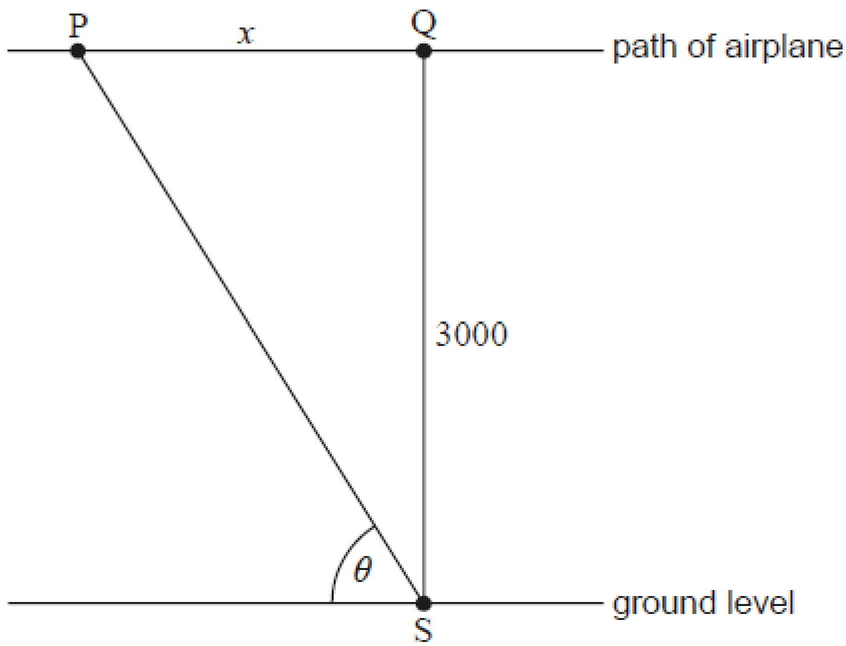
Optimization + related rates [65 marks]

1. [Maximum mark: 9] 23M.1.AHL.TZ1.17

An airplane, P, is flying at a constant altitude of $3000\,\mathrm{m}$ at a speed of $250\,\mathrm{m\,s^{-1}}$. Its path passes over a tracking station, S, at ground level. Let Q be the point $3000\,\mathrm{m}$ directly above the tracking station.

At a particular time, T, as the airplane is flying towards Q, the angle of elevation, θ, of the airplane from S is increasing at a rate of 0.075 radians per second. The distance from Q to P is given by x.



(a) Use related rates to show that, at time T, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{10\,000}{3}$. [2]

(b) Find $x(\theta)$, x as a function of θ. [1]

(c) Find an expression for $\frac{\mathrm{d}x}{\mathrm{d}\theta}$ in terms of $\sin\theta$. [3]

(d) Hence find the horizontal distance from the station to the plane at time T. [3]

2. [Maximum mark: 5] 23M.1.AHL.TZ2.12

A spherical balloon is being inflated such that its volume is increasing at a rate of $15 \text{ cm}^3 \text{ s}^{-1}$.

(a) Find the radius of the balloon when its volume is $288\pi \text{ cm}^3$. [2]

(b) Hence or otherwise, find the rate of change of the radius at this instant. [3]

3. [Maximum mark: 16] 23M.2.AHL.TZ2.3

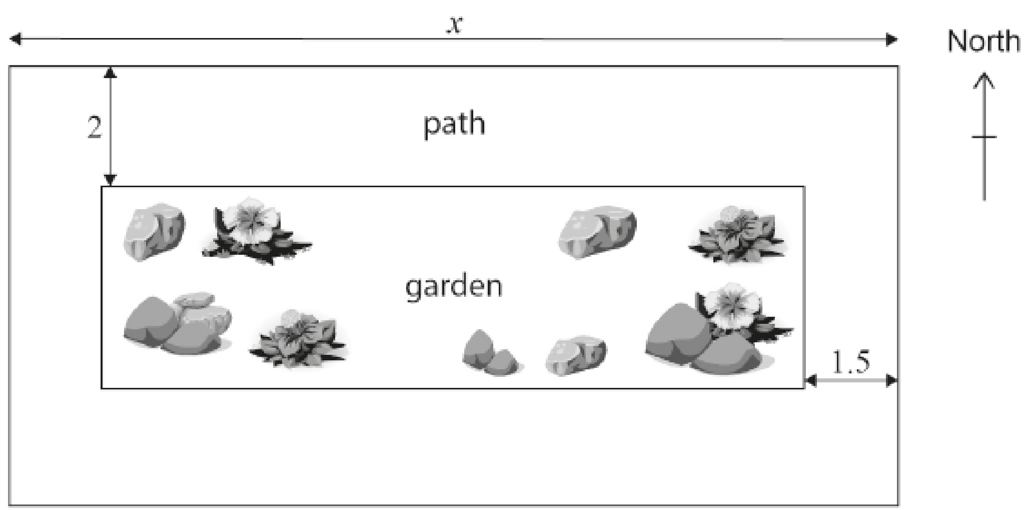
A particular park consists of a rectangular garden, of area $A\,\mathrm{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\mathrm{m}^2$.

The width of the path at the north and south side of the park is $2\,\mathrm{m}$.

The width of the path at the west and east side of the park is $1.5\,\mathrm{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale



(a) Show that $A = 1212 - 4x - \frac{3600}{x}$. [5]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\mathrm{m}^2$. [4]

(c) Find an expression for $\frac{\mathrm{d}A}{\mathrm{d}x}$. [3]

(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden. [2]

(e) Find the maximum possible area of the garden. [2]

4. [Maximum mark: 7] 22M.1.AHL.TZ1.16

The wind chill index W is a measure of the temperature, in $^{\circ}\mathrm{C}$, felt when taking into account the effect of the wind.

When Frieda arrives at the top of a hill, the relationship between the wind chill index and the speed of the wind v in kilometres per hour $\left(\mathrm{km\,h^{-1}}\right)$ is given by the equation

$$W = 19.34 - 7.405v^{0.16}$$

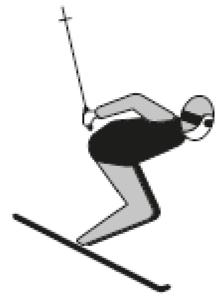
(a) Find an expression for $\frac{\mathrm{d}W}{\mathrm{d}v}$. [2]

(b) When Frieda arrives at the top of a hill, the speed of the wind is 10 kilometres per hour and increasing at a rate of $5\,\mathrm{km\,h^{-1}\,minute^{-1}}$.

Find the rate of change of W at this time. [5]

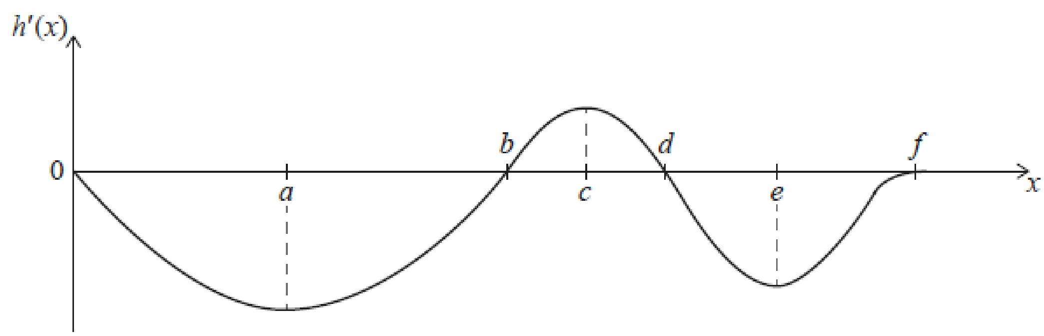
5. [Maximum mark: 5] 21N.1.AHL.TZ0.8

Juri skis from the top of a hill to a finishing point at the bottom of the hill. She takes the shortest route, heading directly to the finishing point (F).



Let $h(x)$ define the height of the hill above F at a horizontal distance x from the starting point at the top of the hill.

The graph of the **derivative** of $h(x)$ is shown below. The graph of $h\prime(x)$ has local minima and maxima when x is equal to $a,\ c$ and e. The graph of $h\prime(x)$ intersects the x-axis when x is equal to $b,\ d$, and f.

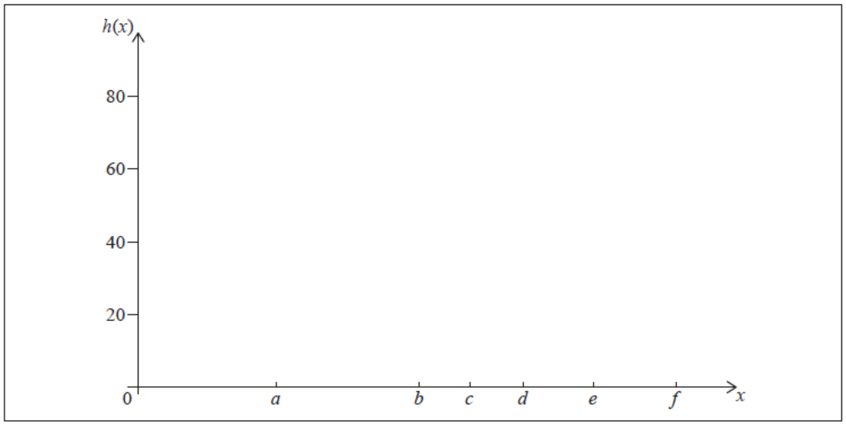


(a.i) Identify the x value of the point where $|h\prime(x)|$ has its maximum value. [1]

(a.ii) Interpret this point in the given context. [1]

(b) Juri starts at a height of 60 metres and finishes at F, where $x=f$.

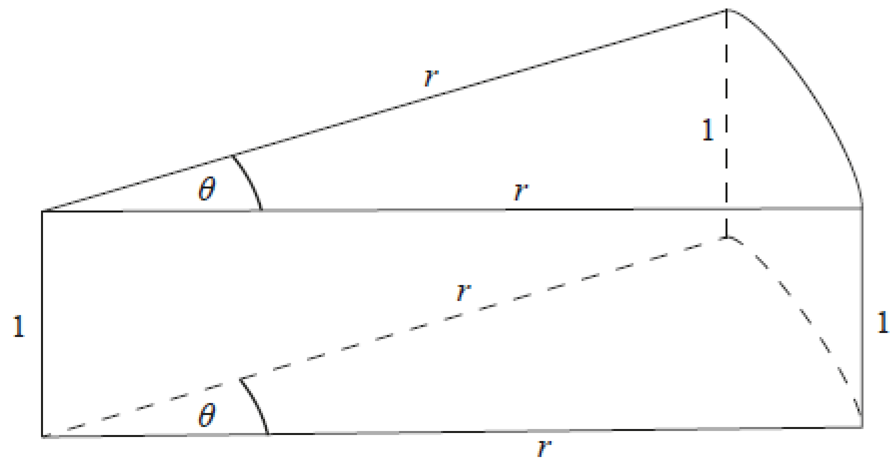
Sketch a possible diagram of the hill on the following pair of coordinate axes.



[3]

6. [Maximum mark: 9] 21N.1.AHL.TZ0.15

The following diagram shows a frame that is made from wire. The total length of wire is equal to $15\,\mathrm{cm}$. The frame is made up of two identical sectors of a circle that are parallel to each other. The sectors have angle θ radians and radius $r\,\mathrm{cm}$. They are connected by $1\,\mathrm{cm}$ lengths of wire perpendicular to the sectors. This is shown in the diagram below.



(a) Show that $r = \frac{6}{2+\theta}$. [2]

The faces of the frame are covered by paper to enclose a volume, V.

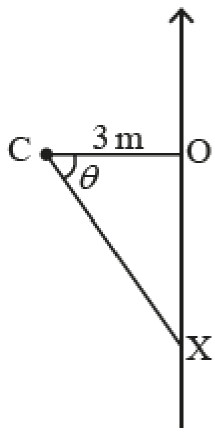
(b.i) Find an expression for V in terms of θ. [2]

(b.ii) Find the expression $\frac{\mathrm{d}V}{\mathrm{d}\theta}$. [3]

(b.iii) Solve algebraically $\frac{\mathrm{d}V}{\mathrm{d}\theta} = 0$ to find the value of θ that will maximize the volume, V. [2]

7. [Maximum mark: 6] 19M.1.AHL.TZ1.H_5

A camera at point C is 3 m from the edge of a straight section of road as shown in the following diagram. The camera detects a car travelling along the road at $t = 0$. It then rotates, always pointing at the car, until the car passes O, the point on the edge of the road closest to the camera.

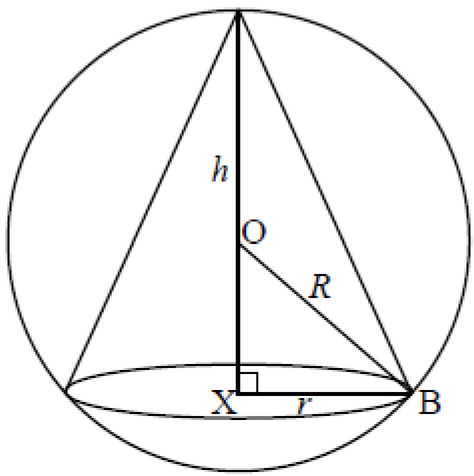


A car travels along the road at a speed of 24 ms^{-1}. Let the position of the car be X and let $\text{O}\hat{\text{C}}\text{X} = \theta$.

Find $\frac{\text{d}\theta}{\text{d}t}$, the rate of rotation of the camera, in radians per second, at the instant the car passes the point O . [6]

8. [Maximum mark: 8] 19M.1.AHL.TZ2.H_8

A right circular cone of radius r is inscribed in a sphere with centre O and radius R as shown in the following diagram. The perpendicular height of the cone is h, X denotes the centre of its base and B a point where the cone touches the sphere.



(a) Show that the volume of the cone may be expressed by $V = \frac{\pi}{3}\left(2Rh^2 - h^3\right)$. [4]

(b) Given that there is one inscribed cone having a maximum volume, show that the volume of this cone is $\frac{32\pi R^3}{81}$. [4]