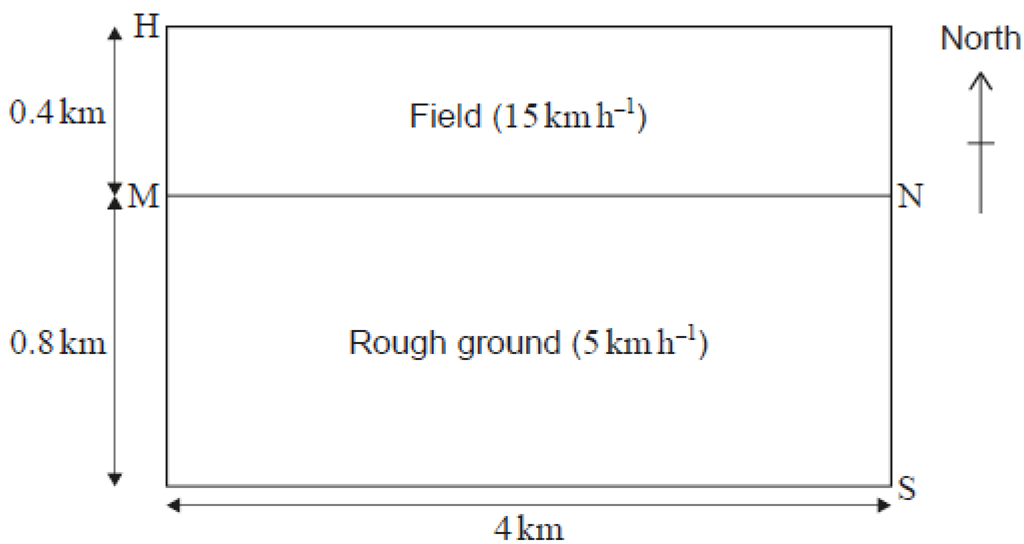
Paper 3 - optimization [26 marks]

1. [Maximum mark: 26] 23M.3.AHL.TZ2.1

This question considers the optimal route between two points, separated by several regions where different speeds are possible.

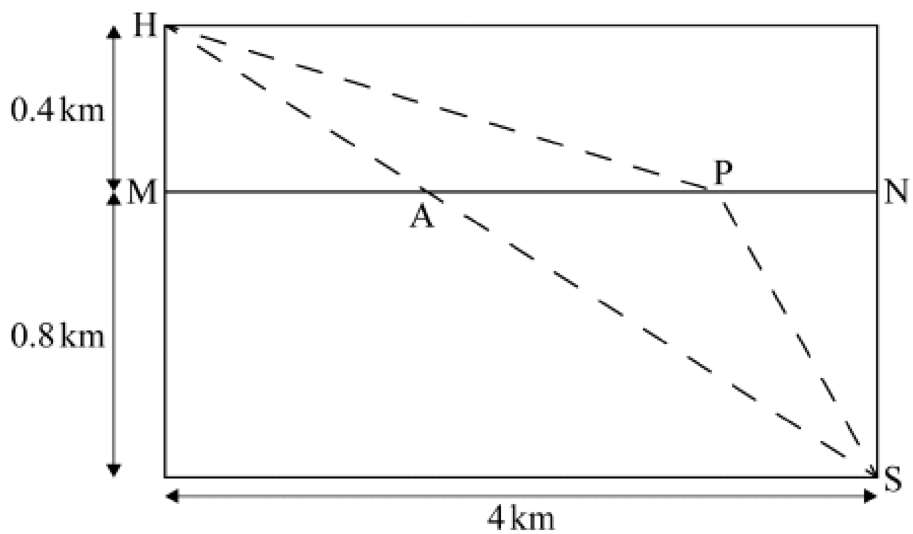
Huw lives in a house, H, and he attends a school, S, where H and S are marked on the following diagram. The school is situated $1.2\ \mathrm{km}$ south and $4\ \mathrm{km}$ east of Huw's house. There is a boundary $[\mathrm{MN}]$, going from west to east, $0.4\ \mathrm{km}$ south of his house. The land north of $[\mathrm{MN}]$ is a field over which Huw runs at 15 kilometres per hour ($\mathrm{km\ h^{-1}}$). The land south of $[\mathrm{MN}]$ is rough ground over which Huw walks at $5\ \mathrm{km\ h^{-1}}$. The two regions are shown in the following diagram.

diagram not to scale



(a) Huw travels in a straight line from H to S. Calculate the time that Huw takes to complete this journey. Give your answer correct to the nearest minute. [6]

Markscheme



METHOD 1

$\mathrm{M\hat{H}S} = \left(\tan^{-1} \frac{4}{1.2}\right) = 73.3007\ldots^{\circ}$ OR $1.27933\ldots$ ***(A1)***

use of trigonometry to find HA or AS ***(M1)***

$\mathrm{HA} = \frac{0.4}{\cos \mathrm{M\hat{H}S}}$ **AND** $\mathrm{AS} = \frac{0.8}{\cos \mathrm{M\hat{H}S}}$ ***(A1)***

($\mathrm{HA} = 1.39204\ldots$ and $\mathrm{AS} = 2.78408\ldots$)

use of $\mathrm{time} = \frac{\mathrm{distance}}{\mathrm{speed}}$ for either of their distances ***(M1)***

time taken $= \left(\frac{\mathrm{AH}}{15} + \frac{\mathrm{AS}}{5}\right)$

$0.649618\ldots$ (hours) ***(A1)***

($38.99712\ldots$ minutes)

therefore 39 (mins) ***A1FT***

Note: Allow ***FT***, within the question part, from their time in hours for the final ***A1***.

METHOD 2

EITHER

use of similar triangles to identify either length MA or AN ***(M1)***

$\left(\frac{4}{3} \text{ or } \frac{8}{3}\right)$

attempt to use Pythagoras for either triangle AMH or ANS ***(M1)***

$\mathrm{AH}^2 = 0.4^2 + \left(\frac{4}{3}\right)^2$ **AND** $\mathrm{AS}^2 = 0.8^2 + \left(\frac{8}{3}\right)^2$ ***(A1)***

OR

attempt to use Pythagoras for larger triangle ***(M1)***

$\mathrm{SH}^2 = 4^2 + 1.2^2$

$\mathrm{AH} = \frac{1}{3}\sqrt{4^2 + 1.2^2}$ **AND** $\mathrm{AS} = \frac{2}{3}\sqrt{4^2 + 1.2^2}$ ***(M1)(A1)***

THEN

($\mathrm{HA} = 1.39204\ldots$ and $\mathrm{AS} = 2.78408\ldots$)

use of $\text{time} = \frac{\text{distance}}{\text{speed}}$ for either of THEIR distances ***(M1)***

time taken $= \left(\frac{\mathrm{AH}}{15} + \frac{\mathrm{AS}}{5}\right)$

$0.649618\ldots$ (hours) ***(A1)***

($38.99712\ldots$ minutes)

therefore 39 (mins) ***A1FT***

Note: Allow ***FT***, within the question part, from their time in hours for the final ***A1***.

[6 marks]

(b) Huw realizes that his journey time could be reduced by taking a less direct route. He therefore defines a point P on $[\mathrm{MN}]$ that is $x\ \mathrm{km}$ east of M. Huw decides to run from H to P and then walk from P to S. Let $T(x)$ represent the time, in hours, taken by Huw to complete the journey along this route.

(b.i) Show that $T\left(x\right)=\frac{\sqrt{0.4^2+x^2}+3\sqrt{0.8^2+(4-x)^2}}{15}$. [3]

Markscheme

$\mathrm{PH}^2=0.4^2+x^2$ **AND** $\mathrm{PS}^2=0.8^2+(4-x)^2$ ***A1***

Note: This ***A1*** can be implied by a clear expression for the time in each region coming from distance / speed below.

$T(x)=\frac{\mathrm{PH}}{15}+\frac{\mathrm{PS}}{5}$ ***(M1)***

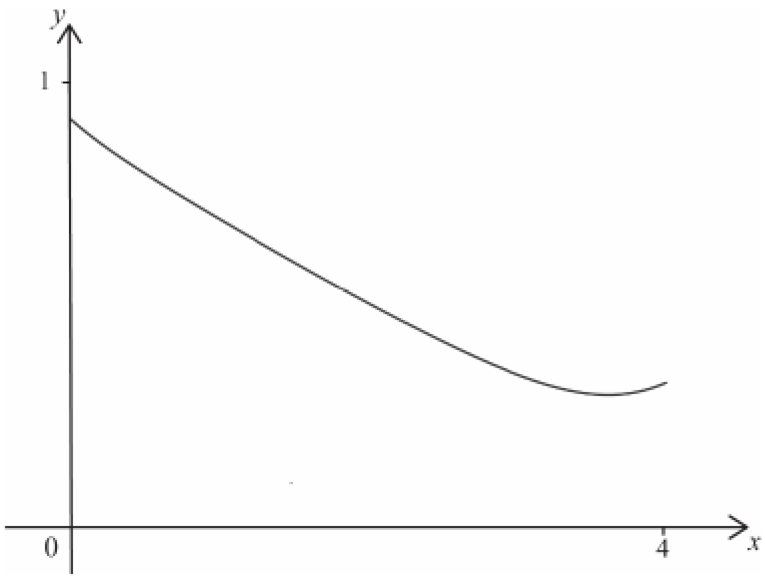
$T\left(x\right)=\frac{\sqrt{0.4^2+x^2}}{15}+\frac{\sqrt{0.8^2+(4-x)^2}}{5}$ ***A1***

$T\left(x\right)=\frac{\sqrt{0.4^2+x^2}+3\sqrt{0.8^2+(4-x)^2}}{15}$ ***AG***

[3 marks]

(b.ii) Sketch the graph of $y = T(x)$. [2]

Markscheme



correct shape with minimum point nearer $x = 4$ than $x = 0$ ***A1***

correct (approximate) y-intercept, $0.843\ldots$ (must be clearly below 1) ***A1***

[2 marks]

(b.iii) Hence determine the value of x that minimizes $T(x)$. [1]

Markscheme

using the GDC, at the minimum $x = 3.72\ (3.71898\ldots)$ ***A1***

Note: Do not accept coordinates of the minimum point.

[1 mark]

(b.iv) Find by how much Huw's journey time is reduced when he takes this optimal route, compared to travelling in a straight line from H to S. Give your answer correct to the nearest minute. [2]

Markscheme

finding their $T(x)$ for their value of x ***M1***

$T(x) = 0.418946\ldots$

so time saved ($= 38.97712\ldots - 25.1367\ldots$ mins) = 14 (mins) ***A1***

[2 marks]

(c.i) Determine an expression for the derivative $T\prime(x)$. [3]

Markscheme

attempt at chain rule ***M1***

$T\prime\left(x\right) = \frac{1}{15}\left(\frac{x}{\sqrt{0.4^2+x^2}} - \frac{3(4-x)}{\sqrt{0.8^2+(4-x)^2}}\right)$ ***A1A1***

Note: Award ***A1*** for each correct term. Accept any equivalent form i.e. condone fractions not simplified.

[3 marks]

(c.ii) Hence show that $T(x)$ is minimized when

$$\frac{x}{\sqrt{0.16+x^2}} = \frac{3(4-x)}{\sqrt{0.64+(4-x)^2}}.$$ [1]

Markscheme

setting their $T\prime(x) = 0$ ***M1***

Note: This requires more than just a statement that the derivative equals zero – they must use their attempt at $T\prime(x)$.

$$\frac{1}{15}\left(\frac{x}{\sqrt{0.4^2+x^2}} - \frac{3(4-x)}{\sqrt{0.8^2+(4-x)^2}}\right) = 0$$

$$\frac{x}{\sqrt{0.16+x^2}} = \frac{3(4-x)}{\sqrt{0.64+(4-x)^2}}$$ ***AG***

[1 mark]

(c.iii) For the optimal route, verify that the equation in part (c)(ii) satisfies the following result:

$$\frac{\cos \mathrm{H\widehat{P}M}}{\cos \mathrm{S\widehat{P}N}} = \frac{\text{speed over field}}{\text{speed over rough ground}}.$$ [2]

Markscheme

METHOD 1

$\cos \mathrm{H\widehat{P}M} = \frac{x}{\sqrt{0.16+x^2}}$ **AND** $\cos \mathrm{S\widehat{P}N} = \frac{4-x}{\sqrt{0.64+(4-x)^2}}$ ***A1***

substituting in the above equation and rearranging ***M1***

$\cos \mathrm{H\widehat{P}M} = 3\cos \mathrm{S\widehat{P}N}$ leading to $\frac{\cos \mathrm{H\widehat{P}M}}{\cos \mathrm{S\widehat{P}N}} = 3 = \left(\frac{15}{5}\right)$

verifying the result ***AG***

METHOD 2

$$\frac{x}{\sqrt{0.16+x^2}} = \frac{3(4-x)}{\sqrt{0.64+(4-x)^2}}$$

attempt to rearrange into a quotient ***M1***

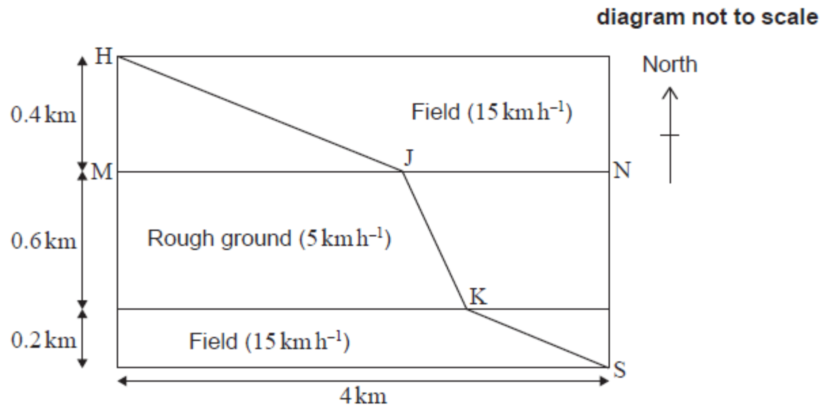
$$\left(\frac{15}{5} = 3 =\right)\frac{\frac{x}{\sqrt{0.16+x^2}}}{\frac{4-x}{\sqrt{0.64+(4-x)^2}}}$$

$= \frac{\cos \mathrm{H\widehat{P}M}}{\cos \mathrm{S\widehat{P}N}}$ ***A1***

verifying the result ***AG***

[2 marks]

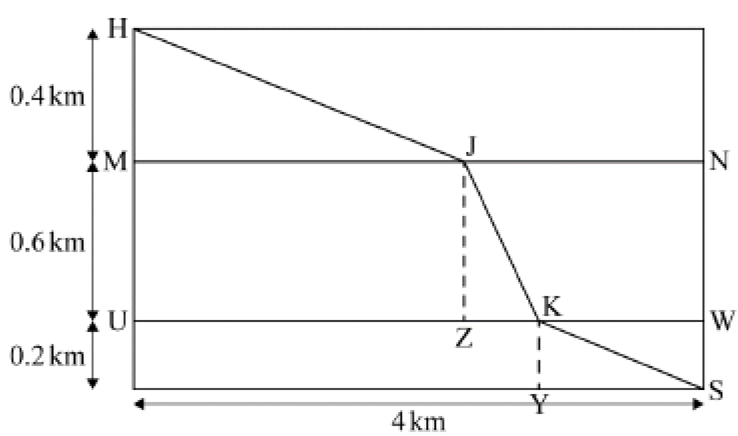
(d) The owner of the rough ground converts the southern quarter into a field over which Huw can run at $15\ \mathrm{km\ h^{-1}}$. The following diagram shows the optimal route, HJKS, in this new situation. You are given that $[\mathrm{HJ}]$ is parallel to $[\mathrm{KS}]$.



Using a similar result to that given in part (c)(iii), at the point J, determine MJ. [6]

Markscheme

METHOD 1



let $\mathrm{MJ} = y\ \mathrm{km}$ and W and Z be the points on the new boundary directly below N and J

attempt to find ZK in terms of MJ ***M1***

$$(\mathrm{KW} = 0.5y)$$

$$\mathrm{ZK} = (4 - 1.5y)\ \mathrm{km}$$ ***A1***

attempt to use the result from (c)(iii) at J ***M1***

$$\frac{\cos\,\mathrm{H\hat{J}M}}{\cos\,\mathrm{Z\hat{K}J}} = \frac{y}{\sqrt{y^2+0.4^2}} \div \frac{(4-1.5y)}{\sqrt{(4-1.5y)^2+0.6^2}} = \frac{15}{5}$$ ***A1***

Note: Accept $\cos\,\mathrm{N\hat{J}K}$ in place of $\cos\,\mathrm{Z\hat{K}J}$.

$$\left(\text{leading to } \frac{y}{\sqrt{y^2+0.16}} \div \frac{3(4-1.5y)}{\sqrt{(4-1.5y)^2+0.36}}\right)$$

valid method for solving this equation, eg drawing graphs of both sides of the equation, using SOLVER, etc. ***(M1)***

solution is $y = 2.53$ ***A1***

METHOD 2

combining the field into one region with height $0.6\ \mathrm{km}$ ***M1***

$$\cos\,\mathrm{H\hat{P}M} = \frac{x}{\sqrt{0.36+x^2}}$$

$$\cos\,\mathrm{S\hat{P}N} = \frac{4-x}{\sqrt{0.36+(4-x)^2}}$$ ***A1***

Note: Both expressions, or their ratio, are required for the ***A1*** to be awarded.

therefore

$\frac{x\sqrt{0.36+(4-x)^2}}{\left(4-x\right)\sqrt{0.36+x^2}} = 3$ ***A1***

valid method for solving ***(M1)***

attempting to find MJ in terms of x e.g. $\mathrm{MJ} = \frac{2}{3}x$ ***M1***

so $\mathrm{MJ} = 2.53$ ***A1***

[6 marks]