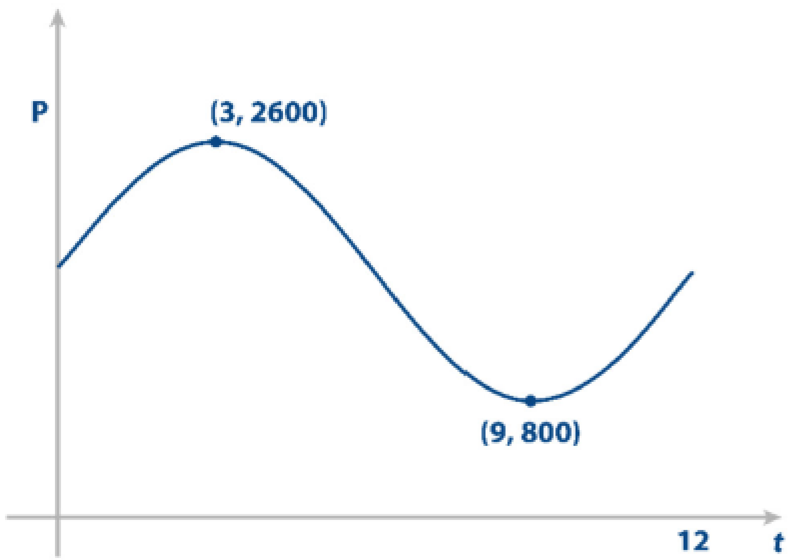
1. [Maximum mark: 7] EXN.1.SL.TZ0.6

The size of the population (P) of migrating birds in a particular town can be approximately modelled by the equation

$P = a\ \sin(bt) + c,\quad a,\ b,\ c \in \mathbb{R}^+$, where t is measured in months from the time of the initial measurements.

In a 12 month period the maximum population is 2600 and occurs when $t = 3$ and the minimum population is 800 and occurs when $t = 9$.

This information is shown on the graph below.



(a.i) Find the value of a. [2]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$\frac{2600-800}{2} = 900$ **(M1)A1**

[2 marks]

(a.ii) Find the value of b. [2]

Markscheme

$\frac{360}{12} = 30$ **(M1)A1**

Note: Accept $\frac{2\pi}{12} = 0.524\ (0.523598\ldots)$.

[2 marks]

(a.iii) Find the value of c. [1]

Markscheme

$\frac{2600+800}{2} = 1700$ **A1**

[1 mark]

(b) Find the value of t at which the population first reaches 2200. [2]

Markscheme

Solve $900\sin(30t) + 1700 = 2200$ **(M1)**

$t = 1.12\ (1.12496\ldots)$ **A1**

[2 marks]

2. [Maximum mark: 5] EXN.1.SL.TZ0.2

A factory produces engraved gold disks. The cost C of the disks is directly proportional to the cube of the radius r of the disk.

A disk with a radius of 0.8 cm costs 375 US dollars (USD).

(a) Find an equation which links C and r. [3]

Markscheme

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$C = kr^3$ **(M1)**

$375 = k \times 0.8^3 \Rightarrow k = 732\ (732.421\ldots)$ **(M1)**

$C = 732r^3$ **A1**

[3 marks]

(b) Find, to the nearest USD, the cost of disk that has a radius of 1.1 cm. [2]

Markscheme

$C = 732.42\ldots \times 1.1^3$ **(M1)**

$C = \$975\ (974.853\ldots)$ **A1**

Note: accept $\$974$ from use of $C = 732r^3$.

[2 marks]

3. [Maximum mark: 13] EXM.2.SL.TZ0.3

Urvashi wants to model the height of a moving object. She collects the following data showing the height, h metres, of the object at time t seconds.

t (seconds)	2	5	7
h (metres)	34	38	24

She believes the height can be modeled by a quadratic function, $h\left(t\right)=at^{2}+bt+c$, where $a,\ b,\ c\in\mathbb{R}$.

(a) Show that $4a+2b+c=34$. [1]

Markscheme

$t=2,\ h=34\Rightarrow\ \ 34=a2^{2}+2b+c$ ***M1***

$\Rightarrow\ 34=4a+2b+c$ ***AG***

[1 mark]

(b) Write down two more equations for a, b and c. [3]

Markscheme

attempt to substitute either (5, 38) or (7, 24) ***M1***

$25a+5b+c=38$ ***A1***

$49a+7b+c=24$ ***A1***

[3 marks]

(c) Solve this system of three equations to find the value of a, b and c. [4]

Markscheme

$a = -\frac{5}{3},\ b = 13,\ c = \frac{44}{3}$ ***M1A1A1A1***

[3 marks]

Hence find

(d.i) when the height of the object is zero. [3]

Markscheme

$-\frac{5}{3}t^2 + 13t + \frac{44}{3} = 0$ ***M1***

$t = 8.8$ seconds ***M1A1***

[3 marks]

(d.ii) the maximum height of the object. [2]

Markscheme

attempt to find maximum height, e.g. sketch of graph ***M1***

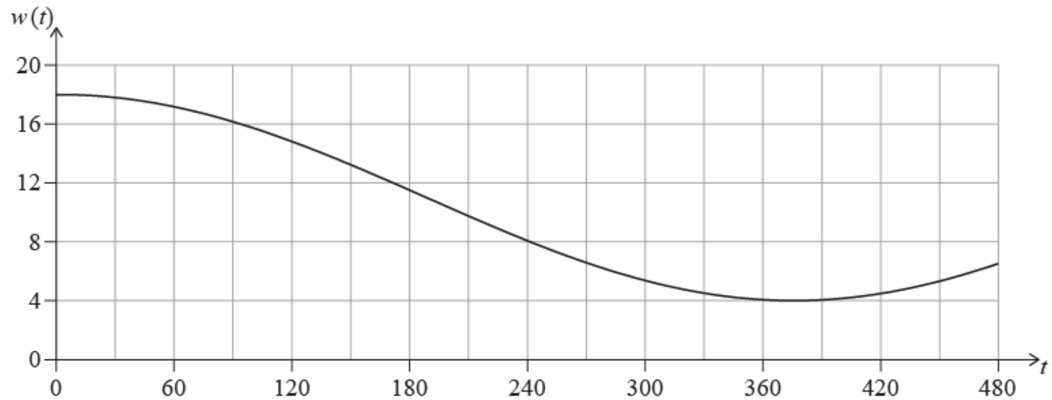
$h = 40.0$ metres ***A1***

[2 marks]

4. [Maximum mark: 15] 23M.2.SL.TZ1.3

The depth of water, w metres, in a particular harbour can be modelled by the function $w(t) = a\cos(bt^{\circ}) + d$ where t is the length of time, in minutes, after $06:00$.

On 20 January, the first high tide occurs at $06:00$, at which time the depth of water is $18\,\mathrm{m}$. The following low tide occurs at $12:15$ when the depth of water is $4\,\mathrm{m}$. This is shown in the diagram.



(a) Find the value of a. [2]

Markscheme

$\frac{18-4}{2}$ *(M1)*

$(a =)\ 7$ ***A1***

[2 marks]

(b) Find the value of d. [2]

Markscheme

$\frac{18+4}{2}$ **OR** $18-7$ **OR** $4+7$ *(M1)*

$(d =)\ 11$ ***A1***

[2 marks]

(c) Find the period of the function in minutes. [3]

Markscheme

(time between high and low tide is) 6h15m **OR** 375 minutes ***(A1)***

multiplying by 2 ***(M1)***

750 minutes ***A1***

[3 marks]

(d) Find the value of b. [2]

Markscheme

EITHER

$\frac{360^{\circ}}{b} = 750$ ***(A1)***

OR

$7\cos(b \times 375) + 11 = 4$ ***(A1)***

THEN

$(b =)\ 0.48$ ***A1***

Note: Award ***A1A0*** for an answer of $\frac{2\pi}{750}\ \left(=\frac{\pi}{375}=0.00837758\ldots\right)$.

[2 marks]

Naomi is sailing to the harbour on the morning of 20 January. Boats can enter or leave the harbour only when the depth of water is at least $6\ \mathrm{m}$.

(e) Find the latest time before $12:00$, to the nearest minute, that Naomi can enter the harbour. [4]

Markscheme

equating their cos function to 6 **OR** graphing their cos function and 6 ***(M1)***

$7\cos(0.48t)+11=6$

$\Rightarrow t=282.468\ldots$ (minutes) ***(A1)***

$=4.70780\ldots$ (hr) **OR** 4hr 42 mins (4hr $42.4681\ldots$ mins) ***(A1)***

so the time is $10:42$ ***A1***

[4 marks]

(f) Find the length of time (in minutes) between $06:00$ and $15:00$ on 20 January during which Naomi **cannot** enter or leave the harbour. [2]

Markscheme

next solution is $t\ =\ 467.531\ldots$ **(A1)**

$467.531\ldots - 282.468\ldots$

185 (mins) $(185.063\ldots)$ ***A1***

Note: Accept an (unsupported) answer of 186 (from correct 3 sf values for t)

[2 marks]

5. [Maximum mark: 6] 22N.1.SL.TZ0.5

Celeste heated a cup of coffee and then let it cool to room temperature. Celeste found the coffee's temperature, T, measured in $^\circ\mathrm{C}$, could be modelled by the following function,

$$T(t) = 71\mathrm{e}^{-0.0514t} + 23,\ t \geq 0,$$

where t is the time, in minutes, after the coffee started to cool.

(a) Find the coffee's temperature 16 minutes after it started to cool. [2]

Markscheme

$71\mathrm{e}^{-0.0514(16)} + 23$ ***(M1)***

$54.2\,^\circ\mathrm{C}\ \ (54.1956\ldots)$ ***A1***

[2 marks]

The graph of T has a horizontal asymptote.

(b) Write down the equation of the horizontal asymptote. [1]

Markscheme

$T = 23$ ***A1***

Note: Condone $y = 23$.

[1 mark]

(c) Write down the room temperature. [1]

Markscheme

$23\,^{\circ}\mathrm{C}$ *A1*

[1 mark]

(d) Given that $T^{-1}(50) = k$, find the value of k. [2]

Markscheme

$50 = 71\mathrm{e}^{-0.0514(k)} + 23$ *(M1)*

$k = 18.8 \quad \left(\frac{-5000}{257}\ln\left(\frac{27}{71}\right),\ 18.8101\ldots\right)$ *A1*

Note: Award ***M1*** for a sketch showing a point of intersection between the exponential function and $y = 50$.

[2 marks]

6. [Maximum mark: 6] 22N.1.SL.TZ0.10

Stars are classified by their brightness. The brightest stars in the sky have a magnitude of 1. The magnitude, m, of another star can be modelled as a function of its brightness, b, relative to a star of magnitude 1, as shown by the following equation.

$$m = 1 - 2.5 \log_{10}(b)$$

The star called Acubens has a brightness of 0.0525.

(a) Find the magnitude of Acubens. [2]

Markscheme

$m = 1 - 2.5 \log_{10}(0.0525)$ ***(M1)***

$= 4.20 \ \ (4.19960\ldots)$ ***A1***

[2 marks]

Ceres has a magnitude of 7 and is the least bright star visible without magnification.

(b) Find the brightness of Ceres. [2]

Markscheme

attempt to solve $7 = 1 - 2.5 \log_{10}(b)$ ***(M1)***

Note: Accept a sketch from their GDC as an attempt to solve $7 = 1 - 2.5 \log_{10}(b)$.

$b = 0.00398 \ \ (0.00398107\ldots)$ ***A1***

[2 marks]

(c) Find how many times brighter Acubens is compared to Ceres. [2]

Markscheme

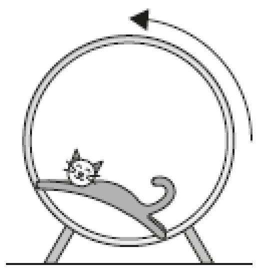
$\frac{0.0525}{0.00398107}$ ***(M1)***

$= 13.2 \ \ (13.1874\ldots)$ ***A1***

[2 marks]

7. [Maximum mark: 6] 22N.1.SL.TZ0.12

A cat runs inside a circular exercise wheel, making the wheel spin at a constant rate in an anticlockwise direction. The height, $h\,\mathrm{cm}$, of a fixed point, P, on the wheel can be modelled by $h(t) = a\sin(bt) + c$ where t is the time in seconds and $a,\ b,\ c \in \mathbb{R}^{+}$.



When $t = 0$, point P is at a height of $78\,\mathrm{cm}$.

(a) Write down the value of c. [1]

Markscheme

78 ***A1***

[1 mark]

When $t = 4$, point P first reaches its maximum height of $143\,\mathrm{cm}$.

(b.i) Find the value of a. [1]

Markscheme

65 ***A1***

[1 mark]

(b.ii) Find the value of b. [2]

Markscheme

EITHER

(period =) 16 (could be seen on sketch) ***(M1)***

$b = \frac{2\pi}{16}$ **OR** $b = \frac{360^\circ}{16}$

$(b =)\ 0.393\ \left(0.392699\ldots,\ \frac{\pi}{8}\right)$ **OR** $(b =)\ 22.5^\circ$

A1

OR

$143 = 65\sin(4b) + 78$ ***(M1)***

$(\sin(4b) = 1)$

$(4b = \frac{\pi}{2}$ **OR** $4b = 90^\circ)$

$(b =)\ 0.393\ \left(0.392699\ldots,\ \frac{\pi}{8}\right)$ **OR** $(b =)\ 22.5^\circ$

A1

[2 marks]

(c) Write down the minimum height of point P. [1]

Markscheme

13 ***A1***

Note: Apply follow through marking only if their final answer is positive.

[1 mark]

Later, the cat is tired, and it takes twice as long for point P to complete one revolution at a new constant rate.

(d) Write down the new value of b. [1]

Markscheme

$(b =)\ 0.196\ \left(0.196349\ldots,\ \frac{\pi}{16}\right)$ **OR** $(b =)\ 11.3^{\circ}\ \ (11.25^{\circ})$ ***A1***

[1 mark]

8. [Maximum mark: 5] 22M.1.SL.TZ1.3

The height of a baseball after it is hit by a bat is modelled by the function

$$h(t) = -4.8t^2 + 21t + 1.2$$

where $h(t)$ is the height in metres above the ground and t is the time in seconds after the ball was hit.

(a) Write down the height of the ball above the ground at the instant it is hit by the bat. [1]

Markscheme

1.2 metres ***A1***

[1 mark]

(b) Find the value of t when the ball hits the ground. [2]

Markscheme

$-4.8t^2 + 21t + 1.2 = 0$ ***(M1)***

$(t =)\ 4.43\text{s} \quad (4.431415\ldots\ \text{s})$ ***A1***

Note: If both values for t are seen do not award the ***A1*** mark unless the negative is explicitly excluded.

[2 marks]

(c) State an appropriate domain for t in this model. [2]

Markscheme

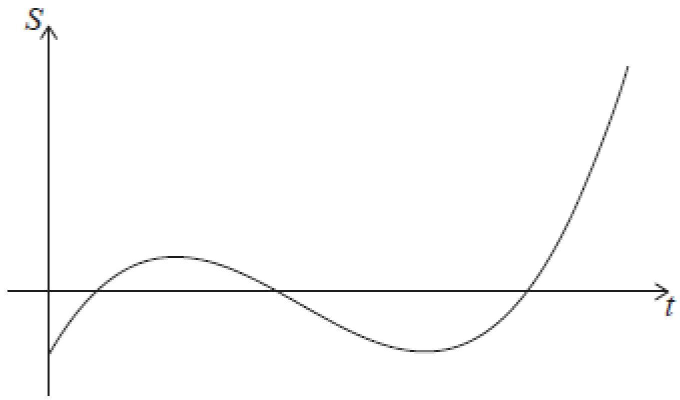
$0 \le t \le 4.43$ **OR** $[0,\ 4.43]$ ***A1A1***

Note: Award ***A1*** for correct endpoints and ***A1*** for expressing answer with correct notation. Award at most ***A1A0*** for use of x instead of t.

[2 marks]

9. [Maximum mark: 8] 22M.1.SL.TZ2.9

The graph below shows the average savings, S thousand dollars, of a group of university graduates as a function of t, the number of years after graduating from university.



(a) Write down one feature of this graph which suggests a cubic function might be appropriate to model this scenario. [1]

Markscheme

Accept any one of the following (or equivalent):

one minimum and one maximum point
three x-intercepts or three roots (or zeroes)
one point of inflexion ***R1***

Note: Do not accept "S shape" as a justification.

[1 mark]

The equation of the model can be expressed in the form $S = at^3 + bt^2 + ct + d$, where $a,\ b,\ c$ and d are real constants.

The graph of the model must pass through the following four points.

t	0	1	2	3
S	–5	3	–1	–5

(b.i) Write down the value of d. [1]

Markscheme

$(d =) -5$ ***A1***

[1 mark]

(b.ii) Write down three simultaneous equations for $a,\ b$ and c. [2]

Markscheme

$8 = a + b + c$

$4 = 8a + 4b + 2c$

$0 = 27a + 9b + 3c$ ***A2***

Note: Award ***A2*** if all three equations are correct.
Award ***A1*** if at least one is correct. Award ***A1*** for three correct equations that include the letter "d".

[2 marks]

(b.iii) Hence, or otherwise, find the values of $a,\ b$ and c. [1]

Markscheme

$a = 2,\ b = -12,\ c = 18$ ***A1***

[1 mark]

A negative value of S indicates that a graduate is expected to be in debt.

(c) Use the model to determine the total length of time, in years, for which a graduate is expected to be in debt after graduating from university. [3]

Markscheme

equating found expression to zero ***(M1)***

$0 = 2t^3 - 12t^2 + 18t - 5$

$t = 0.358216\ldots,\ 1.83174\ldots,\ 3.81003\ldots$ ***(A1)***

(so total time in debt is $3.81003\ldots - 1.83174\ldots + 0.358216 \approx$)

$2.34\ \ (2.33650\ldots)$ years ***A1***

[3 marks]