Name:
Result:

**1.** *(5 points)*

The normal to the curve $y = a\sqrt{x} + \frac{b}{x}$ at $x = 1$ is given by the equation $2x - y + 3 = 0$. Find the values of $a$ and $b$.

Using the equation of the normal we get that at $x = 1$ we have $y = 5$. The gradient of the normal is 2, so the gradient of the tangent is $-\frac{1}{2}$.

So we have $y(1) = 5$ and $y'(1) = -\frac{1}{2}$.

$y' = \frac{a}{2\sqrt{x}} - \frac{b}{x^2}$.

We get

$$\begin{cases} a + b = 5 \\ \frac{a}{2} - b = -\frac{1}{2} \end{cases}$$

This gives $a = 3$ and $b = 2$.

**2.** *(5 points)*

Let $f(x) = \frac{1}{3}x^3 + ax^2 + bx - 2$. The graph of $f(x)$ has a local maximum at $x = -4$ and a local minimum at $x = 1$.

(a) Find the values of $a$ and $b$. [3]

(b) Find the $x$-coordinate of the point of inflexion of the graph of $f(x)$. [2]

(a) Using the stationary points we get that $f'(-4) = f'(1) = 0$.

We have $f'(x) = x^2 + 2ax + b$.

The system we get is:

$$\begin{cases} 16 - 8a + b = 0 \\ 1 + 2a + b = 0 \end{cases}$$

This gives $a = 1.5$ and $b = -4$.

(b) $f''(x) = 2x + 3$, so the point of inflexion ($f''(x)$ changes sign) at $x = -1.5$.

**3.** *(4 points)*

The following table shows value of $f(x)$, $g(x)$, $f'(x)$ and $g'(x)$ for various values of $x$:

| $x$ | $f(x)$ | $g(x)$ | $f'(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 0 | 2 | 1 | 2 | 5 |
| 1 | 3 | 4 | 1 | 2 |
| 2 | 3 | 1 | -1 | -4 |
| 3 | 1 | 0 | -1 | -1 |
| 4 | 0 | 2 | -5 | 1 |

Let $h(x) = (f \circ g)(x)$. Find the equation of the normal to the graph of $h(x)$ at $x = 2$. Write your answer in the form $Ax + By + C = 0$, where $A, B, C \in \mathbb{Z}$.

$h(x) = f(g(x))$, so $h'(x) = f'(g(x)) \times g'(x)$.

$h'(2) = f'(g(2)) \times g'(2) = f'(1) \times (-4) = -4$

$h(2) = f(g(2)) = f(1) = 3$

The equation of the normal is then $y - 3 = \frac{1}{4}(x - 2)$, rearranging we get:

$$x - 4y + 10 = 0$$

**4.** *(5 points)*

Consider the curve $y = \dfrac{1}{x}$ with $x > 0$. The tangent to this curve at the point where $x = a$ intersects the axes at points $A$ and $B$. Find the area of the triangle $OAB$ where $O$ is the origin.

The point of tangency is $(a, \frac{1}{a})$.

$y' = -\frac{1}{x^2}$, so $m_T = -\frac{1}{a^2}$ and the equation of the tangent is:

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a)$$

The $x$-intercept is $2a$, the $y$-intercept is $\frac{2}{a}$, so the area is:

$$A = \frac{1}{2} \times 2a \times \frac{2}{a} = 2$$

A constant! Does not depend on where we draw the tangent! Incredible!

P.S. 4 points are for finding the area, the $5^{\text{th}}$ point is for expressing the excitement that it does not depend on $a$.

**5.** *(13 points)*

Consider the function $f(x) = \sqrt{x + \frac{1}{x-a}}$, where $x > a$ and $a > 0$.

(a) Find $f'(x)$ in terms of $a$. [3]

(b) The function is increasing for $x > 3$. Find $a$. [3]

(c) State the coordinates of the minimum of the graph of $f(x)$. [1]

(d) Using the graph of $f'(x)$, or otherwise, find the exact coordinates of the point of inflexion of the graph of $f(x)$. [4]

(e) Sketch the graph of $f(x)$ indicating all the features you've found. [2]

(a) $f'(x) = \frac{1 - \frac{1}{(x-a)^2}}{2\sqrt{x + \frac{1}{x-a}}}$. This gets all marks, but it can be nicely simplified to

$$f'(x) = \frac{(x-a)^2 - 1}{2(x-a)^2\sqrt{x + \frac{1}{x-a}}} = \frac{(x-a-1)(x-a+1)}{2(x-a)^2\sqrt{x + \frac{1}{x-a}}}$$

(b) By analysing the sign of $f'(x)$ (and finding stationary points) we get that $f(x)$ is increasing for $x > a + 1$, so $a + 1 = 3$, which gives $a = 2$.

(c) The stationary points occurs at $x = a + 1 = 3$ and $f(3) = 2$, so the minimum is $(3, 2)$.

(d) We want to find a maximum on the graph of $f'(x)$ (as this correspond to the point of inflexion on the graph of $f(x)$). The maximum occurs at $x = 5$. $f(5) = \sqrt{5 + \frac{1}{3}} = \frac{4}{\sqrt{3}}$, so the inflexion point is $(5, \frac{4}{\sqrt{3}})$.