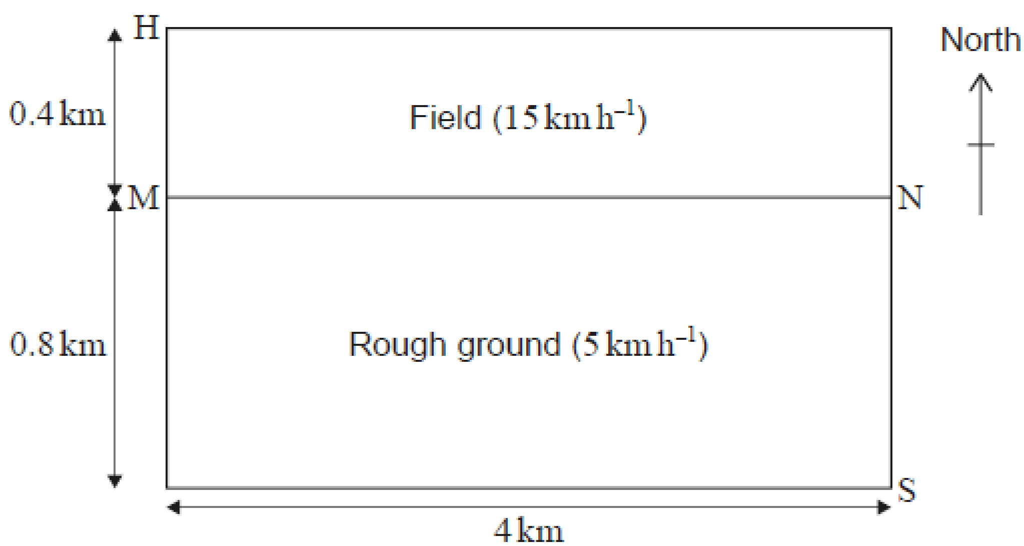
Paper 3 - optimization [26 marks]

1. [Maximum mark: 26] 23M.3.AHL.TZ2.1

This question considers the optimal route between two points, separated by several regions where different speeds are possible.

Huw lives in a house, H, and he attends a school, S, where H and S are marked on the following diagram. The school is situated $1.2\ \mathrm{km}$ south and $4\ \mathrm{km}$ east of Huw's house. There is a boundary $[\mathrm{MN}]$, going from west to east, $0.4\ \mathrm{km}$ south of his house. The land north of $[\mathrm{MN}]$ is a field over which Huw runs at 15 kilometres per hour ($\mathrm{km\ h^{-1}}$). The land south of $[\mathrm{MN}]$ is rough ground over which Huw walks at $5\ \mathrm{km\ h^{-1}}$. The two regions are shown in the following diagram.

diagram not to scale



(a) Huw travels in a straight line from H to S. Calculate the time that Huw takes to complete this journey. Give your answer correct to the nearest minute. [6]

(b) Huw realizes that his journey time could be reduced by taking a less direct route. He therefore defines a point P on $[\mathrm{MN}]$ that is $x\ \mathrm{km}$ east of M. Huw decides to run from H to P and then walk from P to S. Let $T(x)$ represent the time, in hours, taken by Huw to complete the journey along this route.

(b.i) Show that $T\left(x\right) = \frac{\sqrt{0.4^2+x^2}+3\sqrt{0.8^2+(4-x)^2}}{15}$. [3]

(b.ii) Sketch the graph of $y = T(x)$. [2]

(b.iii) Hence determine the value of x that minimizes $T(x)$. [1]

(b.iv) Find by how much Huw's journey time is reduced when he takes this optimal route, compared to travelling in a straight line from H to S. Give your answer correct to the nearest minute. [2]

(c.i) Determine an expression for the derivative $T'(x)$. [3]

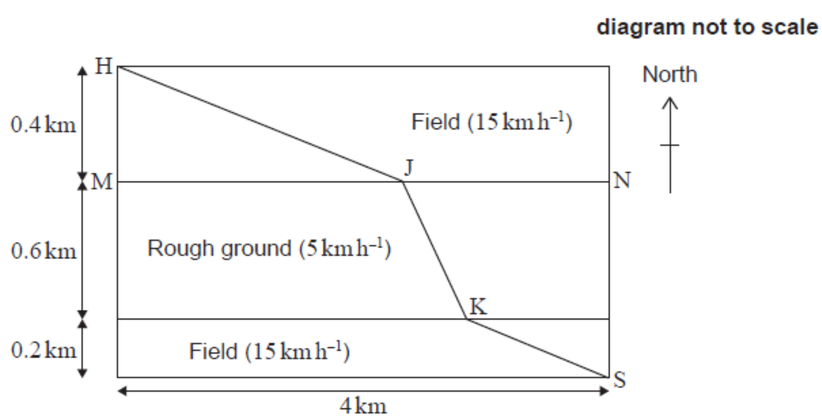
(c.ii) Hence show that $T(x)$ is minimized when

$$\frac{x}{\sqrt{0.16+x^2}} = \frac{3(4-x)}{\sqrt{0.64+(4-x)^2}}.$$ [1]

(c.iii) For the optimal route, verify that the equation in part (c)(ii) satisfies the following result:

$$\frac{\cos \mathrm{H\hat{P}M}}{\cos \mathrm{S\hat{P}N}} = \frac{\text{speed over field}}{\text{speed over rough ground}}.$$ [2]

(d) The owner of the rough ground converts the southern quarter into a field over which Huw can run at $15\ \mathrm{km\ h^{-1}}$. The following diagram shows the optimal route, HJKS, in this new situation. You are given that $[\mathrm{HJ}]$ is parallel to $[\mathrm{KS}]$.



Using a similar result to that given in part (c)(iii), at the point J, determine MJ. [6]