Functions - review (TL) [170 marks]

1. [Maximum mark: 5] SPM.1.SL.TZ0.5

The functions f and g are defined such that $f(x) = \frac{x+3}{4}$ and $g(x) = 8x + 5$.

(a) Show that $(g \circ f)(x) = 2x + 11$. [2]

Markscheme

attempt to form composition ***M1***

correct substitution $g\left(\frac{x+3}{4}\right) = 8\left(\frac{x+3}{4}\right) + 5$ ***A1***

$(g \circ f)(x) = 2x + 11$ ***AG***

[2 marks]

(b) Given that $(g \circ f)^{-1}(a) = 4$, find the value of a. [3]

Markscheme

attempt to substitute 4 (seen anywhere) ***(M1)***

correct equation $a = 2 \times 4 + 11$ ***(A1)***

a = 19 ***A1***

[3 marks]

2. [Maximum mark: 6] EXN.1.SL.TZ0.5

The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = x - 2$ and $g(x) = ax + b$, where $a,\ b \in \mathbb{R}$.

Given that $(f \circ g)(2) = -3$ and $(g \circ f)(1) = 5$, find the value of a and the value of b. [6]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$(f \circ g)(x) = ax + b - 2$ **(M1)**

$(f \circ g)(2) = -3 \Rightarrow 2a + b - 2 = -3\ (2a + b = -1)$ **A1**

$(g \circ f)(x) = a(x - 2) + b$ **(M1)**

$(g \circ f)(1) = 5 \Rightarrow -a + b = 5$ **A1**

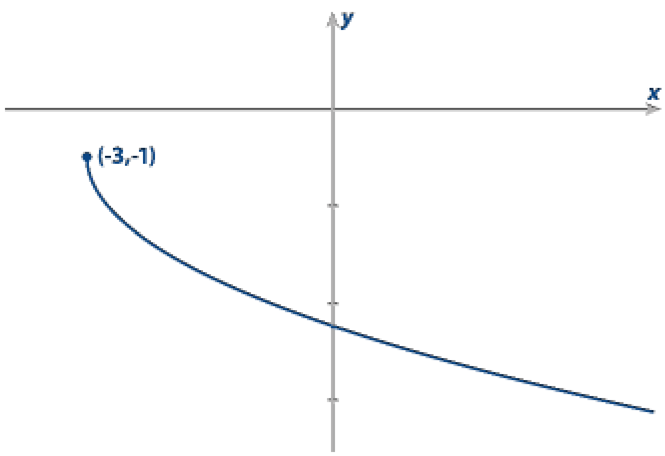
a valid attempt to solve their two linear equations for a and b **M1**

so $a = -2$ and $b = 3$ **A1**

[6 marks]

3. [Maximum mark: 14] EXN.1.SL.TZ0.8

The following diagram shows the graph of $y = -1 - \sqrt{x+3}$ for $x \geq -3$.



(a) Describe a sequence of transformations that transforms the graph of $y = \sqrt{x}$ for $x \geq 0$ to the graph of $y = -1 - \sqrt{x+3}$ for $x \geq -3$. [3]

Markscheme

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for example,

a reflection in the x-axis (in the line $y = 0$) **A1**

a horizontal translation (shift) 3 units to the left **A1**

a vertical translation (shift) down by 1 unit **A1**

Note: Award **A1** for each correct transformation applied in a correct position in the sequence. Do not accept use of the "move" for a translation.

Note: Award **A1A1A1** for a correct alternative sequence of transformations. For example,

a vertical translation (shift) down by 1 unit, followed by a horizontal translation (shift) 3 units to the left and then a reflection in the line $y = -1$.

[3 marks]

A function f is defined by $f(x) = -1 - \sqrt{x+3}$ for $x \geq -3$.

(b) State the range of f. [1]

Markscheme

range is $f(x) \leq -1$ **A1**

Note: Correct alternative notations include $]-\infty, -1]$, $(-\infty, -1]$ or $y \leq -1$.

[1 mark]

(c) Find an expression for $f^{-1}(x)$, stating its domain. [5]

Markscheme

$-1 - \sqrt{y+3} = x$ **M1**

Note: Award **M1** for interchanging x and y (can be done at a later stage).

$\sqrt{y+3} = -x - 1(= -(x+1))$ **A1**

$y + 3 = (x+1)^2$ **A1**

so $f^{-1}(x) = (x+1)^2 - 3$ $\left(f^{-1}(x) = x^2 + 2x - 2\right)$ **A1**

domain is $x \leq -1$ **A1**

Note: Correct alternative notations include $]-\infty, -1]$ or $(-\infty, -1]$.

[5 marks]

(d) Find the coordinates of the point(s) where the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect. [5]

Markscheme

the point of intersection lies on the line $y = x$

EITHER

$(x+1)^2 - 3 = x$ **M1**

attempts to solve their quadratic equation **M1**

for example, $(x+2)(x-1) = 0$ or $x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2}$ $\left(x = \frac{-1 \pm 3}{2}\right)$

OR

$-1-\sqrt{x+3}=x$ **M1**

$\left(-1-\sqrt{x+3}\right)^2=x^2 \Rightarrow 2\sqrt{x+3}+x+4=x^2$

substitutes $2\sqrt{x+3}=-2(x+1)$ to obtain
$-2(x+1)+x+4=x^2$

attempts to solve their quadratic equation **M1**

for example, $(x+2)(x-1)=0$ or
$x=\frac{-1\pm\sqrt{1^2-4(1)(-2)}}{2}$ $\left(x=\frac{-1\pm3}{2}\right)$

THEN

$x=-2, 1$ **A1**

as $x\le-1$, the only solution is $x=-2$ **R1**

so the coordinates of the point of intersection are $(-2,\ -2)$ **A1**

Note: Award **R0A1** if $(-2,\ -2)$ is stated without a valid reason given for rejecting $(1,\ 1)$.

[5 marks]

4. [Maximum mark: 17] EXN.2.SL.TZ0.9

The temperature $T\ ^\circ\mathrm{C}$ of water t minutes after being poured into a cup can be modelled by $T = T_0\mathrm{e}^{-kt}$ where $t \geq 0$ and $T_0,\ k$ are positive constants.

The water is initially boiling at $100\ ^\circ\mathrm{C}$. When $t = 10$, the temperature of the water is $70\ ^\circ\mathrm{C}$.

(a) Show that $T_0 = 100$. [1]

Markscheme

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when $t = 0,\ T = 100 \Rightarrow 100 = T_0\mathrm{e}^0$ **A1**

so $T_0 = 100$ **AG**

[1 mark]

(b) Show that $k = \frac{1}{10}\ln\frac{10}{7}$. [3]

Markscheme

correct substitution of $t = 10,\ T = 70$ **M1**

$70 = 100\mathrm{e}^{-10k}$ or $\mathrm{e}^{-10k} = \frac{7}{10}$

EITHER

$-10k = \ln\frac{7}{10}$ **A1**

$\ln \frac{7}{10} = -\ln \frac{10}{7}$ or $-\ln \frac{7}{10} = \ln \frac{10}{7}$ **A1**

OR

$e^{10k} = \frac{10}{7}$ **A1**

$10k = \ln \frac{10}{7}$ **A1**

THEN

$k = \frac{1}{10} \ln \frac{10}{7}$ **AG**

[3 marks]

(c) Find the temperature of the water when $t = 15$. [2]

Markscheme

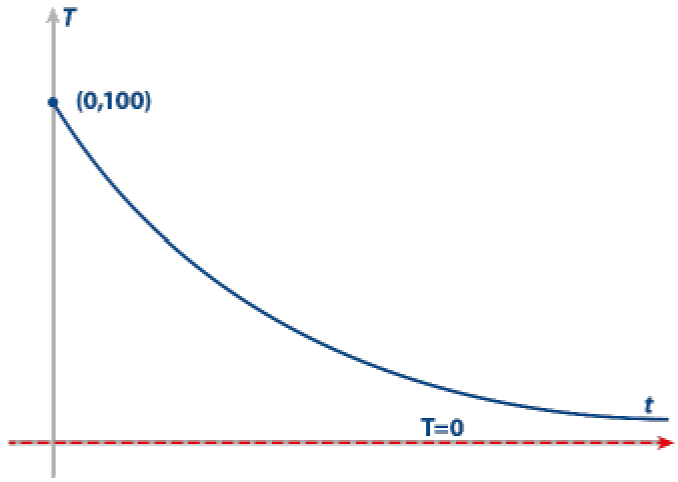
substitutes $t = 15$ into T **(M1)**

$T = 58.6\,(^\circ\mathrm{C})$ **A1**

[2 marks]

(d) Sketch the graph of T versus t, clearly indicating any asymptotes with their equations and stating the coordinates of any points of intersection with the axes. [4]

Markscheme



a decreasing exponential **A1**

starting at $(0,\ 100)$ labelled on the graph or stated **A1**

$T \to 0$ as $t \to \infty$ **A1**

horizontal asymptote $T = 0$ labelled on the graph or stated **A1**

Note: Award **A0** for stating $y = 0$ as the horizontal asymptote.

[4 marks]

(e) Find the time taken for the water to have a temperature of $50\ ^\circ\mathrm{C}$. Give your answer correct to the nearest second. [4]

Markscheme

$100\mathrm{e}^{-kt} = 50$ where $k = \frac{1}{10}\ln\frac{10}{7}$ **A1**

EITHER

uses an appropriate graph to attempt to solve for t **(M1)**

OR

manipulates logs to attempt to solve for t e.g. $\ln \frac{1}{2} = \left(-\frac{1}{10}\ln \frac{10}{7}\right)t$ **(M1)**

$t = \frac{\ln 2}{\frac{1}{10}\ln \frac{10}{7}} = 19.433\ldots$ **A1**

THEN

temperature will be $50\ ^{\circ}\mathrm{C}$ after 19 minutes and 26 seconds **A1**

[4 marks]

(f) The model for the temperature of the water can also be expressed in the form $T = T_0 a^{\frac{t}{10}}$ for $t \geq 0$ and a is a positive constant.

Find the exact value of a. [3]

Markscheme

METHOD 1

substitutes $T_0 = 100, t = 10$ and $T = 70$ into $T = T_0 a^{\frac{t}{10}}$ **(M1)**

$70 = 100a^{\frac{10}{10}}$ **A1**

$a = \frac{7}{10}$ **A1**

METHOD 2

$100a^{\frac{t}{10}} = 100\mathrm{e}^{-kt}$ where $k = \frac{1}{10}\ln\frac{10}{7}$

EITHER

$\mathrm{e}^{-k} = a^{\frac{1}{10}} \Rightarrow a = \mathrm{e}^{-10k}$ **(M1)**

OR

$a = \left(\mathrm{e}^{\left(-\frac{1}{10}\ln\frac{10}{7}\right)t}\right)^{\frac{10}{t}}$ **(M1)**

THEN

$a = \mathrm{e}^{-\ln\frac{10}{7}} \left(= \mathrm{e}^{\ln\frac{7}{10}}\right)$ **A1**

$a = \frac{7}{10}$ **A1**

[3 marks]

5. [Maximum mark: 7] 23M.1.SL.TZ1.2

The function f is defined by $f\left(x\right) = \frac{7x+7}{2x-4}$ for $x \in \mathbb{R}, x \neq 2$.

(a) Find the zero of $f(x)$. [2]

Markscheme

recognizing $f(x) = 0$ ***(M1)***

$x = -1$ ***A1***

[2 marks]

(b) For the graph of $y = f(x)$, write down the equation of

(b.i) the vertical asymptote; [1]

Markscheme

$x = 2$ (must be an equation with x) ***A1***

[1 mark]

(b.ii) the horizontal asymptote. [1]

Markscheme

$y = \frac{7}{2}$ (must be an equation with y) ***A1***

[1 mark]

(c) Find $f^{-1}(x)$, the inverse function of $f(x)$. [3]

Markscheme

EITHER

interchanging x and y ***(M1)***

$2xy - 4x = 7y + 7$

correct working with y terms on the same side: $2xy - 7y = 4x + 7$ ***(A1)***

OR

$2yx - 4y = 7x + 7$

correct working with x terms on the same side: $2yx - 7x = 4y + 7$ ***(A1)***

interchanging x and y OR making x the subject $x = \frac{4y+7}{2y-7}$ ***(M1)***

THEN

$f^{-1}(x) = \frac{4x+7}{2x-7}$ (or equivalent) $\left(x \neq \frac{7}{2}\right)$ ***A1***

[3 marks]

6. [Maximum mark: 5] 23M.1.SL.TZ2.3

A function f is defined by $f(x) = 1 - \frac{1}{x-2}$, where $x \in \mathbb{R}, x \neq 2$.

(a) The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

Write down the equation of

(a.i) the vertical asymptote; [1]

Markscheme

$x = 2$ ***A1***

[1 mark]

(a.ii) the horizontal asymptote. [1]

Markscheme

$y = 1$ ***A1***

[1 mark]

(b) Find the coordinates of the point where the graph of $y = f(x)$ intersects

(b.i) the y-axis; [1]

Markscheme

$\left(0, \frac{3}{2}\right)$ ***A1***

[1 mark]

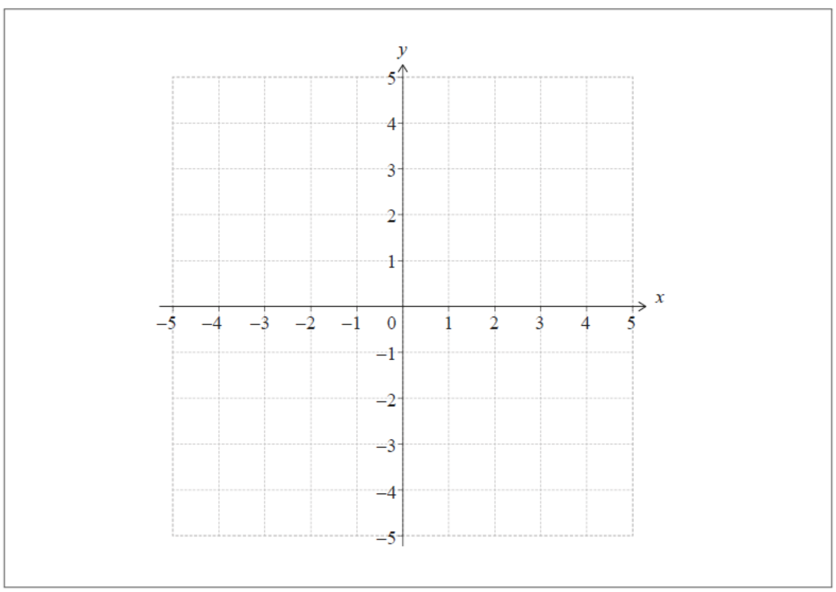
(b.ii) the x-axis. [1]

Markscheme

$(3, 0)$ ***A1***

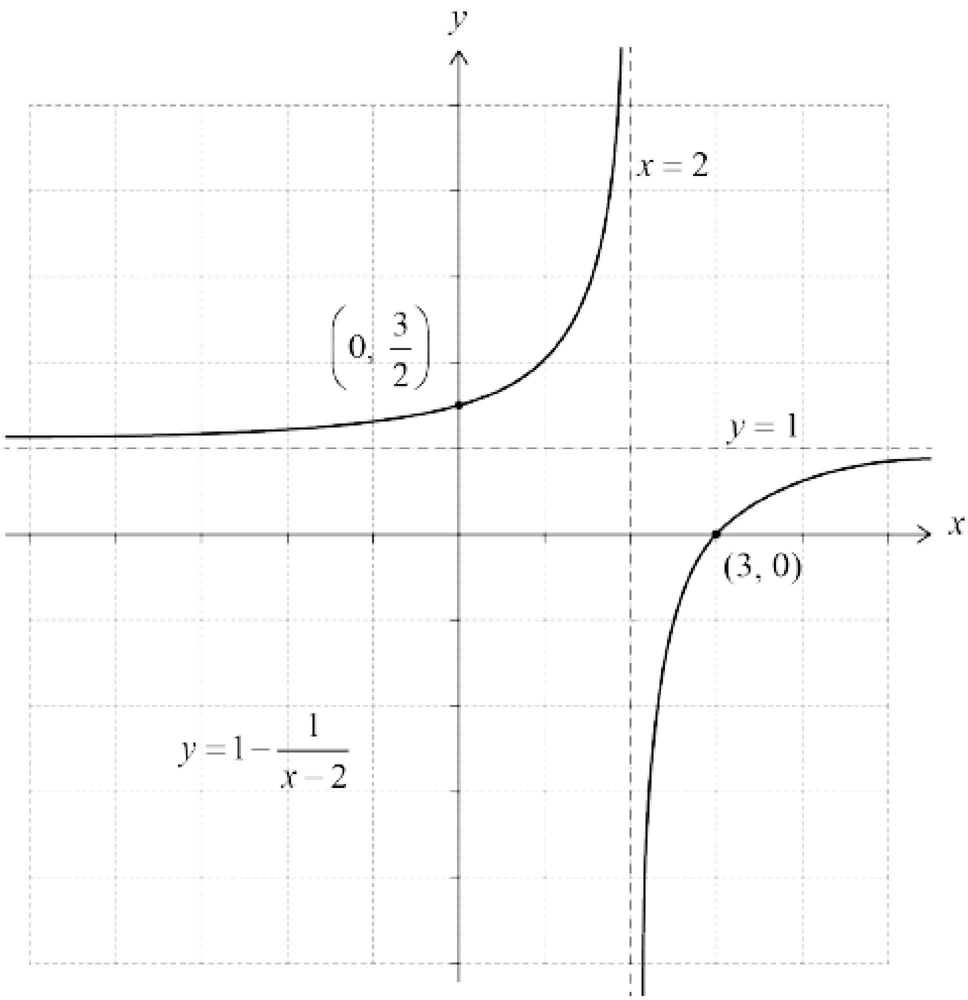
[1 mark]

(c) On the following set of axes, sketch the graph of $y = f(x)$, showing all the features found in parts (a) and (b).



[1]

Markscheme



two correct branches with correct asymptotic behaviour and intercepts clearly shown ***A1***

[1 mark]

7. [Maximum mark: 7] 23M.2.SL.TZ1.7

The temperature of a cup of tea, t minutes after it is poured, can be modelled by $H(t) = 21 + 75e^{-0.08t},\ t \geq 0$. The temperature is measured in degrees Celsius ($^\circ\mathrm{C}$).

(a.i) Find the initial temperature of the tea. [1]

Markscheme

$96\,(^\circ)$ (exact) ***A1***

[1 mark]

(a.ii) Find the temperature of the tea three minutes after it is poured. [1]

Markscheme

$79.9970\ldots$

$80.0\,(^\circ)$ (accept 80) ***A1***

[1 mark]

(b) After k minutes, the tea will be below $67\ ^\circ\mathrm{C}$ and cool enough to drink.

Find the least possible value of k, where $k \in \mathbb{Z}^+$. [3]

Markscheme

METHOD 1

valid attempt to solve $H(t) = 67$ (accept an inequality) ***(M1)***

eg intersection of graphs, use of logarithms.

6.11058… ***(A1)***

7 (min) ***A1***

METHOD 2

valid attempt to find crossover values ***(M1)***

(6, 67.4087…) and (7, 63.8406…) ***(A1)***

7 (min) ***A1***

[3 marks]

As the tea cools, $H(t)$ approaches the temperature of the room, which is constant.

(c) Find the temperature of the room. [2]

Markscheme

recognition that $t \to \infty$ ***(M1)***

$21\ (^\circ\mathrm{C})$ ***A1***

[2 marks]

8. [Maximum mark: 7] 22N.2.SL.TZ0.5

The population of a town t years after 1 January 2014 can be modelled by the function

$P(t) = 15\,000e^{kt}$, where $k < 0$ and $t \geq 0$.

It is known that between 1 January 2014 and 1 January 2022 the population decreased by 11%.

Use this model to estimate the population of this town on 1 January 2041. [7]

Markscheme

recognition that initial population is 15000 (seen anywhere) ***(A1)***

$P(0) = 15000$ OR 0.11×15000 OR 0.89×15000

population after 11% decrease is $15000 \times 0.89 (= 13350)$ ***(A1)***

recognizing that $t = 8$ on 1 January 2022 (seen anywhere) ***(A1)***

substitution of their value of t for 1 January 2022 and their value of $P(8)$ into the model ***(M1)***

$15000 \times 0.89 = 15000e^{8k}$ OR $13350 = 15000e^{8k}$

$k = \frac{\ln 0.89}{8} (-0.014566)$ ***(A1)***

substitution of $t = 2041 - 2014 (= 27)$ and their value for k into the model ***(M1)***

$P(27) = 15000e^{-0.0145\ldots \times 27}$

$10122.3\ldots$

$P(27) = 10100 \; (10122)$ ***A1***

[7 marks]

9. [Maximum mark: 5] 22M.1.SL.TZ2.1

The following table shows values of $f(x)$ and $g(x)$ for different values of x.

Both f and g are one-to-one functions.

x	−2	0	3	4
$f(x)$	8	4	0	−3
$g(x)$	−5	−2	4	0

(a) Find $g(0)$. [1]

Markscheme

$g(0) = -2$ ***A1***

[1 mark]

(b) Find $(f \circ g)(0)$. [2]

Markscheme

evidence of using composite function ***(M1)***

$f(g(0))$ OR $f(-2)$

$(f \circ g)(0) = 8$ ***A1***

[2 marks]

(c) Find the value of x such that $f(x) = 0$. [2]

Markscheme

$x = 3$ ***A2***

[2 marks]

10. [Maximum mark: 8] 22M.1.AHL.TZ2.3

A function f is defined by $f(x) = \frac{2x-1}{x+1}$, where $x \in \mathbb{R},\ x \neq -1$.

The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

(a.i) Write down the equation of the vertical asymptote. [1]

Markscheme

$x = -1$ ***A1***

[1 mark]

(a.ii) Write down the equation of the horizontal asymptote. [1]

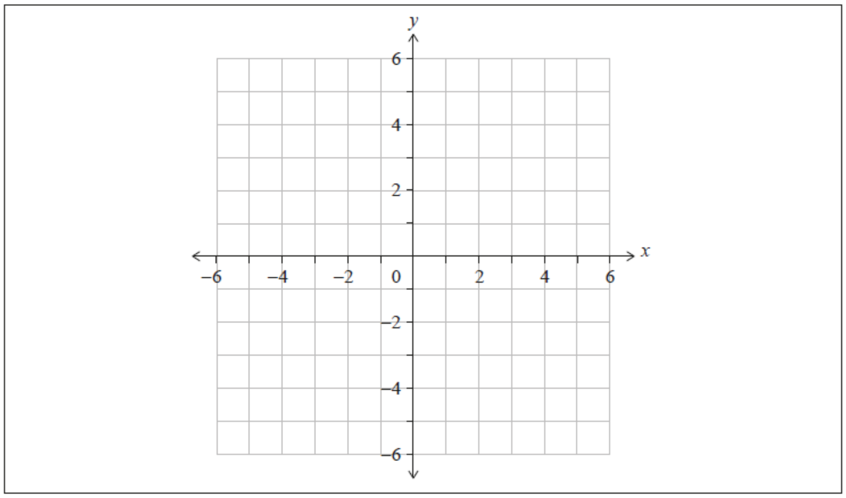
Markscheme

$y = 2$ ***A1***

[1 mark]

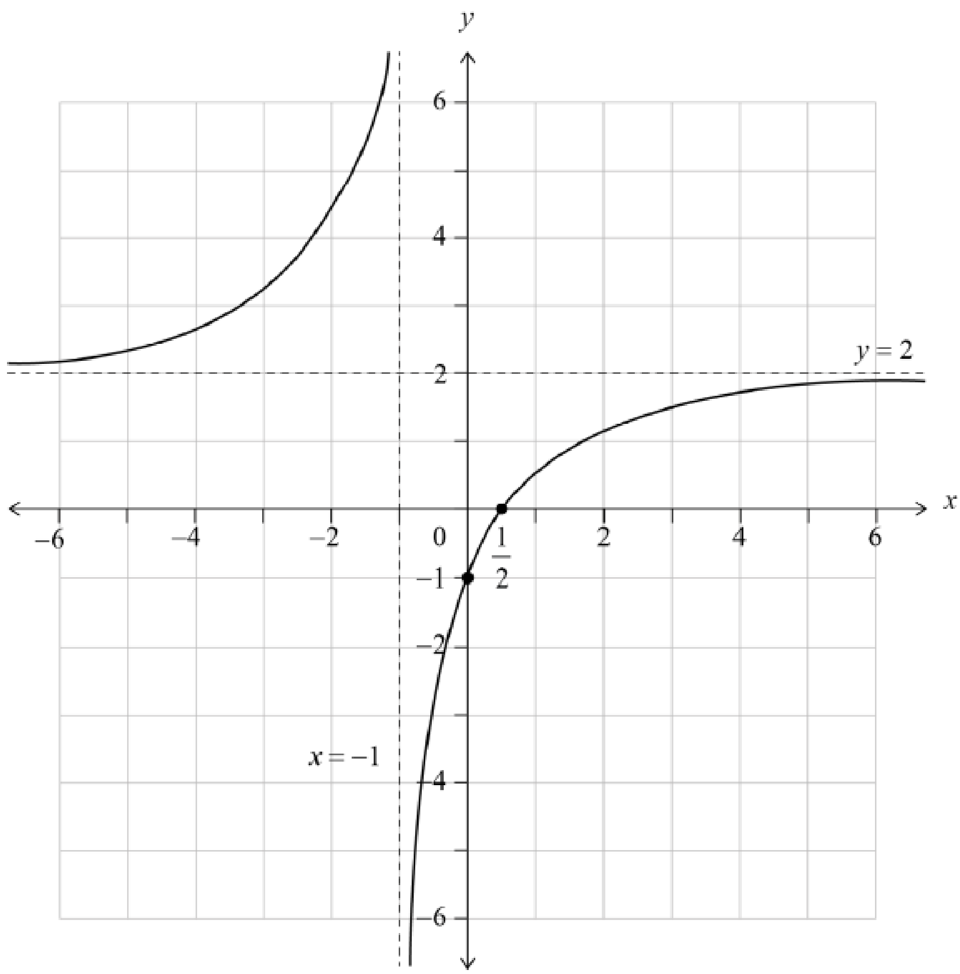
(b) On the set of axes below, sketch the graph of $y = f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes.



[3]

Markscheme



rational function shape with two branches in opposite quadrants, with two correctly positioned asymptotes and asymptotic behaviour shown ***A1***

axes intercepts clearly shown at $x = \frac{1}{2}$ and $y = -1$ ***A1A1***

[3 marks]

(c) Hence, solve the inequality $0 < \frac{2x-1}{x+1} < 2$. [1]

Markscheme

$x > \frac{1}{2}$ ***A1***

Note: Accept correct alternative correct notation, such as $\left(\frac{1}{2},\ \infty\right)$ and $\left]\frac{1}{2}, \infty\right[$.

[1 mark]

(d) Solve the inequality $0 < \frac{2|x|-1}{|x|+1} < 2$. [2]

Markscheme

EITHER

attempts to sketch $y = \frac{2|x|-1}{|x|+1}$ ***(M1)***

OR

attempts to solve $2|x| - 1 = 0$ ***(M1)***

Note: Award the ***(M1)*** if $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ are identified.

THEN

$x < -\frac{1}{2}$ or $x > \frac{1}{2}$ ***A1***

Note: Accept the use of a comma. Condone the use of 'and'. Accept correct alternative notation.

[2 marks]

11. [Maximum mark: 15] 21N.1.SL.TZ0.8

Consider the function $f(x) = a^x$ where $x,\ a \in \mathbb{R}$ and $x > 0,\ a > 1.$

The graph of f contains the point $\left(\frac{2}{3}, 4\right)$.

(a) Show that $a = 8.$ [2]

Markscheme

$f\left(\frac{2}{3}\right) = 4$ OR $a^{\frac{2}{3}} = 4$ ***(M1)***

$a = 4^{\frac{3}{2}}$ OR $a = \left(2^2\right)^{\frac{3}{2}}$ OR $a^2 = 64$ OR $\sqrt[3]{a} = 2$ ***A1***

$a = 8$ ***AG***

[2 marks]

(b) Write down an expression for $f^{-1}(x)$. [1]

Markscheme

$f^{-1}(x) = \log_8 x$ ***A1***

Note: Accept $f^{-1}(x) = \log_a x$.
Accept any equivalent expression for f^{-1} e.g. $f^{-1}(x) = \frac{\ln x}{\ln 8}$.

[1 mark]

(c) Find the value of $f^{-1}\left(\sqrt{32}\right)$. [3]

Markscheme

correct substitution *(A1)*

$\log_8 \sqrt{32}$ OR $8^x = 32^{\frac{1}{2}}$

correct working involving log/index law *(A1)*

$\frac{1}{2}\log_8 32$ OR $\frac{5}{2}\log_8 2$ OR $\log_8 2 = \frac{1}{3}$ OR $\log_2 2^{\frac{5}{2}}$ OR $\log_2 8 = 3$ OR $\frac{\ln 2^{\frac{5}{2}}}{\ln 2^3}$ OR $2^{3x} = 2^{\frac{5}{2}}$

$f^{-1}\left(\sqrt{32}\right) = \frac{5}{6}$ ***A1***

[3 marks]

Consider the arithmetic sequence $\log_8 27\ ,\ \log_8 p\ ,\ \log_8 q\ ,\ \log_8 125$, where $p > 1$ and $q > 1$.

(d.i) Show that $27,\ p,\ q$ and 125 are four consecutive terms in a geometric sequence. [4]

Markscheme

METHOD 1

equating a pair of differences *(M1)*

$u_2 - u_1 = u_4 - u_3 (= u_3 - u_2)$

$\log_8 p - \log_8 27 = \log_8 125 - \log_8 q$

$\log_8 125 - \log_8 q = \log_8 q - \log_8 p$

$\log_8\left(\frac{p}{27}\right) = \log_8\left(\frac{125}{q}\right),\ \log_8\left(\frac{125}{q}\right) = \log_8\left(\frac{q}{p}\right)$ ***A1A1***

$\frac{p}{27} = \frac{125}{q}$ and $\frac{125}{q} = \frac{q}{p}$ ***A1***

$27,\ p,\ q$ and 125 are in geometric sequence ***AG***

Note: If candidate assumes the sequence is geometric, award no marks for part (i). If $r = \frac{5}{3}$ has been found, this will be awarded marks in part (ii).

METHOD 2

expressing a pair of consecutive terms, in terms of d ***(M1)***

$p = 8^d \times 27$ and $q = 8^{2d} \times 27$ OR $q = 8^{2d} \times 27$ and $125 = 8^{3d} \times 27$

two correct pairs of consecutive terms, in terms of d ***A1***

$\frac{8^d \times 27}{27} = \frac{8^{2d} \times 27}{8^d \times 27} = \frac{8^{3d} \times 27}{8^{2d} \times 27}$ (must include 3 ratios) ***A1***

all simplify to 8^d ***A1***

$27,\ p,\ q$ and 125 are in geometric sequence ***AG***

[4 marks]

(d.ii) Find the value of p and the value of q. [5]

Markscheme

METHOD 1 (geometric, finding r)

$u_4 = u_1 r^3$ OR $125 = 27(r)^3$ ***(M1)***

$r = \frac{5}{3}$ (seen anywhere) ***A1***

$p = 27r$ OR $\frac{125}{q} = \frac{5}{3}$ ***(M1)***

$p = 45,\ q = 75$ ***A1A1***

METHOD 2 (arithmetic)

$u_4 = u_1 + 3d$ OR $\log_8 125 = \log_8 27 + 3d$ ***(M1)***

$d = \log_8\left(\frac{5}{3}\right)$ (seen anywhere) ***A1***

$\log_8 p = \log_8 27 + \log_8\left(\frac{5}{3}\right)$ OR
$\log_8 q = \log_8 27 + 2\log_8\left(\frac{5}{3}\right)$ ***(M1)***

$p = 45,\ q = 75$ ***A1A1***

METHOD 3 (geometric using proportion)

recognizing proportion ***(M1)***

$pq = 125 \times 27$ OR $q^2 = 125p$ OR $p^2 = 27q$

two correct proportion equations ***A1***

attempt to eliminate either p or q ***(M1)***

$q^2 = 125 \times \frac{125 \times 27}{q}$ OR $p^2 = 27 \times \frac{125 \times 27}{p}$

$p = 45,\ q = 75$ ***A1A1***

[5 marks]

12. [Maximum mark: 9] 21N.1.AHL.TZ0.2

The function f is defined by $f(x) = \frac{2x+4}{3-x}$, where $x \in \mathbb{R},\ x \neq 3$.

Write down the equation of

(a.i) the vertical asymptote of the graph of f. [1]

Markscheme

$x = 3$ ***A1***

[1 mark]

(a.ii) the horizontal asymptote of the graph of f. [1]

Markscheme

$y = -2$ ***A1***

[1 mark]

Find the coordinates where the graph of f crosses

(b.i) the x-axis. [1]

Markscheme

$(-2, 0)$ (accept $x = -2$) ***A1***

[1 mark]

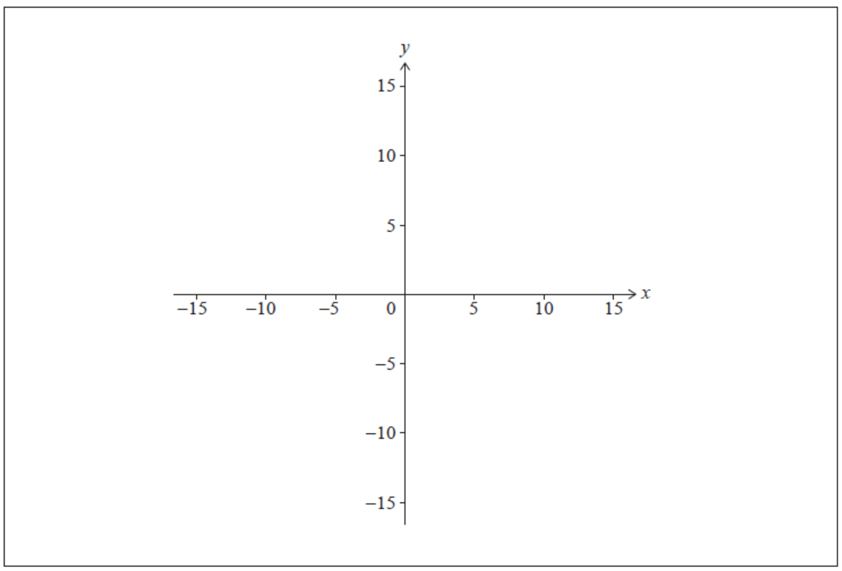
(b.ii) the y-axis. [1]

Markscheme

$\left(0,\ \frac{4}{3}\right)$ (accept $y = \frac{4}{3}$ and $f(0) = \frac{4}{3}$) ***A1***

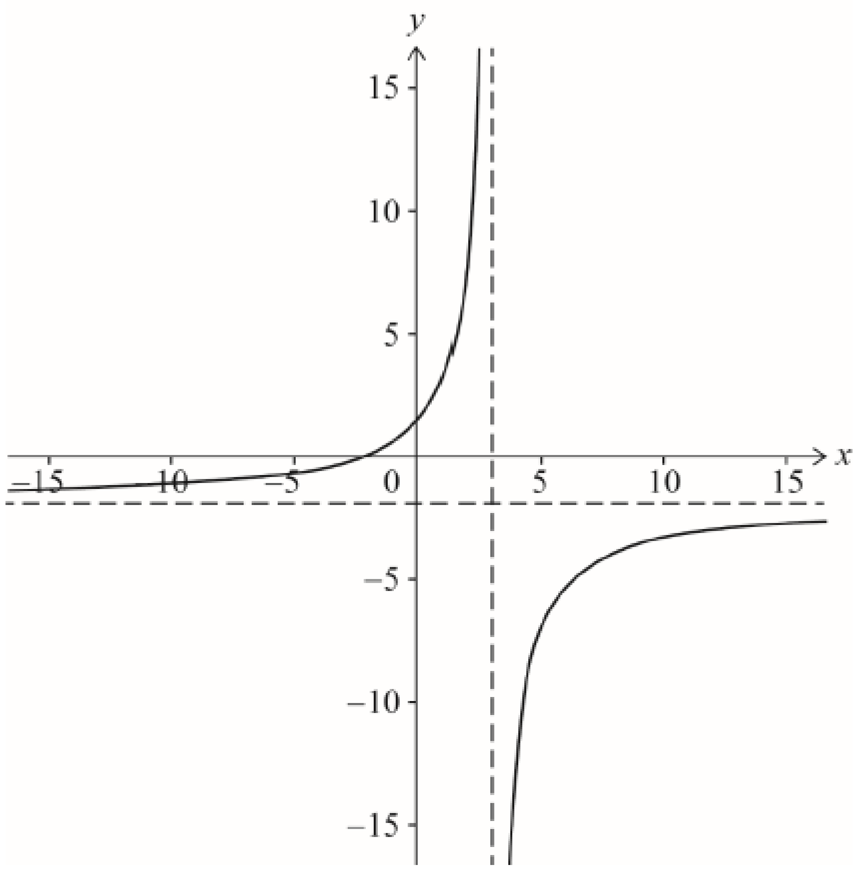
[1 mark]

(c) Sketch the graph of f on the axes below.



[1]

Markscheme



A1

Note: Award ***A1*** for completely correct shape: two branches in correct quadrants with asymptotic behaviour.

[1 mark]

(d) The function g is defined by $g(x) = \frac{ax+4}{3-x}$, where $x \in \mathbb{R},\ x \neq 3$ and $a \in \mathbb{R}$.

Given that $g(x) = g^{-1}(x)$, determine the value of a. [4]

Markscheme

METHOD 1

$(g(x) =)y = \frac{ax+4}{3-x}$

attempt to find x in terms of y ***(M1)***

OR exchange x and y and attempt to find y in terms of x

$3y - xy = ax + 4$ ***A1***

$ax + xy = 3y - 4$

$x(a + y) = 3y - 4$

$x = \frac{3y-4}{y+a}$

$g^{-1}(x) = \frac{3x-4}{x+a}$ ***A1***

Note: Condone use of $y =$

$g(x) \equiv g^{-1}(x)$

$\frac{ax+4}{3-x} \equiv \frac{3x-4}{x+a}$

$\Rightarrow a = -3$ ***A1***

METHOD 2

$g(x) = \frac{ax+4}{3-x}$

attempt to find an expression for $g\left(g(x)\right)$ and equate to x ***(M1)***

$gg(x) = \frac{a\left(\frac{ax+4}{3-x}\right)+4}{3-\left(\frac{ax+4}{3-x}\right)} = x$ ***A1***

$\frac{a(ax+4)+4(3-x)}{(9-3x)-(ax+4)} = x$

$$\frac{a(ax+4)+4(3-x)}{5-(3+a)x} = x$$

$$a(ax+4)+4(3-x) = x(5-(3+a)x)$$ ***A1***

equating coefficients of x^2 (or similar)

$$a = -3$$ ***A1***

[4 marks]

13. [Maximum mark: 5] 21N.1.AHL.TZ0.3

Solve the equation $\log_3 \sqrt{x} = \frac{1}{2 \log_2 3} + \log_3 (4x^3)$, where $x > 0$. [5]

Markscheme

attempt to use change the base ***(M1)***

$$\log_3 \sqrt{x} = \frac{\log_3 2}{2} + \log_3 (4x^3)$$

attempt to use the power rule ***(M1)***

$$\log_3 \sqrt{x} = \log_3 \sqrt{2} + \log_3 (4x^3)$$

attempt to use product or quotient rule for logs, $\ln a + \ln b = \ln ab$ ***(M1)***

$$\log_3 \sqrt{x} = \log_3 \left(4\sqrt{2}x^3\right)$$

Note: The ***M*** marks are for attempting to use the relevant log rule and may be applied in any order and at any time during the attempt seen.

$$\sqrt{x} = 4\sqrt{2}x^3$$

$$x = 32x^6$$

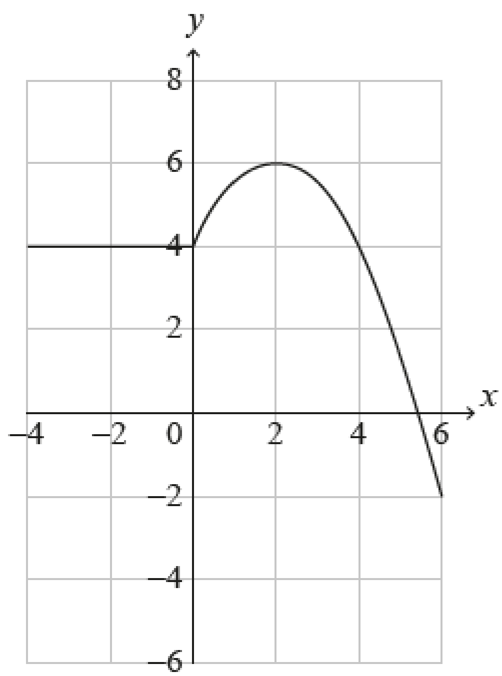
$x^5 = \frac{1}{32}$ ***(A1)***

$x = \frac{1}{2}$ ***A1***

[5 marks]

14. [Maximum mark: 5] 21M.1.SL.TZ1.1

The graph of $y = f(x)$ for $-4 \leq x \leq 6$ is shown in the following diagram.



(a.i) Write down the value of $f(2)$. [1]

Markscheme

$f(2) = 6$ ***A1***

[1 mark]

(a.ii) Write down the value of $(f \circ f)(2)$. [1]

Markscheme

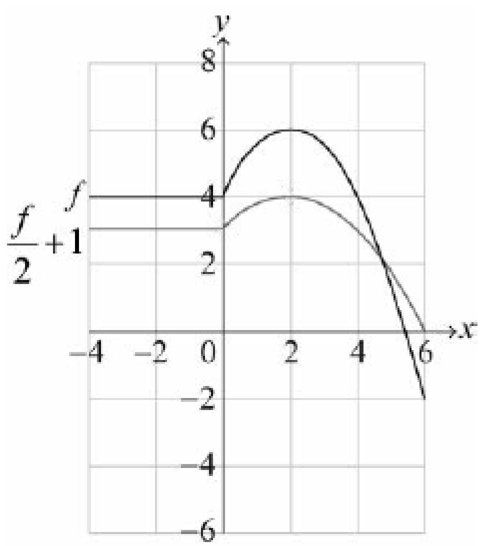
$(f \circ f)(2) = -2$ ***A1***

[1 mark]

(b) Let $g(x) = \frac{1}{2}f(x) + 1$ for $-4 \le x \le 6$. On the axes above, sketch the graph of g.

[3]

Markscheme



M1A1A1

Note: Award ***M1*** for an attempt to apply any vertical stretch or vertical translation, ***A1*** for a correct horizontal line segment between -4 and 0 (located roughly at $y = 3$),
A1 for a correct concave down parabola including max point at $(2, 4)$ and for correct end points at $(0, 3)$ and $(6, 0)$ (within circles). Points do not need to be labelled.

[3 marks]

15. [Maximum mark: 7] 21M.2.SL.TZ2.6

All living plants contain an isotope of carbon called carbon-14. When a plant dies, the isotope decays so that the amount of carbon-14 present in the remains of the plant decreases. The time since the death of a plant can be determined by measuring the amount of carbon-14 still present in the remains.

The amount, A, of carbon-14 present in a plant t years after its death can be modelled by $A = A_0 \mathrm{e}^{-kt}$ where $t \geq 0$ and $A_0,\ k$ are positive constants.

At the time of death, a plant is defined to have 100 units of carbon-14.

(a) Show that $A_0 = 100$. [1]

Markscheme

$100 = A_0 \mathrm{e}^0$ ***A1***

$A_0 = 100$ ***AG***

[1 mark]

The time taken for half the original amount of carbon-14 to decay is known to be 5730 years.

(b) Show that $k = \frac{\ln 2}{5730}$. [3]

Markscheme

correct substitution of values into exponential equation ***(M1)***

$50 = 100\mathrm{e}^{-5730k}$ OR $\mathrm{e}^{-5730k} = \frac{1}{2}$

EITHER

$-5730k = \ln \frac{1}{2}$ ***A1***

$\ln \frac{1}{2} = -\ln 2$ OR $-\ln \frac{1}{2} = \ln 2$ ***A1***

OR

$e^{5730k} = 2$ ***A1***

$5730k = \ln 2$ ***A1***

THEN

$k = \frac{\ln 2}{5730}$ ***AG***

Note: There are many different ways of showing that $k = \frac{\ln 2}{5730}$ which involve showing different steps. Award full marks for at least two correct algebraic steps seen.

[3 marks]

(c) Find, correct to the nearest 10 years, the time taken after the plant's death for 25% of the carbon-14 to decay. [3]

Markscheme

if 25% of the carbon-14 has decayed, 75% remains ie, 75 units remain ***(A1)***

$75 = 100e^{-\frac{\ln 2}{5730}t}$

EITHER

using an appropriate graph to attempt to solve for t *(M1)*

OR

manipulating logs to attempt to solve for t *(M1)*

$\ln 0.75 = -\frac{\ln 2}{5730} t$

$t = 2378.164\ldots$

THEN

$t = 2380$ (years) (correct to the nearest 10 years) ***A1***

[3 marks]

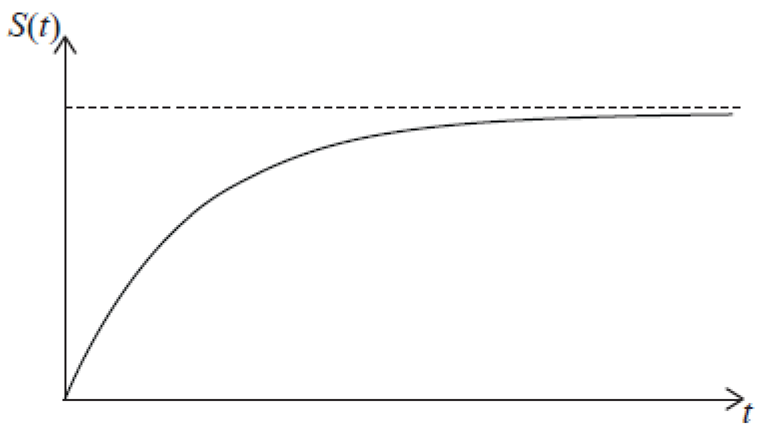
16. [Maximum mark: 6] 20N.1.SL.TZ0.T_12

Jean-Pierre jumps out of an airplane that is flying at constant altitude. Before opening his parachute, he goes through a period of freefall.

Jean-Pierre's vertical speed during the time of freefall, S, in $\mathrm{m\,s^{-1}}$, is modelled by the following function.

$$S(t) = K - 60(1.2^{-t})\ ,\ t \geq 0$$

where t, is the number of seconds after he jumps out of the airplane, and K is a constant. A sketch of Jean-Pierre's vertical speed against time is shown below.



Jean-Pierre's initial vertical speed is $0\,\mathrm{m\,s^{-1}}$.

(a) Find the value of K. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure. It appeared in a paper that permitted the use of a calculator, and so might not be suitable for all forms of practice.

$0 = K - 60(1.2^0)$ ***(M1)***

Note: Award ***(M1)*** for correctly substituted function equated to zero.

$(K =)\ 60$ ***(A1) (C2)***

[2 marks]

(b) In the context of the model, state what the horizontal asymptote represents. [1]

Markscheme

the (vertical) speed that Jean-Pierre is approaching (as t increases) ***(A1) (C1)***
OR
the limit of the (vertical) speed of Jean-Pierre ***(A1) (C1)***

Note: Accept "maximum speed" or "terminal speed".

[1 mark]

(c) Find Jean-Pierre's vertical speed after 10 seconds. Give your answer in $\mathrm{km\,h^{-1}}$. [3]

Markscheme

$(S =)\ 60 - 60\left(1.2^{-10}\right)$ ***(M1)***

Note: Award ***(M1)*** for correctly substituted function.

$(S =)\ 50.3096\ldots\ \left(\mathrm{m\,s^{-1}}\right)$ ***(A1)*(ft)**

Note: Follow through from part (a).

$181\ (\text{km h}^{-1})\ \ (181.114\ldots\ (\text{km h}^{-1}))$ ***(A1)*(ft)** *(C3)*

Note: Award the final ***(A1)*(ft)** for correct conversion of their speed to km h^{-1}.

[3 marks]

17. [Maximum mark: 6] 20N.1.SL.TZ0.S_4

Let $f(x) = a\ \log_3(x - 4)$, for $x > 4$, where $a > 0$.

Point $\mathrm{A}(13,\ 7)$ lies on the graph of f.

(a) Find the value of a. [3]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to substitute coordinates (in any order) into f ***(M1)***

eg $a\ \log_3(13 - 4) = 7\ ,\ a\ \log_3(7 - 4) = 13\ ,\ a\ \log\ 9 = 7$

finding $\log_3 9 = 2$ (seen anywhere) ***(A1)***

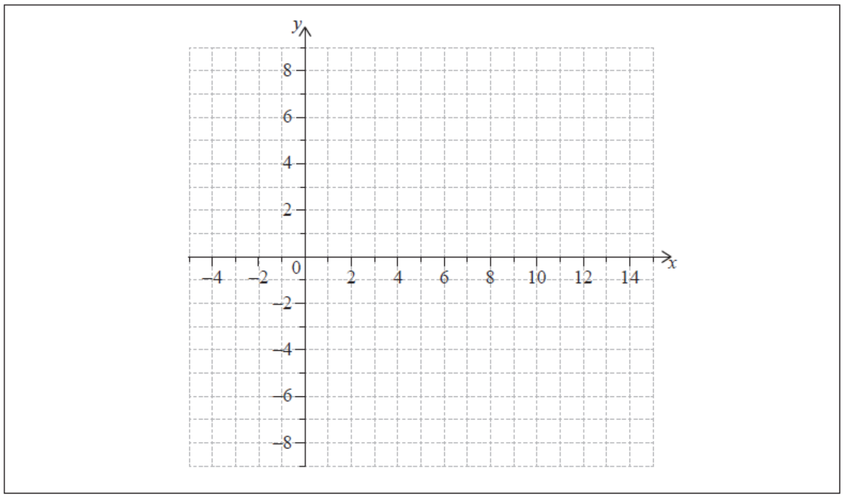
eg $\log_3 9 = 2\ ,\ 2a = 7$

$a = \frac{7}{2}$ ***A1 N2***

[3 marks]

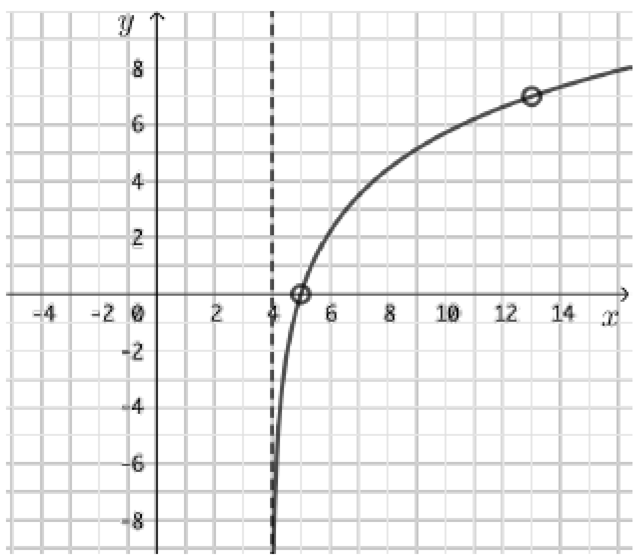
(b) The x-intercept of the graph of f is $(5,\ 0)$.

On the following grid, sketch the graph of f.



[3]

Markscheme



A1A1A1 N3

Note: Award ***A1*** for correct shape of logarithmic function (must be increasing and concave down).
Only if the shape is correct, award the following:
A1 for being asymptotic to $x = 4$
A1 for curve including both points in circles.

[3 marks]

18. [Maximum mark: 15] 20N.1.AHL.TZ0.H_12

Consider the function defined by $f(x) = \frac{kx-5}{x-k}$, where $x \in \mathbb{R}\setminus\{k\}$ and $k^2 \neq 5$.

(a) State the equation of the vertical asymptote on the graph of $y = f(x)$. [1]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$x = k$ ***A1***

[1 mark]

(b) State the equation of the horizontal asymptote on the graph of $y = f(x)$. [1]

Markscheme

$y = k$ ***A1***

[1 mark]

(c) Use an algebraic method to determine whether f is a self-inverse function. [4]

Markscheme

METHOD 1

$$(f \circ f)(x) = \frac{k\left(\frac{kx-5}{x-k}\right)-5}{\left(\frac{kx-5}{x-k}\right)-k}$$ ***M1***

$$= \frac{k(kx-5)-5(x-k)}{kx-5-k(x-k)}$$ ***A1***

$$= \frac{k^2x-5k-5x+5k}{kx-5-kx+k^2}$$

$$= \frac{k^2x-5x}{k^2-5}$$ ***A1***

$$= \frac{x\left(k^2-5\right)}{k^2-5}$$

$$= x$$

$(f \circ f)(x) = x$, (hence f is self-inverse) ***R1***

Note: The statement $f(f(x)) = x$ could be seen anywhere in the candidate's working to award ***R1***.

METHOD 2

$$f(x) = \frac{kx-5}{x-k}$$

$$x = \frac{ky-5}{y-k}$$ ***M1***

Note: Interchanging x and y can be done at any stage.

$$x(y-k) = ky-5$$ ***A1***

$$xy - xk = ky - 5$$

$$xy - ky = xk - 5$$

$$y(x-k) = kx-5$$ ***A1***

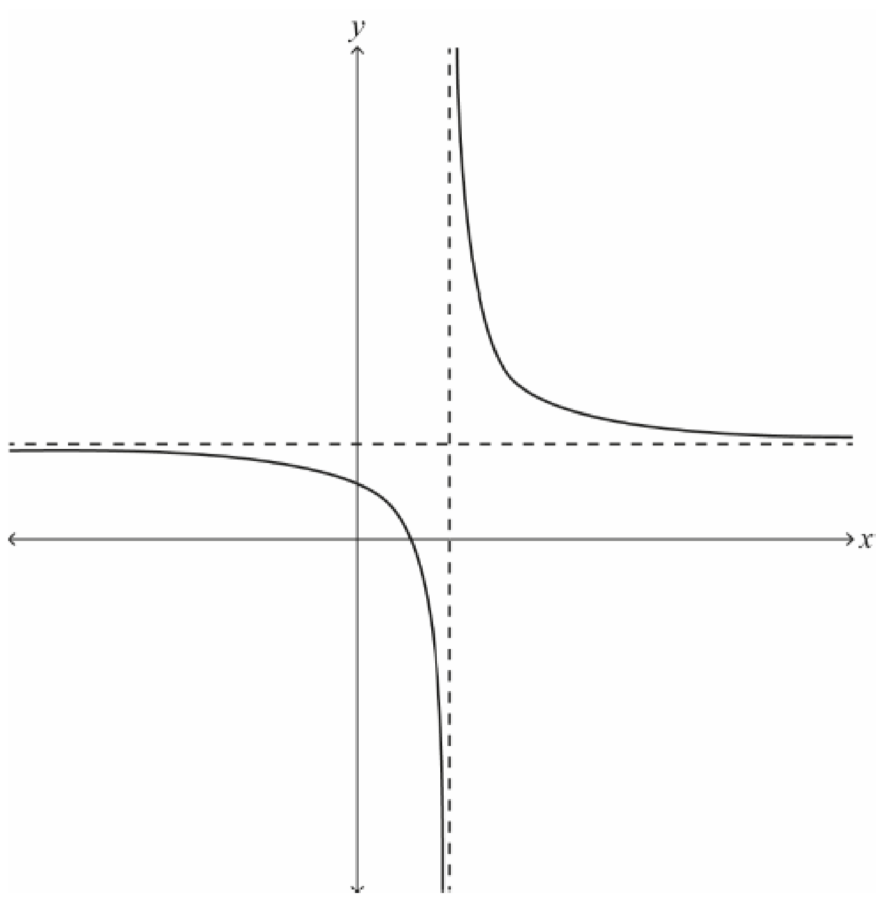
$y = f^{-1}(x) = \frac{kx-5}{x-k}$ (hence f is self-inverse) ***R1***

[4 marks]

Consider the case where $k = 3$.

(d) Sketch the graph of $y = f(x)$, stating clearly the equations of any asymptotes and the coordinates of any points of intersections with the coordinate axes. [3]

Markscheme



attempt to draw both branches of a rectangular hyperbola ***M1***

$x = 3$ and $y = 3$ ***A1***

$\left(0,\ \frac{5}{3}\right)$ and $\left(\frac{5}{3},\ 0\right)$ ***A1***

[3 marks]

(e) The region bounded by the x-axis, the curve $y = f(x)$, and the lines $x = 5$ and $x = 7$ is rotated through 2π about the x-axis. Find the volume of the solid generated, giving your answer in the form $\pi(a + b\ \ln 2)$, where $a,\ b \in \mathbb{Z}$. [6]

Markscheme

METHOD 1

$\text{volume} = \pi \int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$ ***(M1)***

EITHER

attempt to express $\frac{3x-5}{x-3}$ in the form $p + \frac{q}{x-3}$ ***M1***

$\frac{3x-5}{x-3} = 3 + \frac{4}{x-3}$ ***A1***

OR

attempt to expand $\left(\frac{3x-5}{x-3}\right)^2$ or $(3x-5)^2$ and divide out ***M1***

$\left(\frac{3x-5}{x-3}\right)^2 = 9 + \frac{24x-56}{(x-3)^2}$ ***A1***

THEN

$\left(\frac{3x-5}{x-3}\right)^2 = 9 + \frac{24}{x-3} + \frac{16}{(x-3)^2}$ ***A1***

$$\text{volume} = \pi \int_5^7 \left(9 + \frac{24}{x-3} + \frac{16}{(x-3)^2}\right) \mathrm{d}x$$

$= \pi\left[9x + 24\ \ln\ (x-3) - \frac{16}{x-3}\right]_5^7$ ***A1***

$= \pi\lfloor(63 + 24\ln 4 - 4) - (45 + 24\ln 2 - 8)\rfloor$

$= \pi(22 + 24\ln 2)$ ***A1***

METHOD 2

$\text{volume} = \pi\int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$ ***(M1)***

substituting $u = x - 3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$ ***A1***

$3x - 5 = 3(u + 3) - 5 = 3u + 4$

$\text{volume} = \pi\int_2^4 \left(\frac{3u+4}{u}\right)^2 \mathrm{d}u$ ***M1***

$= \pi\int_2^4 9 + \frac{16}{u^2} + \frac{24}{u}\,\mathrm{d}u$ ***A1***

$= \pi\left[9u - \frac{16}{u} + 24\ln u\right]_2^4$ ***A1***

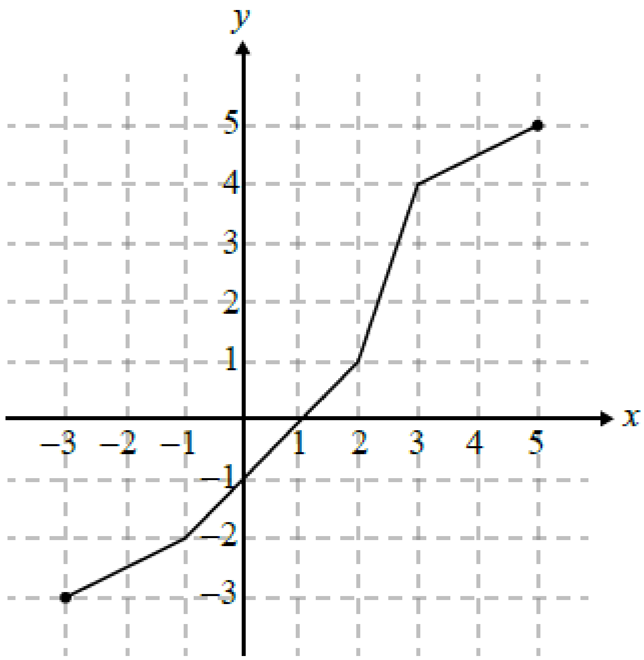
Note: Ignore absence of or incorrect limits seen up to this point.

$= \pi(22 + 24\ln 2)$ ***A1***

[6 marks]

19. [Maximum mark: 6] 19N.2.AHL.TZ0.H_3

The following diagram shows the graph of $y = f(x)$, $-3 \leqslant x \leqslant 5$.



(a) Find the value of $(f \circ f)(1)$. [2]

Markscheme

$f(1) = 0$ ***(A1)***

$f(0) = -1$ ***A1***

[2 marks]

(b) Given that $f^{-1}(a) = 3$, determine the value of a. [2]

Markscheme

$a = f(3)$ ***(M1)***

$\Rightarrow a = 4$ ***A1***

[2 marks]

(c) Given that $g\left(x\right) = 2f\left(x - 1\right)$, find the domain and range of g. [2]

Markscheme

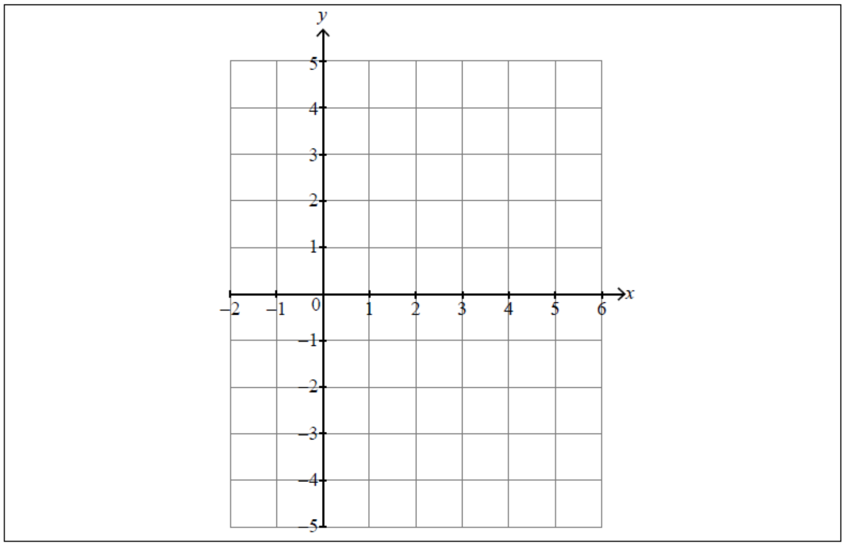
domain is $-2 \leqslant x \leqslant 6$ ***A1***

range is $-6 \leqslant y \leqslant 10$ ***A1***

[2 marks]

20. [Maximum mark: 5] 19M.1.AHL.TZ2.H_5

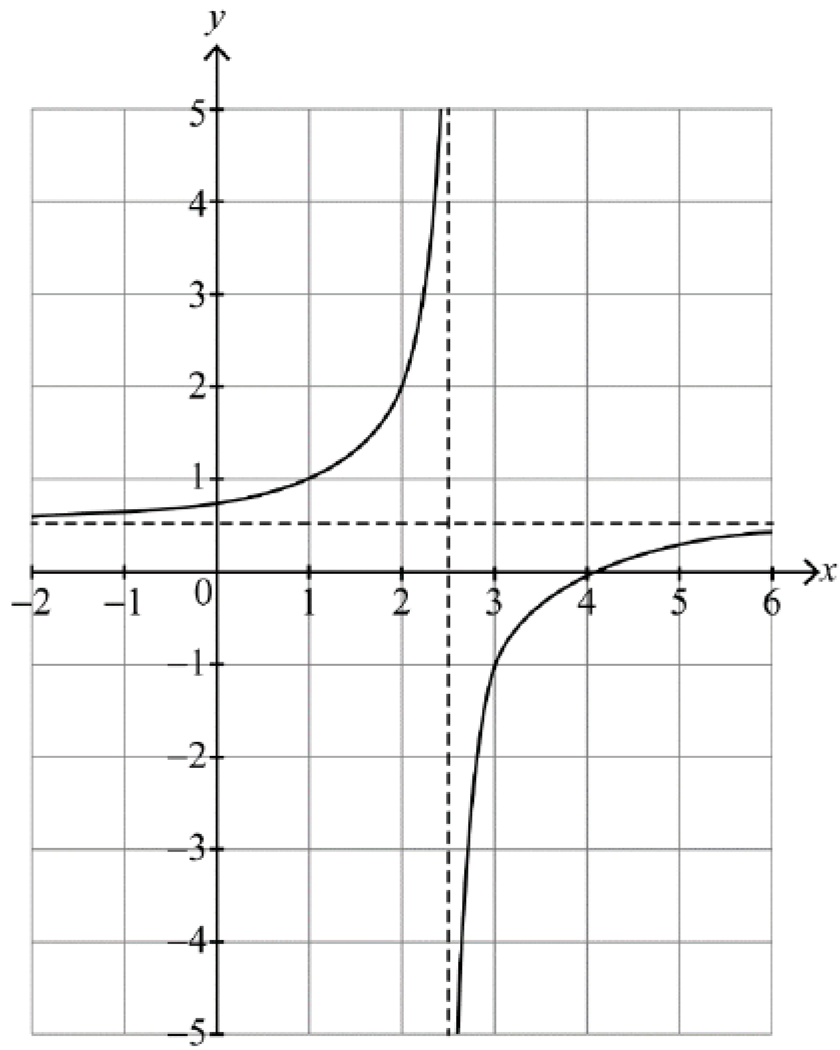
(a) Sketch the graph of $y = \frac{x-4}{2x-5}$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes.



[5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.



correct shape: two branches in correct quadrants with asymptotic behaviour ***A1***

crosses at (4, 0) and $\left(0,\ \frac{4}{5}\right)$ ***A1A1***

asymptotes at $x = \frac{5}{2}$ and $y = \frac{1}{2}$ ***A1A1***

[5 marks]

21. [Maximum mark: 10] 19M.1.AHL.TZ2.H_11

Consider the functions f and g defined by $f(x) = \ln|x|, x \in \mathbb{R} \setminus \{0\}$, and $g(x) = \ln|x+k|, x \in \mathbb{R} \setminus \{-k\}$, where $k \in \mathbb{R}, k > 2$.

(a) Describe the transformation by which $f(x)$ is transformed to $g(x)$. [1]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

translation k units to the left (or equivalent) ***A1***

[1 mark]

(b) State the range of g. [1]

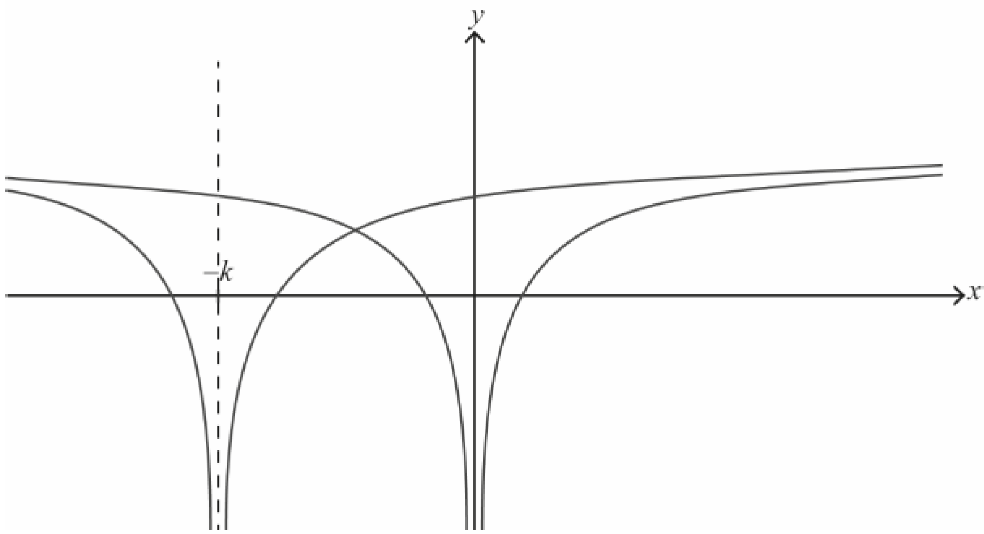
Markscheme

range is $(g(x) \in)\mathbb{R}$ ***A1***

[1 mark]

(c) Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same axes, clearly stating the points of intersection with any axes. [6]

Markscheme



correct shape of $y = f\left(x\right)$ ***A1***

their $f\left(x\right)$ translated k units to left (possibly shown by $x = -k$ marked on x-axis) ***A1***

asymptote included and marked as $x = -k$ ***A1***

$f\left(x\right)$ intersects x-axis at $x = -1, x = 1$ ***A1***

$g\left(x\right)$ intersects x-axis at $x = -k - 1, x = -k + 1$ ***A1***

$g\left(x\right)$ intersects y-axis at $y = \ln k$ ***A1***

Note: Do not penalise candidates if their graphs "cross" as $x \to \pm\infty$.

Note: Do not award ***FT*** marks from the candidate's part (a) to part (c).

[6 marks]

The graphs of f and g intersect at the point P .

(d) Find the coordinates of P. [2]

Markscheme

at P $\ln(x+k) = \ln(-x)$

attempt to solve $x+k=-x$ (or equivalent) ***(M1)***

$x = -\frac{k}{2} \Rightarrow y = \ln\left(\frac{k}{2}\right)$ (or $y = \ln\left|\frac{k}{2}\right|$) ***A1***

P$\left(-\frac{k}{2},\ \ln\frac{k}{2}\right)$ (or P$\left(-\frac{k}{2},\ \ln\left|\frac{k}{2}\right|\right)$)

[2 marks]