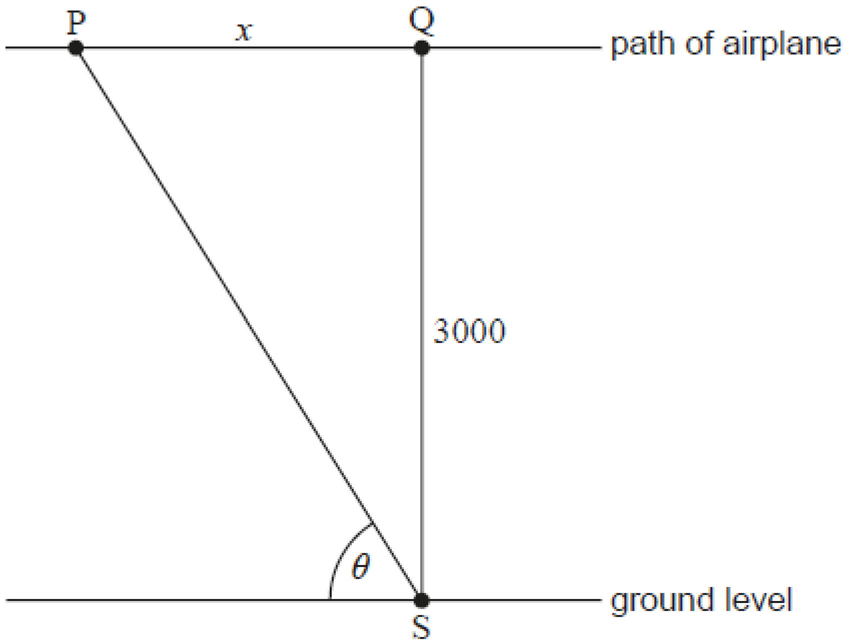
Optimization + related rates [65 marks]

1. [Maximum mark: 9] 23M.1.AHL.TZ1.17

An airplane, P, is flying at a constant altitude of $3000\,\mathrm{m}$ at a speed of $250\,\mathrm{m\,s^{-1}}$. Its path passes over a tracking station, S, at ground level. Let Q be the point $3000\,\mathrm{m}$ directly above the tracking station.

At a particular time, T, as the airplane is flying towards Q, the angle of elevation, θ, of the airplane from S is increasing at a rate of 0.075 radians per second. The distance from Q to P is given by x.



(a) Use related rates to show that, at time T, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{10000}{3}$. [2]

Markscheme

attempt to use the chain rule to set up a related rate ***(M1)***

correct expression ***A1***

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}x}{\mathrm{d}t} \div \frac{\mathrm{d}\theta}{\mathrm{d}t} \textbf{ OR } \frac{-250}{0.075}$$

$$= -\frac{10000}{3}$$ ***AG***

[2 marks]

(b) Find $x(\theta), x$ as a function of θ. [1]

Markscheme

$x(\theta) = \frac{3000}{\tan\theta}$ ***A1***

[1 mark]

(c) Find an expression for $\frac{\mathrm{d}x}{\mathrm{d}\theta}$ in terms of $\sin\theta$. [3]

Markscheme

attempt to use chain rule **OR** quotient rule ***(M1)***

$\frac{-3000}{\tan^2\theta\times\cos^2\theta}, \frac{-3000(\sin\theta(-\sin\theta)-\cos^2\theta)}{\sin^2\theta}$ ***(A1)***

$= -\frac{3000}{\sin^2\theta}$ ***A1***

[3 marks]

(d) Hence find the horizontal distance from the station to the plane at time T. [3]

Markscheme

setting their equation in part (c) equal to the given expression in part (a) ***(M1)***

$$-\frac{3000}{2\sin^2\theta} = -\frac{10000}{3}$$

$\theta = 1.24904\ldots$ ***(A1)***

$x(1.24904\ldots) = 1000\,\mathrm{m}$ ***A1***

[3 marks]

2. [Maximum mark: 5] 23M.1.AHL.TZ2.12

A spherical balloon is being inflated such that its volume is increasing at a rate of $15\ \mathrm{cm}^3\ \mathrm{s}^{-1}$.

(a) Find the radius of the balloon when its volume is $288\pi\ \mathrm{cm}^3$. [2]

Markscheme

equating volume of sphere formula to 288π ***(M1)***

$\frac{4}{3}\pi r^3 = 288\pi$

$\Rightarrow r = 6\ (\mathrm{cm})$ ***A1***

[2 marks]

(b) Hence or otherwise, find the rate of change of the radius at this instant. [3]

Markscheme

$\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$ (seen anywhere) ***(A1)***

$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t}$ ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}$

$15 = 4\pi \times 6^2 \times \frac{\mathrm{d}r}{\mathrm{d}t}$

$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{15}{144\pi}\ \left(\mathrm{cm\ s}^{-1}\right)\ (0.0332,\ 0.0331572\ldots)$ ***A1***

[3 marks]

3. [Maximum mark: 16] 23M.2.AHL.TZ2.3

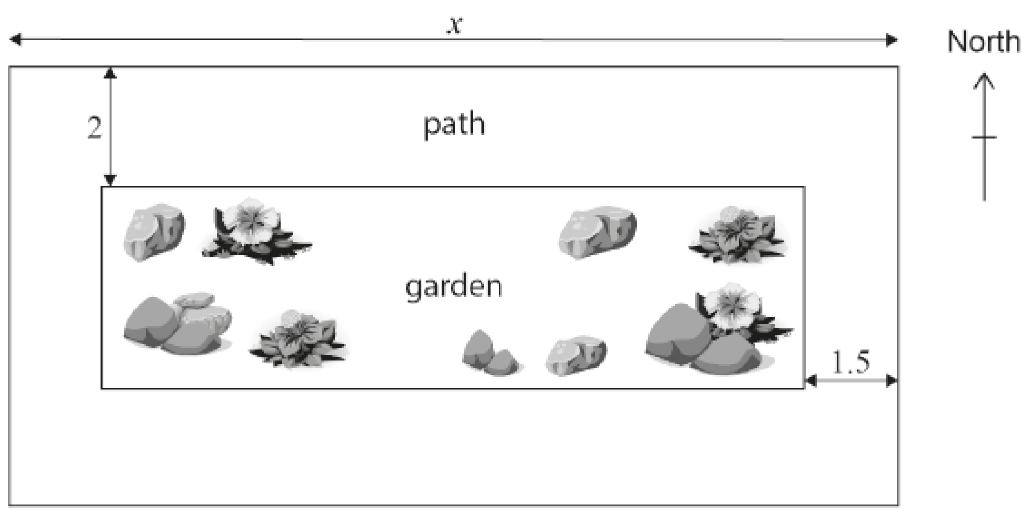
A particular park consists of a rectangular garden, of area $A\,\mathrm{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\mathrm{m}^2$.

The width of the path at the north and south side of the park is $2\,\mathrm{m}$.

The width of the path at the west and east side of the park is $1.5\,\mathrm{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale

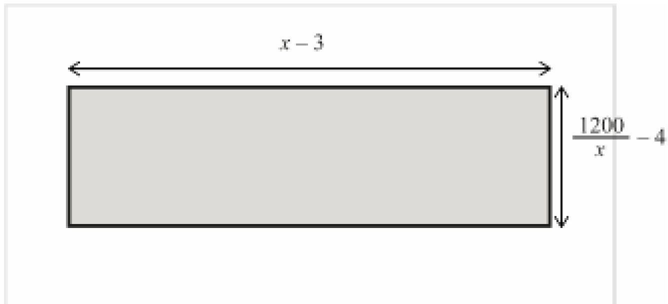


(a) Show that $A = 1212 - 4x - \frac{3600}{x}$. [5]

Markscheme

Note: In methods 1 and 2, full marks are available for candidates who work with a dummy variable, e.g. y, that represents the width of the park and hence is equal to $\frac{1200}{x}$. The substitution to express an answer in only x may come as late as the final line.

METHOD 1 (finding dimensions of garden)



(width of park =) $\frac{1200}{x}$ ***(A1)***

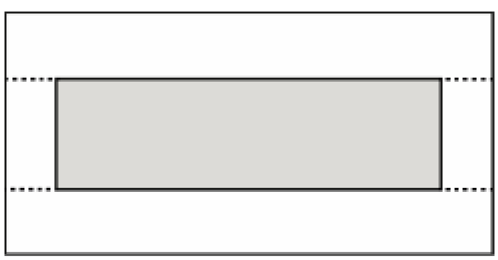
(length of garden =) $x - 3$, (width of garden =) $\frac{1200}{x} - 4$ ***(A1)(A1)***

$A = (x - 3) \times \left(\frac{1200}{x} - 4\right)$ ***A1***

$= 1200 - 4x - \frac{3600}{x} + 12$ ***A1***

$= 1212 - 4x - \frac{3600}{x}$ ***AG***

METHOD 2 (subtracting the area of the path)



width of park $= \frac{1200}{x}$ ***(A1)***

attempt to cut path into 4 (or 8) pieces ***(M1)***

four (or eight) areas of the path expressed in terms of x ***(A1)***

$A = 1200 - 2x - 2x - 1.5\left(\frac{1200}{x} - 4\right) - 1.5\left(\frac{1200}{x} - 4\right)$
A1

correct manipulation leading to given result ***A1***

$= 1212 - 4x - \frac{1800}{x} - \frac{1800}{x}$

$= 1212 - 4x - \frac{3600}{x}$ ***AG***

Note: To award ***(M1)(A1)*** without a diagram the division of the park must be clear.

[5 marks]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\mathrm{m}^2$. [4]

Markscheme

setting $1212 - 4x - \frac{3600}{x} = 800$ (accept a sketch) ***(M1)***

$x = 9.64\ (9.64011\ldots)\ (\mathrm{m})$ **OR**
$x = 9.34\ (93.3598\ldots)\ (\mathrm{m})$ ***A1***

(width =) $124\ (124.479\ldots)\ (\mathrm{m})$ ***A1***

(width =) $12.9\ (12.8534\ldots)\ (\mathrm{m})$ ***A1***

Note: To award the final ***A1*** both values of x **and** both values of the width must be seen. Accept 12.8 for second value of width from candidate dividing 1200 by 3 sf value of 93.4.

[4 marks]

(c) Find an expression for $\frac{\mathrm{d}A}{\mathrm{d}x}$.

[3]

Markscheme

$\left(\frac{\mathrm{d}A}{\mathrm{d}x}=\right)-4+\frac{3600}{x^2}$ **OR** $-4+3600x^{-2}$ ***A1A1A1***

Note: Award ***A1*** for -4, **A1** for $+3600$, and ***A1*** for x^{-2} or x^2 in denominator.

[3 marks]

(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden. [2]

Markscheme

setting *their* $\frac{\mathrm{d}A}{\mathrm{d}x}$ equal to 0 **OR** sketch of *their* $\frac{\mathrm{d}A}{\mathrm{d}x}$ with x-intercept highlighted ***M1***

$(x=)\ 30\ (\mathrm{m})$ ***A1***

Note: To award ***A1FT*** the candidate's value of x must be within the domain given in the problem $(3<x<300)$.

[2 marks]

(e) Find the maximum possible area of the garden. [2]

Markscheme

EITHER

evidence of using GDC to find maximum of graph of $A = 1212 - 4x - \frac{3600}{x}$ ***(M1)***

OR

substitution of *their* x into A ***(M1)***

OR

dividing 1200 by *their* x to find width of park **and** subtracting 3 from *their* x and 4 from the width to find park dimensions ***(M1)***

Note: For the last two methods, only follow through if $3 <$ their $x < 300$.

THEN

$(A =)\ 972\ (\mathrm{m}^2)$ **A1**

[2 marks]

4. [Maximum mark: 7] 22M.1.AHL.TZ1.16

The wind chill index W is a measure of the temperature, in $^\circ\mathrm{C}$, felt when taking into account the effect of the wind.

When Frieda arrives at the top of a hill, the relationship between the wind chill index and the speed of the wind v in kilometres per hour $(\mathrm{km\,h^{-1}})$ is given by the equation

$$W = 19.34 - 7.405v^{0.16}$$

(a) Find an expression for $\frac{\mathrm{d}W}{\mathrm{d}v}$. [2]

Markscheme

use of power rule ***(M1)***

$\frac{\mathrm{d}W}{\mathrm{d}v} = -1.1848v^{-0.84}$ **OR** $-1.18v^{-0.84}$ ***A1***

[2 marks]

(b) When Frieda arrives at the top of a hill, the speed of the wind is 10 kilometres per hour and increasing at a rate of $5\,\mathrm{km\,h^{-1}\,minute^{-1}}$.

Find the rate of change of W at this time. [5]

Markscheme

$\frac{\mathrm{d}v}{\mathrm{d}t} = 5$ ***(A1)***

$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}t} \times \frac{\mathrm{d}W}{\mathrm{d}v}$ ***(M1)***

$\left(\frac{\mathrm{d}W}{\mathrm{d}t} = -5 \times 1.1848v^{-0.84}\right)$

when $v = 10$

$\frac{dW}{dt} = -5 \times 1.1848 \times 10^{-0.84}$ ***(M1)***

$-0.856 \ (-0.856278\ldots)\ ^\circ\mathrm{C\,min}^{-1}$ ***A2***

Note: Accept a negative answer communicated in words, "decreasing at a rate of…".

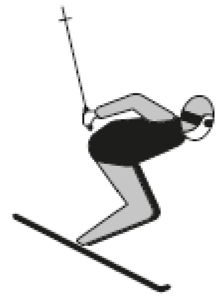
Accept a final answer of $-0.852809\ldots\ ^\circ\mathrm{C\,min}^{-1}$ from use of -1.18.

Accept 51.4 (or 51.2) $^\circ\mathrm{C\,hour}^{-1}$.

[5 marks]

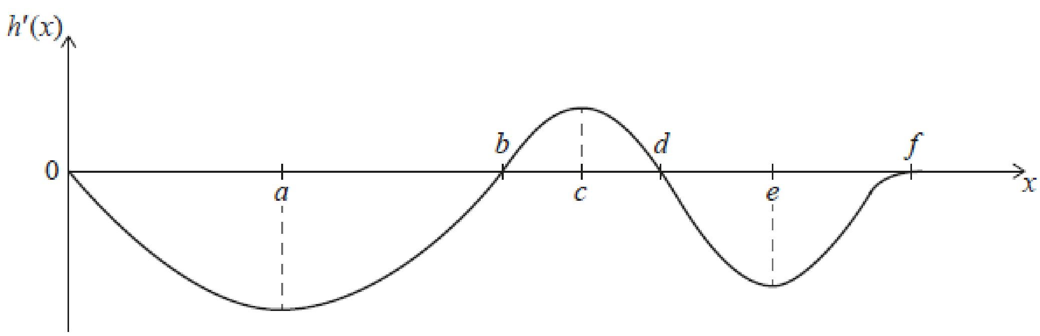
5. [Maximum mark: 5] 21N.1.AHL.TZ0.8

Juri skis from the top of a hill to a finishing point at the bottom of the hill. She takes the shortest route, heading directly to the finishing point (F).



Let $h(x)$ define the height of the hill above F at a horizontal distance x from the starting point at the top of the hill.

The graph of the **derivative** of $h(x)$ is shown below. The graph of $h\prime(x)$ has local minima and maxima when x is equal to $a,\ c$ and e. The graph of $h\prime(x)$ intersects the x-axis when x is equal to $b,\ d$, and f.



(a.i) Identify the x value of the point where $|h\prime(x)|$ has its maximum value. [1]

Markscheme

a ***A1***

[1 mark]

(a.ii) Interpret this point in the given context. [1]

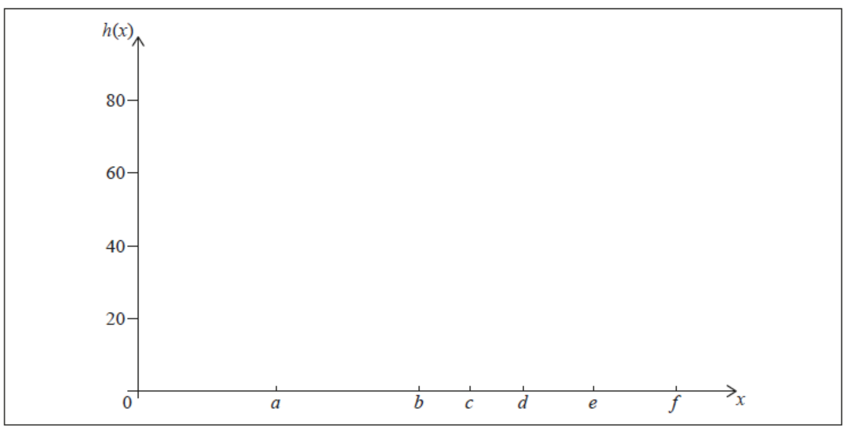
Markscheme

the hill is at its steepest / largest slope of hill ***A1***

[1 mark]

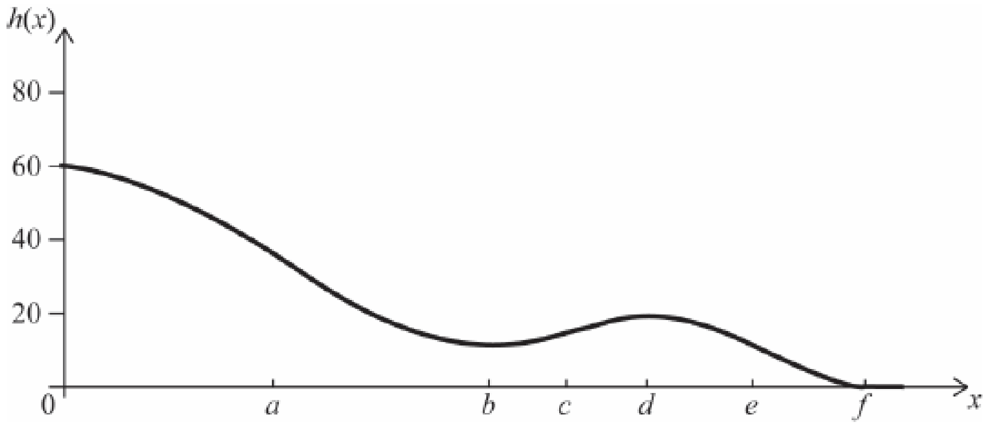
(b) Juri starts at a height of 60 metres and finishes at F, where $x = f$.

Sketch a possible diagram of the hill on the following pair of coordinate axes.



[3]

Markscheme



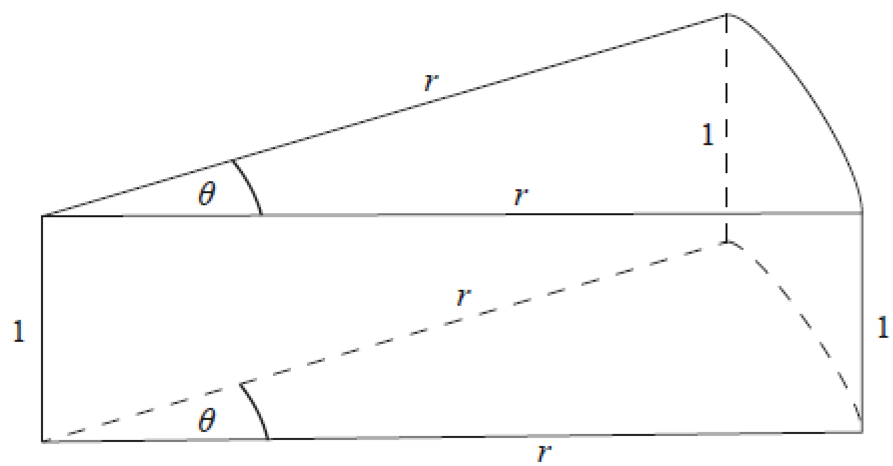
A1A1A1

Note: Award ***(A1)*** for decreasing function from 0 to b and d to f and increasing from b to d; ***(A1)*** for minimum at b and max at d; ***(A1)*** for starting at height of 60 and finishing at a height of 0 at f. If reasonable curvature not evident on graph (i.e. only straight lines used) award ***A1A0A1***.

[3 marks]

6. [Maximum mark: 9] 21N.1.AHL.TZ0.15

The following diagram shows a frame that is made from wire. The total length of wire is equal to 15 cm. The frame is made up of two identical sectors of a circle that are parallel to each other. The sectors have angle θ radians and radius $r\text{ cm}$. They are connected by 1 cm lengths of wire perpendicular to the sectors. This is shown in the diagram below.



(a) Show that $r = \frac{6}{2+\theta}$. [2]

Markscheme

$15 = 3 + 4r + 2r\theta$ ***M1***

$12 = 2r(2 + \theta)$ ***A1***

Note: Award ***A1*** for any reasonable working leading to expected result e,g, factorizing r.

$r = \frac{6}{2+\theta}$ ***AG***

[2 marks]

The faces of the frame are covered by paper to enclose a volume, V.

(b.i) Find an expression for V in terms of θ. [2]

Markscheme

attempt to use sector area to find volume ***(M1)***

volume $= \frac{1}{2}r^2\theta \times 1$

$= \frac{1}{2} \times \frac{36}{(2+\theta)^2} \times \theta \quad \left(= \frac{18\theta}{(2+\theta)^2}\right)$ ***A1***

[2 marks]

(b.ii) Find the expression $\frac{\mathrm{d}V}{\mathrm{d}\theta}$. [3]

Markscheme

$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{(2+\theta)^2 \times 18 - 36\theta(2+\theta)}{(2+\theta)^4}$ ***M1A1A1***

$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{36-18\theta}{(2+\theta)^3}$

[3 marks]

(b.iii) Solve algebraically $\frac{\mathrm{d}V}{\mathrm{d}\theta} = 0$ to find the value of θ that will maximize the volume, V. [2]

Markscheme

$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{36-18\theta}{(2+\theta)^3} = 0$ ***M1***

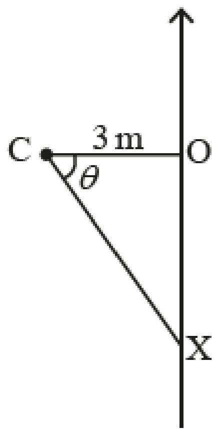
Note: Award this ***M1*** for simplified version equated to zero. The simplified version may have been seen in part (b)(ii).

$\theta = 2$ ***A1***

[2 marks]

7. [Maximum mark: 6] 19M.1.AHL.TZ1.H_5

A camera at point C is 3 m from the edge of a straight section of road as shown in the following diagram. The camera detects a car travelling along the road at $t = 0$. It then rotates, always pointing at the car, until the car passes O, the point on the edge of the road closest to the camera.



A car travels along the road at a speed of 24 ms^{-1}. Let the position of the car be X and let $\mathrm{O\hat{C}X} = \theta$.

Find $\frac{\mathrm{d}\theta}{\mathrm{d}t}$, the rate of rotation of the camera, in radians per second, at the instant the car passes the point O . [6]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

let OX $= x$

METHOD 1

$\frac{\mathrm{d}x}{\mathrm{d}t} = 24$ (or −24) ***(A1)***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \times \frac{\mathrm{d}\theta}{\mathrm{d}x}$ ***(M1)***

$3\tan\theta = x$ ***A1***

EITHER

$3\sec^2\theta = \frac{dx}{d\theta}$ ***A1***

$\frac{d\theta}{dt} = \frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

OR

$\theta = \arctan\left(\frac{x}{3}\right)$

$\frac{d\theta}{dx} = \frac{1}{3} \times \frac{1}{1+\frac{x^2}{9}}$ ***A1***

$\frac{d\theta}{dt} = 24 \times \frac{1}{3\left(1+\frac{x^2}{9}\right)}$

attempt to substitute for $x = 0$ into their differential equation ***M1***

THEN

$\frac{d\theta}{dt} = \frac{24}{3} = 8$ (rad s^{-1}) ***A1***

Note: Accept –8 rad s^{-1}.

METHOD 2

$\frac{dx}{dt} = 24$ (or –24) ***(A1)***

$3\tan\theta = x$ ***A1***

attempt to differentiate implicitly with respect to t ***M1***

$3\sec^2\theta \times \frac{d\theta}{dt} = \frac{dx}{dt}$ ***A1***

$\frac{d\theta}{dt} = \frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3} = 8$ (rad s^{-1}) ***A1***

Note: Accept –8 rad s^{-1}.

Note: Can be done by consideration of CX, use of Pythagoras.

METHOD 3

let the position of the car be at time t be $d - 24t$ from O ***(A1)***

$\tan\theta = \frac{d-24t}{3} \left(= \frac{d}{3} - 8t\right)$ ***M1***

Note: For $\tan\theta = \frac{24t}{3}$ award ***A0M1*** and follow through.

EITHER

attempt to differentiate implicitly with respect to t ***M1***

$\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = -8$ ***A1***

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

OR

$\theta = \arctan\left(\frac{d}{3} - 8t\right)$ ***M1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{8}{1+\left(\frac{d}{3}-8t\right)^2}$ ***A1***

at O, $t = \frac{d}{24}$ ***A1***

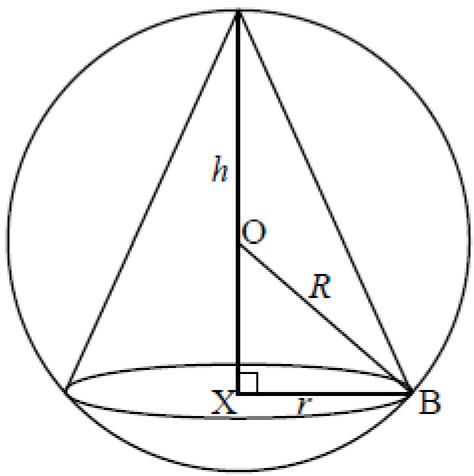
THEN

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -8$ ***A1***

[6 marks]

8. [Maximum mark: 8] 19M.1.AHL.TZ2.H_8

A right circular cone of radius r is inscribed in a sphere with centre O and radius R as shown in the following diagram. The perpendicular height of the cone is h, X denotes the centre of its base and B a point where the cone touches the sphere.



(a) Show that the volume of the cone may be expressed by $V = \frac{\pi}{3}\left(2Rh^2 - h^3\right)$. [4]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to use Pythagoras in triangle OXB ***M1***

$\Rightarrow r^2 = R^2 - (h - R)^2$ ***A1***

substitution of their r^2 into formula for volume of cone $V = \frac{\pi r^2 h}{3}$ ***M1***

$= \frac{\pi h}{3}\left(R^2 - (h - R)^2\right)$

$= \frac{\pi h}{3}\left(R^2 - \left(h^2 + R^2 - 2hR\right)\right)$ ***A1***

Note: This **A** mark is independent and may be seen anywhere for the correct expansion of $(h - R)^2$.

$= \frac{\pi h}{3}\left(2hR - h^2\right)$

$= \frac{\pi}{3}\left(2Rh^2 - h^3\right)$ ***AG***

[4 marks]

(b) Given that there is one inscribed cone having a maximum volume, show that the volume of this cone is $\frac{32\pi R^3}{81}$. [4]

Markscheme

at max, $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$ ***R1***

$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi}{3}\left(4Rh - 3h^2\right)$

$\Rightarrow 4Rh = 3h^2$

$\Rightarrow h = \frac{4R}{3}$ (since $h \neq 0$) ***A1***

EITHER

$V_{\max} = \frac{\pi}{3}\left(2Rh^2 - h^3\right)$ from part (a)

$= \frac{\pi}{3}\left(2R\left(\frac{4R}{3}\right)^2 - \left(\frac{4R}{3}\right)^3\right)$ ***A1***

$= \frac{\pi}{3}\left(2R\frac{16R^2}{9} - \left(\frac{64R^3}{27}\right)\right)$ ***A1***

OR

$r^2 = R^2 - \left(\frac{4R}{3} - R\right)^2$

$r^2 = R^2 - \frac{R^2}{9} = \frac{8R^2}{9}$ ***A1***

$\Rightarrow V_{\max} = \frac{\pi r^2}{3}\left(\frac{4R}{3}\right)$

$= \frac{4\pi R}{9}\left(\frac{8R^2}{9}\right)$ ***A1***

THEN

$= \frac{32\pi R^3}{81}$ ***AG***

[4 marks]