Binomial theorem - revision questions (TL) [53 marks]

1. [Maximum mark: 6] 23N.1.AHL.TZ1.5

The binomial expansion of $(1+kx)^n$ is given by
$1+12x+28k^2x^2+\ldots+k^nx^n$ where $n \in \mathbb{Z}^+$ and $k \in \mathbb{Q}$.

Find the value of n and the value of k. [6]

Markscheme

attempt to apply binomial expansion ***(M1)***

$(1+kx)^n = 1 + {}^nC_1kx + {}^nC_2k^2x^2 + \ldots$ OR ${}^nC_1k = 12$ OR ${}^nC_2 = 28$

$nk = 12$ ***(A1)***

$\frac{n(n-1)}{2} = 28$ OR $\frac{n!}{(n-2)!2!} = 28$ ***(A1)***

$n^2 - n - 56 = 0$ OR $n(n-1) = 56$

valid attempt to solve ***(M1)***

$(n-8)(n+7) = 0$ OR $8(8-1) = 56$ OR finding correct value in Pascal's triangle

$\Rightarrow n = 8$ ***A1***

$\Rightarrow k = \frac{3}{2}$ ***A1***

Note: If candidate finds $n = 8$ with no working shown, award ***M1A0A0M1A1A0.***

If candidate finds $n = 8$ and $k = \frac{3}{2}$ with no working shown, award ***M1A0A0M1A1A1***.

[6 marks]

2. [Maximum mark: 7] 23M.2.SL.TZ1.6

The coefficient of x^6 in the expansion of $\left(ax^3+b\right)^8$ is 448.

The coefficient of x^6 in the expansion of $\left(ax^3+b\right)^{10}$ is 2880.

Find the value of a and the value of b, where $a\,,\ b>0$. [7]

Markscheme

product of a binomial coefficient, a power of ax^3 and a power of b seen ***(M1)***

evidence of correct term chosen

for $n=8$: $r=2$ (or $r=6$) OR for $n=10$: $r=2$ (or $r=8$) ***(A1)***

correct equations (may include powers of x) ***A1A1***

$$_8C_2a^2b^6=448\ \left(28a^2b^6=448\Rightarrow a^2b^6=16\right),$$
$$_{10}C_2a^2b^8=2880\ \left(45a^2b^8=2880\Rightarrow a^2b^8=64\right)$$

attempt to solve their system in a and b algebraically or graphically ***(M1)***

$b=2\,;a=\frac{1}{2}$ ***A1A1***

Note: Award a maximum of ***(M1)(A1)A1A1(M1)A1A0*** for $b=\pm2$ and/or $a=\pm\frac{1}{2}$.

[7 marks]

3. [Maximum mark: 6] 22N.2.SL.TZ0.6

Consider the expansion of $\frac{(ax+1)^9}{21x^2}$, where $a \neq 0$. The coefficient of the term in x^4 is $\frac{8}{7}a^5$.

Find the value of a. [6]

Markscheme

Note: Do not award any marks if there is clear evidence of adding instead of multiplying, for example ${}^9C_r + (ax)^{9-r} + (1)^r$.

valid approach for expansion (must be the product of a binomial coefficient with $n = 9$ and a power of ax) ***(M1)***

${}^9C_r(ax)^{9-r}(1)^r$ OR ${}^9C_{9-r}(ax)^r(1)^{9-r}$ OR
${}^9C_0(ax)^0(1)^9 + {}^9C_1(ax)^1(1)^8 + \ldots$

recognizing that the term in x^6 is needed ***(M1)***

$\frac{\text{Term in } x^6}{21x^2} = kx^4$ OR $r = 6$ OR $r = 3$ OR $9 - r = 6$

correct term or coefficient in binomial expansion (seen anywhere) ***(A1)***

${}^9C_6(ax)^6(1)^3$ OR ${}^9C_3a^6x^6$ OR $84\left(a^6x^6\right)(1)$ OR $84a^6$

EITHER

correct term in x^4 or coefficient (may be seen in equation) ***(A1)***

$\frac{{}^9C_6}{21}a^6x^4$ OR $4a^6x^4$ OR $4a^6$

Set their term in x^4 or coefficient of x^4 equal to $\frac{8}{7}a^5x^4$ or $\frac{8}{7}a^5$ (do not accept other powers of x) ***(M1)***

$\frac{{}^9C_3}{21}a^6x^4 = \frac{8}{7}a^5x^4$ OR $4a^6 = \frac{8}{7}a^5$

OR

correct term in x^6 or coefficient of x^6 (may be seen in equation) ***(A1)***

$84a^6x^6$ OR $84a^6$

set their term in x^6 or coefficient of x^6 equal to $24a^5x^6$ or $24a^5$ (do not accept other powers of x) ***(M1)***

$84a^6x^6 = 24a^5x^6$ OR $84a = 24$

THEN

$a = \frac{2}{7} \approx 0.286\,(0.285714\ldots)$ ***A1***

Note: Award **A0** for the final mark for $a = \frac{2}{7}$ and $a = 0$.

[6 marks]

4. [Maximum mark: 7] 22M.1.SL.TZ2.6

Consider the binomial expansion

$(x+1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + \ldots + 1$ where $x \neq 0$ and $a,\ b \in \mathbb{Z}^+$

.

(a) Show that $b = 21$. [2]

Markscheme

EITHER

recognises the required term (or coefficient) in the expansion ***(M1)***

$bx^5 = {}_7C_2 x^5 1^2$ OR $b = {}_7C_2$ OR ${}_7C_5$

$b = \frac{7!}{2!5!} \left(= \frac{7!}{2!(7-2)!}\right)$

correct working ***A1***

$\frac{7\times6\times5\times4\times3\times2\times1}{2\times1\times5\times4\times3\times2\times1}$ OR $\frac{7\times6}{2!}$ OR $\frac{42}{2}$

OR

lists terms from row 7 of Pascal's triangle ***(M1)***

$1,\ 7,\ 21, \ldots$ ***A1***

THEN

$b = 21$ ***AG***

[2 marks]

(b) The third term in the expansion is the mean of the second term and the fourth term in the expansion.

Find the possible values of x.

[5]

Markscheme

$a = 7$ ***(A1)***

correct equation ***A1***

$21x^5 = \frac{ax^6+35x^4}{2}$ OR $21x^5 = \frac{7x^6+35x^4}{2}$

correct quadratic equation ***A1***

$7x^2 - 42x + 35 = 0$ OR $x^2 - 6x + 5 = 0$ (or equivalent)

valid attempt to solve **their** quadratic ***(M1)***

$(x-1)(x-5) = 0$ OR $x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)}$

$x = 1,\ x = 5$ ***A1***

Note: Award final ***A0*** for obtaining $x = 0,\ x = 1,\ x = 5$.

[5 marks]

5. [Maximum mark: 5] 22M.1.AHL.TZ1.6

Consider the expansion of $\left(8x^3 - \frac{1}{2x}\right)^n$ where $n \in \mathbb{Z}^+$. Determine all possible values of n for which the expansion has a non-zero constant term. [5]

Markscheme

EITHER

attempt to obtain the general term of the expansion

$T_{r+1} = {}_nC_r\left(8x^3\right)^{n-r}\left(-\frac{1}{2x}\right)^r$ OR $T_{r+1} = {}_nC_{n-r}\left(8x^3\right)^r\left(-\frac{1}{2x}\right)^{n-r}$
(M1)

OR

recognize power of x starts at $3n$ and goes down by 4 each time ***(M1)***

THEN

recognizing the constant term when the power of x is zero (or equivalent)
(M1)

$r = \frac{3n}{4}$ or $n = \frac{4}{3}r$ or $3n - 4r = 0$ OR $3r - (n - r) = 0$ (or equivalent) ***A1***

r is a multiple of 3 $(r = 3, 6, 9, \ldots)$ or one correct value of n (seen anywhere)
(A1)

$n = 4k,\ k \in \mathbb{Z}^+$ ***A1***

Note: Accept n is a (positive) multiple of 4 or $n = 4, 8, 12, \ldots$
Do not accept $n = 4, 8, 12$

Note: Award full marks for a correct answer using trial and error approach showing $n = 4, 8, 12, \ldots$ and for recognizing that this pattern continues.

[5 marks]

6. [Maximum mark: 5] 21M.1.SL.TZ2.4

In the expansion of $(x+k)^7$, where $k \in \mathbb{R}$, the coefficient of the term in x^5 is 63.

Find the possible values of k. [5]

Markscheme

EITHER

attempt to use the binomial expansion of $(x+k)^7$ ***(M1)***

${}^7C_0x^7k^0 + {}^7C_1x^6k^1 + {}^7C_2x^5k^2 + \ldots$ (or ${}^7C_0k^7x^0 + {}^7C_1k^5x^1 + {}^7C_2k^5x^2 + \ldots$)

identifying the correct term ${}^7C_2x^5k^2$ (or ${}^7C_5k^2x^5$) ***(A1)***

OR

attempt to use the general term ${}^7C_rx^rk^{7-r}$ (or ${}^7C_rk^rx^{7-r}$) ***(M1)***

$r = 2$ (or $r = 5$) ***(A1)***

THEN

${}^7C_2 = 21$ (or ${}^7C_5 = 21$ (seen anywhere) ***(A1)***

$21x^5k^2 = 63x^5 \ \left(21k^2 = 63\ ,\ k^2 = 3\right)$ ***A1***

$k = \pm\sqrt{3}$ ***A1***

Note: If working shown, award ***M1A1A1A1A0*** for $k = \sqrt{3}$.

[5 marks]

7. [Maximum mark: 5] 21M.2.SL.TZ1.6

Consider the expansion of $(3 + x^2)^{n+1}$, where $n \in \mathbb{Z}^+$.

Given that the coefficient of x^4 is $20\,412$, find the value of n. [5]

Markscheme

METHOD 1

product of a binomial coefficient, a power of 3 (and a power of x^2) seen ***(M1)***

evidence of correct term chosen ***(A1)***

$^{n+1}C_2 \times 3^{n+1-2} \times (x^2)^2 \left(= \frac{n(n+1)}{2} \times 3^{n-1} \times x^4\right)$ **OR** $n - r = 1$

equating their coefficient to 20412 or their term to $20412x^4$ ***(M1)***

EITHER

$^{n+1}C_2 \times 3^{n-1} = 20412$ ***(A1)***

OR

$^{r+2}C_r \times 3^r = 20412 \Rightarrow r = 6$ ***(A1)***

THEN

$n = 7$ ***A1***

METHOD 2

$3^{n+1}\left(1 + \frac{x^2}{3}\right)^{n+1}$

product of a binomial coefficient, and a power of $\frac{x^2}{3}$ **OR** $\frac{1}{3}$ seen ***(M1)***

evidence of correct term chosen ***(A1)***

$$3^{n+1} \times \frac{n(n+1)}{2!} \times \left(\frac{x^2}{3}\right)^2 \left(= \frac{3^{n-1}}{2} n(n+1)x^4\right)$$

equating their coefficient to 20412 or their term to $20412x^4$ ***(M1)***

$3^{n-1} \times \frac{n(n+1)}{2} = 20412$ ***(A1)***

$n = 7$ ***A1***

[5 marks]

8. [Maximum mark: 6] 20N.2.SL.TZ0.S_5

Consider the expansion of $\left(3x^2 - \frac{k}{x}\right)^9$, where $k > 0$.

The coefficient of the term in x^6 is 6048. Find the value of k. [6]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

valid approach for expansion (must have correct substitution for parameters, but accept an incorrect value for r). ***(M1)***

eg

$$\binom{9}{r}\left(3x^2\right)^{9-r}\left(-\frac{k}{x}\right)^r,\ \left(3x^2\right)^9 + \binom{9}{1}\left(3x^2\right)^8\left(-\frac{k}{x}\right)^1 + \binom{9}{2}\left(3x^2\right)^7\left(-\frac{k}{x}\right)^2 + \ldots$$

valid attempt to identify correct term ***(M1)***

eg $2(9 - r) - r = 6,\ \left(x^2\right)^r\left(x^{-1}\right)^{9-r} = x^6$

identifying correct term (may be indicated in expansion) ***(A1)***

eg $r = 4,\ r = 5$

correct term or coefficient in binominal expansion ***(A1)***

eg $\binom{9}{4}\left(3x^2\right)^5\left(-\frac{k}{x}\right)^4,\ 126\left(243x^{10}\right)\left(\frac{k^4}{x^4}\right),\ 30618k^4$

correct equation in k ***(A1)***

eg $\binom{9}{4}(243)\left(k^4\right)x^6 = 6048x^6,\ 30618k^4 = 6048$

$k = \frac{2}{3}$ (exact) 0.667 ***A1 N3***

Note: Do not award ***A1*** if additional answers given.

[6 marks]

9. [Maximum mark: 6] 20N.2.AHL.TZ0.H_4

Find the term independent of x in the expansion of $\frac{1}{x^3}\left(\frac{1}{3x^2}-\frac{x}{2}\right)^9$. [6]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

use of Binomial expansion to find a term in either

$\left(\frac{1}{3x^2}-\frac{x}{2}\right)^9$, $\left(\frac{1}{3x^{7/3}}-\frac{x^{2/3}}{2}\right)^9$, $\left(\frac{1}{3}-\frac{x^3}{2}\right)^9$, $\left(\frac{1}{3x^3}-\frac{1}{2}\right)^9$ or $\left(2-3x^3\right)^9$ ***(M1)(A1)***

Note: Award ***M1*** for a product of three terms including a binomial coefficient and powers of the two terms, and ***A1*** for a correct expression of a term in the expansion.

finding the powers required to be 2 and 7 ***(M1)(A1)***

constant term is ${}_9C_2\times\left(\frac{1}{3}\right)^2\times\left(-\frac{1}{2}\right)^7$ ***(M1)***

Note: Ignore all x's in student's expression.

therefore term independent of x is $-\frac{1}{32}$ $(=-0.03125)$ ***A1***

[6 marks]