Models [71 marks]

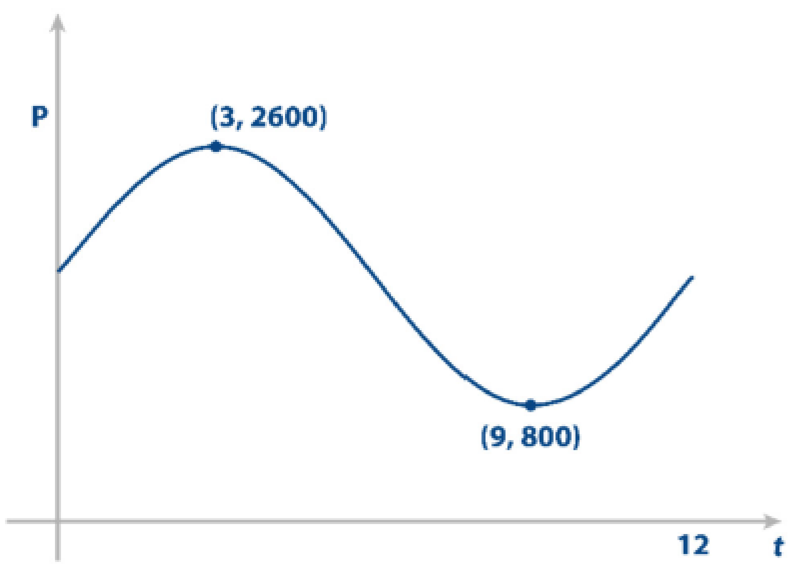
1. [Maximum mark: 7] EXN.1.SL.TZ0.6

The size of the population (P) of migrating birds in a particular town can be approximately modelled by the equation

$P = a\sin(bt) + c, \quad a, b, c \in \mathbb{R}^+$, where t is measured in months from the time of the initial measurements.

In a 12 month period the maximum population is 2600 and occurs when $t = 3$ and the minimum population is 800 and occurs when $t = 9$.

This information is shown on the graph below.



(a.i) Find the value of a. [2]

(a.ii) Find the value of b. [2]

(a.iii) Find the value of c. [1]

(b) Find the value of t at which the population first reaches 2200. [2]

2. [Maximum mark: 5] EXN.1.SL.TZ0.2

A factory produces engraved gold disks. The cost C of the disks is directly proportional to the cube of the radius r of the disk.

A disk with a radius of 0.8 cm costs 375 US dollars (USD).

(a) Find an equation which links C and r. [3]

(b) Find, to the nearest USD, the cost of disk that has a radius of 1.1 cm. [2]

3. [Maximum mark: 13] EXM.2.SL.TZ0.3

Urvashi wants to model the height of a moving object. She collects the following data showing the height, h metres, of the object at time t seconds.

t (seconds)	2	5	7
h (metres)	34	38	24

She believes the height can be modeled by a quadratic function, $h(t) = at^2 + bt + c$, where $a,\ b,\ c \in \mathbb{R}$.

(a) Show that $4a + 2b + c = 34$. [1]

(b) Write down two more equations for a, b and c. [3]

(c) Solve this system of three equations to find the value of a, b and c. [4]

Hence find

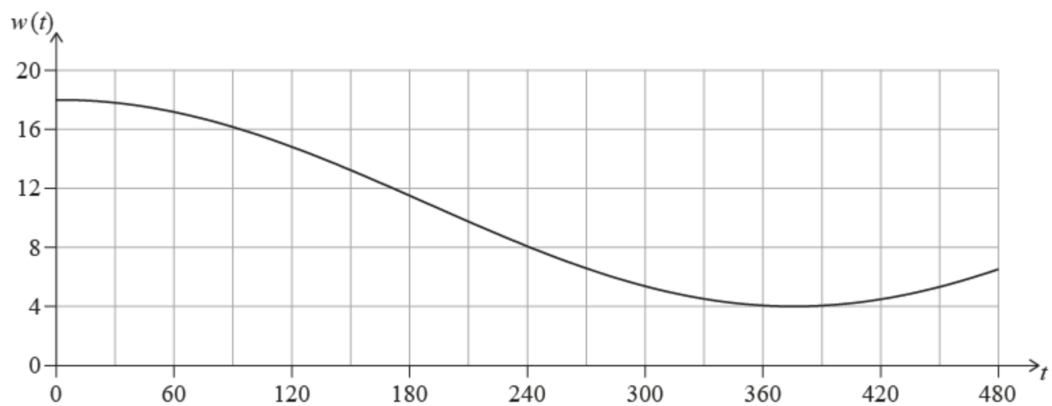
(d.i) when the height of the object is zero. [3]

(d.ii) the maximum height of the object. [2]

4. [Maximum mark: 15] 23M.2.SL.TZ1.3

The depth of water, w metres, in a particular harbour can be modelled by the function $w(t) = a \cos\left(bt^{\circ}\right) + d$ where t is the length of time, in minutes, after $06:00$.

On 20 January, the first high tide occurs at $06:00$, at which time the depth of water is $18\,\mathrm{m}$. The following low tide occurs at $12:15$ when the depth of water is $4\,\mathrm{m}$. This is shown in the diagram.



(a) Find the value of a. [2]

(b) Find the value of d. [2]

(c) Find the period of the function in minutes. [3]

(d) Find the value of b. [2]

Naomi is sailing to the harbour on the morning of 20 January. Boats can enter or leave the harbour only when the depth of water is at least $6\,\mathrm{m}$.

(e) Find the latest time before $12:00$, to the nearest minute, that Naomi can enter the harbour. [4]

(f) Find the length of time (in minutes) between $06:00$ and $15:00$ on 20 January during which Naomi **cannot** enter or leave the harbour. [2]

5. [Maximum mark: 6] 22N.1.SL.TZ0.5

Celeste heated a cup of coffee and then let it cool to room temperature. Celeste found the coffee's temperature, T, measured in $^\circ\mathrm{C}$, could be modelled by the following function,

$$T(t) = 71\mathrm{e}^{-0.0514t} + 23,\ t \geq 0,$$

where t is the time, in minutes, after the coffee started to cool.

(a) Find the coffee's temperature 16 minutes after it started to cool. [2]

The graph of T has a horizontal asymptote.

(b) Write down the equation of the horizontal asymptote. [1]

(c) Write down the room temperature. [1]

(d) Given that $T^{-1}(50) = k$, find the value of k. [2]

6. [Maximum mark: 6] 22N.1.SL.TZ0.10

Stars are classified by their brightness. The brightest stars in the sky have a magnitude of 1. The magnitude, m, of another star can be modelled as a function of its brightness, b, relative to a star of magnitude 1, as shown by the following equation.

$$m = 1 - 2.5\log_{10}(b)$$

The star called Acubens has a brightness of 0.0525.

(a) Find the magnitude of Acubens. [2]

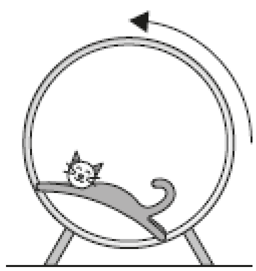
Ceres has a magnitude of 7 and is the least bright star visible without magnification.

(b) Find the brightness of Ceres. [2]

(c) Find how many times brighter Acubens is compared to Ceres. [2]

7. [Maximum mark: 6] 22N.1.SL.TZ0.12

A cat runs inside a circular exercise wheel, making the wheel spin at a constant rate in an anticlockwise direction. The height, $h\,\mathrm{cm}$, of a fixed point, P, on the wheel can be modelled by $h(t) = a\sin(bt) + c$ where t is the time in seconds and $a,\ b,\ c \in \mathbb{R}^+$.



When $t = 0$, point P is at a height of $78\,\mathrm{cm}$.

(a) Write down the value of c. [1]

When $t = 4$, point P first reaches its maximum height of $143\,\mathrm{cm}$.

(b.i) Find the value of a. [1]

(b.ii) Find the value of b. [2]

(c) Write down the minimum height of point P. [1]

Later, the cat is tired, and it takes twice as long for point P to complete one revolution at a new constant rate.

(d) Write down the new value of b. [1]

8. [Maximum mark: 5] 22M.1.SL.TZ1.3

The height of a baseball after it is hit by a bat is modelled by the function

$$h(t) = -4.8t^2 + 21t + 1.2$$

where $h(t)$ is the height in metres above the ground and t is the time in seconds after the ball was hit.

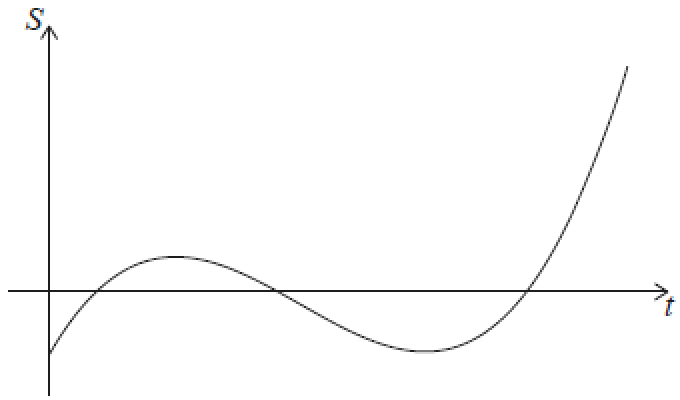
(a) Write down the height of the ball above the ground at the instant it is hit by the bat. [1]

(b) Find the value of t when the ball hits the ground. [2]

(c) State an appropriate domain for t in this model. [2]

9. [Maximum mark: 8] 22M.1.SL.TZ2.9

The graph below shows the average savings, S thousand dollars, of a group of university graduates as a function of t, the number of years after graduating from university.



(a) Write down one feature of this graph which suggests a cubic function might be appropriate to model this scenario. [1]

The equation of the model can be expressed in the form $S = at^3 + bt^2 + ct + d$, where $a,\ b,\ c$ and d are real constants.

The graph of the model must pass through the following four points.

t	0	1	2	3
S	−5	3	−1	−5

(b.i) Write down the value of d. [1]

(b.ii) Write down three simultaneous equations for $a,\ b$ and c. [2]

(b.iii) Hence, or otherwise, find the values of $a,\ b$ and c. [1]

A negative value of S indicates that a graduate is expected to be in debt.

(c) Use the model to determine the total length of time, in years, for which a graduate is expected to be in debt after graduating from university. [3]