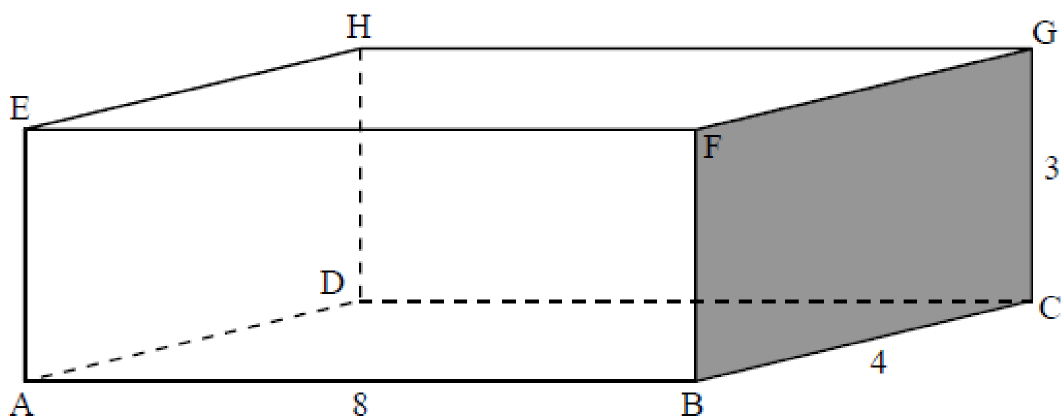
Calculus [109 marks]

1. [Maximum mark: 15] SPM.2.SL.TZ0.4

The Happy Straw Company manufactures drinking straws.

The straws are packaged in small closed rectangular boxes, each with length 8 cm, width 4 cm and height 3 cm. The information is shown in the diagram.



(a) Calculate the surface area of the box in cm^2. [2]

(b) Calculate the length AG. [2]

Each week, the Happy Straw Company sells x boxes of straws. It is known that $\frac{\mathrm{d}P}{\mathrm{d}x} = -2x + 220$, $x \geq 0$, where P is the weekly profit, in dollars, from the sale of x thousand boxes.

(c) Find the number of boxes that should be sold each week to maximize the profit. [3]

(d) Find $P\left(x\right)$. [5]

(e) Find the least number of boxes which must be sold each week in order to make a profit. [3]

2. [Maximum mark: 7] EXN.1.SL.TZ0.7

Consider the curve $y = x^2 - 4x + 2$.

(a) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. [1]

(b) Show that the normal to the curve at the point where $x = 1$ is $2y - x + 3 = 0$. [6]

3. [Maximum mark: 12] EXN.2.SL.TZ0.2

A box of chocolates is to have a ribbon tied around it as shown in the diagram below.

The box is in the shape of a cuboid with a height of 3 cm. The length and width of the box are x and y cm.

After going around the box an extra 10 cm of ribbon is needed to form the bow.

(a) Find an expression for the total length of the ribbon L in terms of x and y. [2]

The volume of the box is $450\,\mathrm{cm}^3$.

(b) Show that $L = 2x + \frac{300}{x} + 22$ [3]

(c) Find $\frac{\mathrm{d}L}{\mathrm{d}x}$ [3]

(d) Solve $\frac{\mathrm{d}L}{\mathrm{d}x} = 0$ [2]

(e) Hence or otherwise find the minimum length of ribbon required. [2]

4. [Maximum mark: 12] EXN.2.SL.TZ0.3

The rate of change of the height (h) of a ball above horizontal ground, measured in metres, t seconds after it has been thrown and until it hits the ground, can be modelled by the equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 11.4 - 9.8t$$

The height of the ball when $t = 0$ is $1.2\,\mathrm{m}$.

(a) Find an expression for the height h of the ball at time t. [6]

(b.i) Find the value of t at which the ball hits the ground. [2]

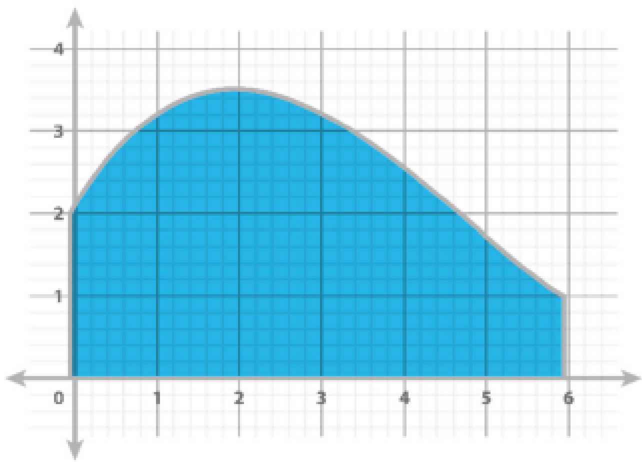
(b.ii) Hence write down the domain of h. [1]

(c) Find the range of h. [3]

5. [Maximum mark: 16] EXN.2.SL.TZ0.6

A theatre set designer is designing a piece of flat scenery in the shape of a hill. The scenery is formed by a curve between two vertical edges of unequal height. One edge is 2 metres high and the other is 1 metre high. The width of the scenery is 6 metres.

A coordinate system is formed with the origin at the foot of the 2 metres high edge. In this coordinate system the highest point of the cross-section is at $(2,\ 3.5)$.



A set designer wishes to work out an approximate value for the area of the scenery $\left(A\,\mathrm{m}^2\right)$.

(a) Explain why $A < 21$. [1]

(b) By dividing the area between the curve and the x-axis into two trapezoids of unequal width show that $A > 14.5$, justifying the direction of the inequality. [4]

In order to obtain a more accurate measure for the area the designer decides to model the curved edge with the polynomial $h(x) = ax^3 + bx^2 + cx + d\ \ a,\ b,\ c,\ d \in \mathbb{R}$ where h metres is the height of the curved edge a horizontal distance $x\,\mathrm{m}$ from the origin.

(c) Write down the value of d. [1]

(d) Use differentiation to show that $12a + 4b + c = 0$. [2]

(e) Determine two other linear equations in a, b and c. [3]

(f) Hence find an expression for $h(x)$. [3]

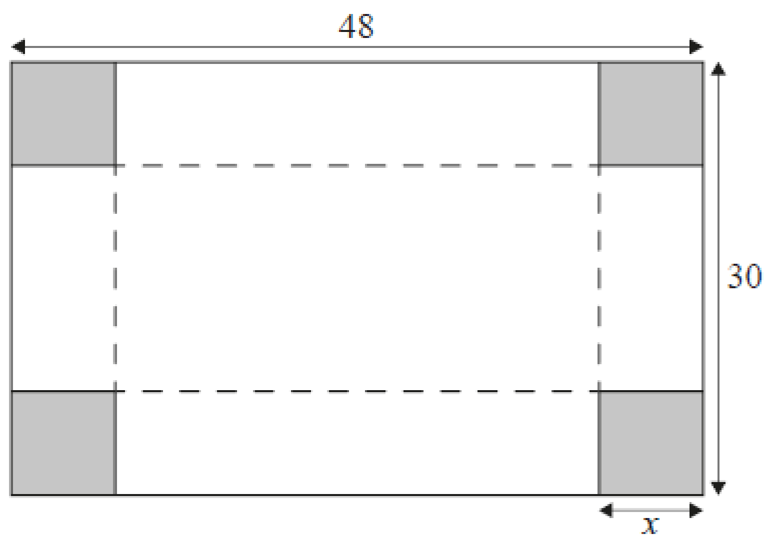
(g) Use the expression found in (f) to calculate a value for A. [2]

6. [Maximum mark: 6] 23M.1.SL.TZ1.7

A rectangular box, with an open top, is to be constructed from a piece of cardboard that measures 48 cm by 30 cm.

Squares of equal size will be cut from the corners of the cardboard, as indicated by the shading in the diagram. The sides will then be folded along the dotted lines to form the box.

diagram not to scale



The volume of the box, in cubic centimetres, can be modelled by the function $V(x) = (48 - 2x)(30 - 2x)(x)$, for $0 < x < k$, where x is the length of the sides of the squares removed in centimetres.

(a) Write down the maximum possible value of k in this context. [1]

(b) Find the value of x that maximizes the volume of the box. [2]

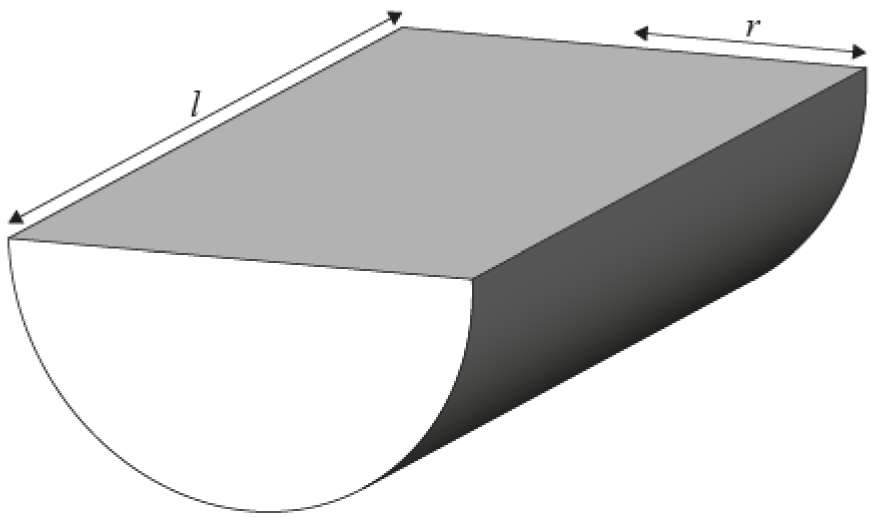
A second piece of 48 cm by 30 cm cardboard is damaged and a strip 2 cm wide must be removed from all four sides. A box will then be constructed in a similar manner from the remaining cardboard.

(c) Calculate the maximum possible volume of the box made from the second piece of cardboard.

7. [Maximum mark: 17] 23M.2.SL.TZ1.5

A large closed container, in the shape of a half cylinder with a rectangular lid, is to be constructed with a volume of $0.8\,\mathrm{m}^3$. The container has a length of l metres and a radius of r metres.

diagram not to scale



(a) Find an exact expression for l in terms of r and π. [2]

The container will be constructed using two different materials. The material for both the curved surface and the rectangular lid of the container costs $\$4.40$ per square metre. The material for the semicircular ends of the container costs $\$p$ per square metre.

The cost, C, of the materials to construct the container can be written in terms of r and p (where $p > 0$ and $r > 0$).

(b) Show that $C = 7.04r^{-1} + \frac{14.08}{\pi}r^{-1} + p\pi r^2$. [4]

(c) Find $\frac{\mathrm{d}C}{\mathrm{d}r}$. [3]

The cost of materials to construct the container is minimized when the radius of the container, r, is $0.7\,\mathrm{m}$.

(d) Find the value of p. [3]

In total, 350 containers will be constructed at this minimum cost.

(e) Calculate the cost of materials, to the nearest dollar, to construct all 350 containers. [3]

The materials for constructing the containers can be purchased at a discount according to the information in the table.

Cost of materials ($\$C$) before discount	**Discount applied to entire order**
$1000 \leq C < 2500$	1%
$2500 \leq C < 5000$	4%
$5000 \leq C < 10\,000$	8%
$C \geq 10\,000$	10%

(f) Determine the cost of materials for 350 containers after the discount is applied. [2]

8. [Maximum mark: 16] 23M.2.SL.TZ2.5

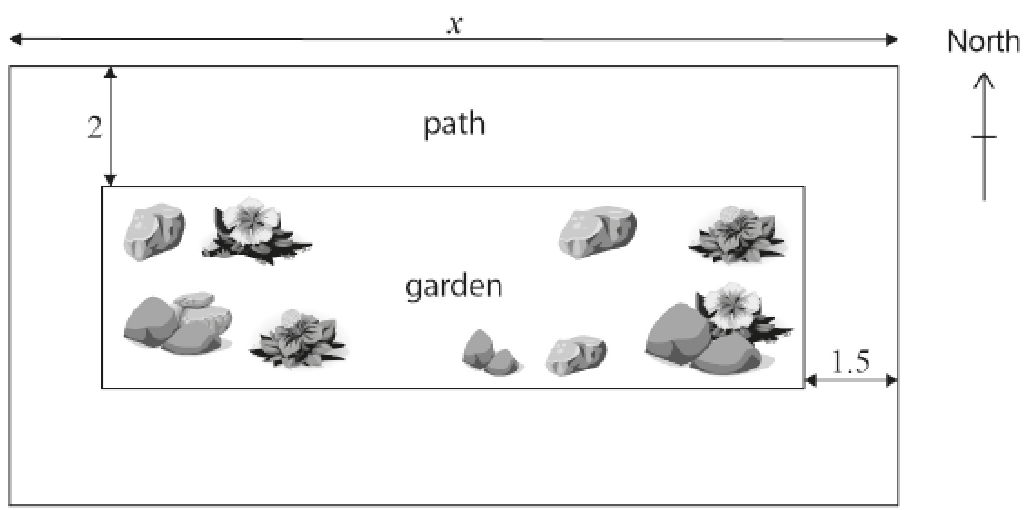
A particular park consists of a rectangular garden, of area $A\,\mathrm{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\mathrm{m}^2$.

The width of the path at the north and south side of the park is $2\,\mathrm{m}$.

The width of the path at the west and east side of the park is $1.5\,\mathrm{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale



(a.i) Write down the length of the garden in terms of x. [1]

(a.ii) Find an expression for the width of the garden in terms of x. [2]

(a.iii) Hence show that $A = 1212 - 4x - \frac{3600}{x}$. [2]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\mathrm{m}^2$. [4]

(c) Find an expression for $\frac{\mathrm{d}A}{\mathrm{d}x}$. [3]

(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden.

[2]

(e) Find the maximum possible area of the garden. [2]

9. [Maximum mark: 8] 22N.1.SL.TZ0.13

Giles charges a customer per hour to hire his boat. It is known that

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 20 - \frac{980}{t^2},\ 0 < t \leq 12$$

where P is the cost per hour, in Norwegian krone (NOK), that the customer is charged and t is the time, in hours, spent on the boat.

The cost per hour has a local minimum when the boat is hired for h hours.

(a) Find the value of h. [2]

Sandra hired Giles' boat for 5 hours and was charged $\mathrm{NOK}\ 328$ per hour.
Yvonne hires Giles' boat for 7 hours.

(b) Show that the cost per hour for Yvonne is $\mathrm{NOK}\ 312$. [6]