Functions - review (TL) [170 marks]

1. [Maximum mark: 5] SPM.1.SL.TZ0.5

The functions f and g are defined such that $f\left(x\right)=\frac{x+3}{4}$ and $g\left(x\right)=8x+5$.

(a) Show that $\left(g\circ f\right)\left(x\right)=2x+11$. [2]

(b) Given that $\left(g\circ f\right)^{-1}\left(a\right)=4$, find the value of a. [3]

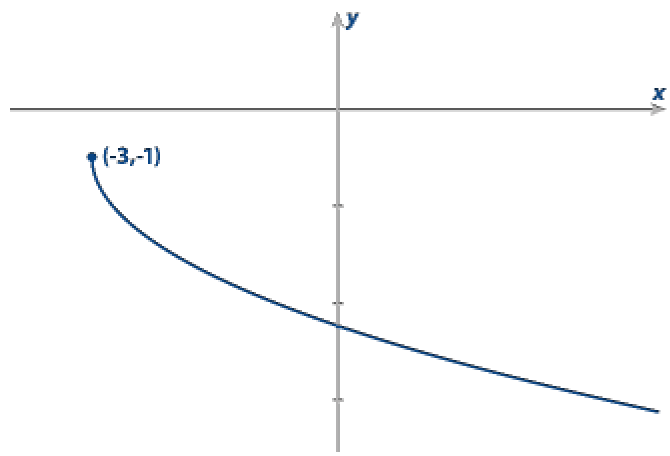
2. [Maximum mark: 6] EXN.1.SL.TZ0.5

The functions f and g are defined for $x\in\mathbb{R}$ by $f(x)=x-2$ and $g(x)=ax+b$, where $a,\ b\in\mathbb{R}$.

Given that $(f\circ g)(2)=-3$ and $(g\circ f)(1)=5$, find the value of a and the value of b. [6]

3. [Maximum mark: 14] EXN.1.SL.TZ0.8

The following diagram shows the graph of $y = -1 - \sqrt{x+3}$ for $x \geq -3$.



(a) Describe a sequence of transformations that transforms the graph of $y = \sqrt{x}$ for $x \geq 0$ to the graph of $y = -1 - \sqrt{x+3}$ for $x \geq -3$. [3]

A function f is defined by $f(x) = -1 - \sqrt{x+3}$ for $x \geq -3$.

(b) State the range of f. [1]

(c) Find an expression for $f^{-1}(x)$, stating its domain. [5]

(d) Find the coordinates of the point(s) where the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect. [5]

4. [Maximum mark: 17] EXN.2.SL.TZ0.9

The temperature $T\ ^\circ\mathrm{C}$ of water t minutes after being poured into a cup can be modelled by $T = T_0\mathrm{e}^{-kt}$ where $t \geq 0$ and $T_0,\ k$ are positive constants.

The water is initially boiling at $100\ ^\circ\mathrm{C}$. When $t = 10$, the temperature of the water is $70\ ^\circ\mathrm{C}$.

(a) Show that $T_0 = 100$. [1]

(b) Show that $k = \frac{1}{10}\ln\frac{10}{7}$. [3]

(c) Find the temperature of the water when $t = 15$. [2]

(d) Sketch the graph of T versus t, clearly indicating any asymptotes with their equations and stating the coordinates of any points of intersection with the axes. [4]

(e) Find the time taken for the water to have a temperature of $50\ ^\circ\mathrm{C}$. Give your answer correct to the nearest second. [4]

(f) The model for the temperature of the water can also be expressed in the form $T = T_0a^{\frac{t}{10}}$ for $t \geq 0$ and a is a positive constant.

Find the exact value of a. [3]

5. [Maximum mark: 7] 23M.1.SL.TZ1.2

The function f is defined by $f(x) = \frac{7x+7}{2x-4}$ for $x \in \mathbb{R}, x \neq 2$.

(a) Find the zero of $f(x)$. [2]

(b) For the graph of $y = f(x)$, write down the equation of

(b.i) the vertical asymptote; [1]

(b.ii) the horizontal asymptote. [1]

(c) Find $f^{-1}(x)$, the inverse function of $f(x)$. [3]

6. [Maximum mark: 5] 23M.1.SL.TZ2.3

A function f is defined by $f(x) = 1 - \frac{1}{x-2}$, where $x \in \mathbb{R}, x \neq 2$.

(a) The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

Write down the equation of

(a.i) the vertical asymptote; [1]

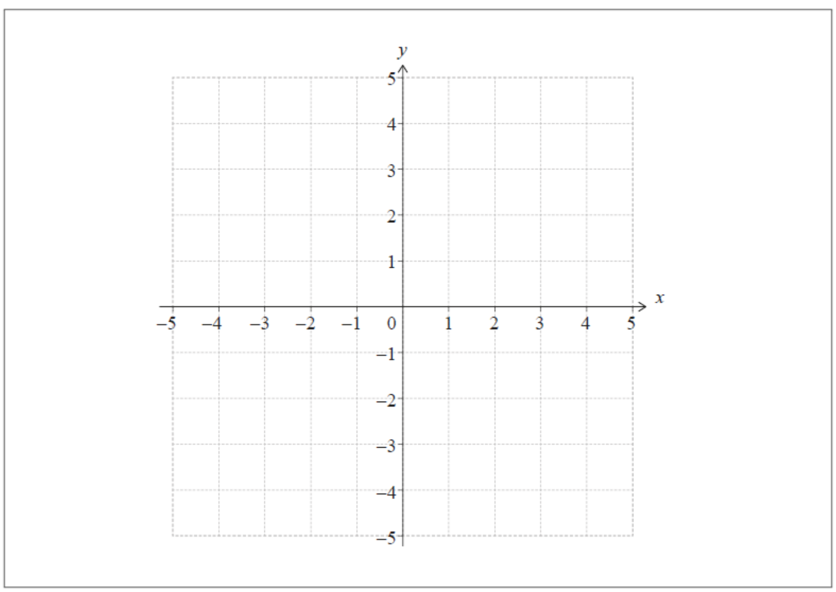
(a.ii) the horizontal asymptote. [1]

(b) Find the coordinates of the point where the graph of $y = f(x)$ intersects

(b.i) the y-axis; [1]

(b.ii) the x-axis. [1]

(c) On the following set of axes, sketch the graph of $y = f(x)$, showing all the features found in parts (a) and (b).



[1]

7. [Maximum mark: 7] 23M.2.SL.TZ1.7

The temperature of a cup of tea, t minutes after it is poured, can be modelled by $H(t) = 21 + 75\mathrm{e}^{-0.08t},\ t \geq 0$. The temperature is measured in degrees Celsius ($^\circ\mathrm{C}$).

(a.i) Find the initial temperature of the tea. [1]

(a.ii) Find the temperature of the tea three minutes after it is poured. [1]

(b) After k minutes, the tea will be below $67\ ^\circ\mathrm{C}$ and cool enough to drink.

Find the least possible value of k, where $k \in \mathbb{Z}^+$. [3]

As the tea cools, $H(t)$ approaches the temperature of the room, which is constant.

(c) Find the temperature of the room. [2]

8. [Maximum mark: 7] 22N.2.SL.TZ0.5

The population of a town t years after 1 January 2014 can be modelled by the function

$P(t) = 15\,000\mathrm{e}^{kt}$, where $k < 0$ and $t \geq 0$.

It is known that between 1 January 2014 and 1 January 2022 the population decreased by 11%.

Use this model to estimate the population of this town on 1 January 2041. [7]

9. [Maximum mark: 5] 22M.1.SL.TZ2.1

The following table shows values of $f(x)$ and $g(x)$ for different values of x.

Both f and g are one-to-one functions.

x	−2	0	3	4
$f(x)$	8	4	0	−3
$g(x)$	−5	−2	4	0

(a) Find $g(0)$. [1]

(b) Find $(f \circ g)(0)$. [2]

(c) Find the value of x such that $f(x) = 0$. [2]

10. [Maximum mark: 8] 22M.1.AHL.TZ2.3

A function f is defined by $f(x) = \frac{2x-1}{x+1}$, where $x \in \mathbb{R},\ x \neq -1$.

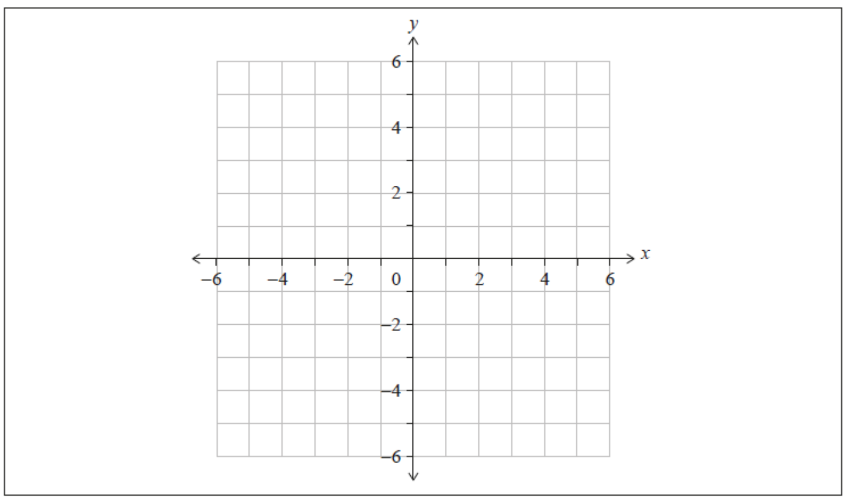
The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

(a.i) Write down the equation of the vertical asymptote. [1]

(a.ii) Write down the equation of the horizontal asymptote. [1]

(b) On the set of axes below, sketch the graph of $y = f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes.



[3]

(c) Hence, solve the inequality $0 < \frac{2x-1}{x+1} < 2$. [1]

(d) Solve the inequality $0 < \frac{2|x|-1}{|x|+1} < 2$. [2]

11. [Maximum mark: 15] 21N.1.SL.TZ0.8

Consider the function $f(x) = a^x$ where $x,\ a \in \mathbb{R}$ and $x > 0,\ a > 1$.

The graph of f contains the point $\left(\frac{2}{3}, 4\right)$.

(a) Show that $a = 8$. [2]

(b) Write down an expression for $f^{-1}(x)$. [1]

(c) Find the value of $f^{-1}\left(\sqrt{32}\right)$. [3]

Consider the arithmetic sequence
$\log_8 27\ ,\ \log_8 p\ ,\ \log_8 q\ ,\ \log_8 125$, where $p > 1$ and $q > 1$.

(d.i) Show that $27,\ p,\ q$ and 125 are four consecutive terms in a geometric sequence. [4]

(d.ii) Find the value of p and the value of q. [5]

12. [Maximum mark: 9] 21N.1.AHL.TZ0.2

The function f is defined by $f(x) = \frac{2x+4}{3-x}$, where $x \in \mathbb{R},\ x \neq 3$.

Write down the equation of

(a.i) the vertical asymptote of the graph of f. [1]

(a.ii) the horizontal asymptote of the graph of f. [1]

Find the coordinates where the graph of f crosses

(b.i) the x-axis. [1]

(b.ii) the y-axis. [1]

(c) Sketch the graph of f on the axes below.

[1]

(d) The function g is defined by $g(x) = \frac{ax+4}{3-x}$, where $x \in \mathbb{R},\ x \neq 3$ and $a \in \mathbb{R}$.

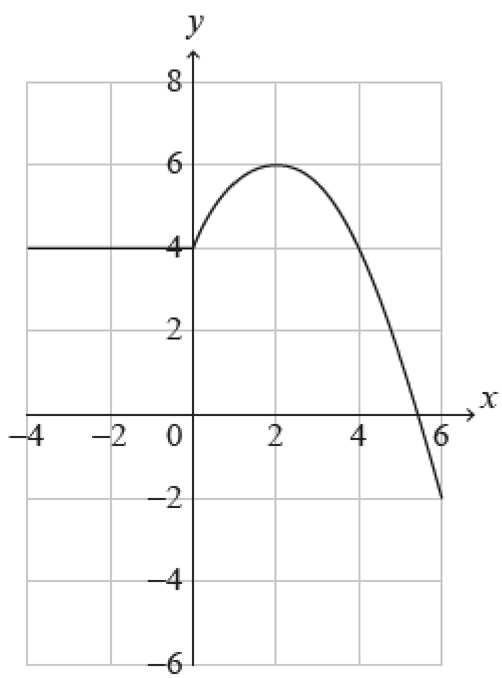
Given that $g(x) = g^{-1}(x)$, determine the value of a. [4]

13. [Maximum mark: 5] 21N.1.AHL.TZ0.3

Solve the equation $\log_3 \sqrt{x} = \frac{1}{2 \log_2 3} + \log_3 (4x^3)$, where $x > 0$. [5]

14. [Maximum mark: 5] 21M.1.SL.TZ1.1

The graph of $y = f(x)$ for $-4 \leq x \leq 6$ is shown in the following diagram.



(a.i) Write down the value of $f(2)$. [1]

(a.ii) Write down the value of $(f \circ f)(2)$. [1]

(b) Let $g(x) = \frac{1}{2} f(x) + 1$ for $-4 \leq x \leq 6$. On the axes above, sketch the graph of g.

[3]

15. [Maximum mark: 7] 21M.2.SL.TZ2.6

All living plants contain an isotope of carbon called carbon-14. When a plant dies, the isotope decays so that the amount of carbon-14 present in the remains of the plant decreases. The time since the death of a plant can be determined by measuring the amount of carbon-14 still present in the remains.

The amount, A, of carbon-14 present in a plant t years after its death can be modelled by $A = A_0 e^{-kt}$ where $t \geq 0$ and $A_0,\ k$ are positive constants.

At the time of death, a plant is defined to have 100 units of carbon-14.

(a) Show that $A_0 = 100$. [1]

The time taken for half the original amount of carbon-14 to decay is known to be 5730 years.

(b) Show that $k = \frac{\ln 2}{5730}$. [3]

(c) Find, correct to the nearest 10 years, the time taken after the plant's death for 25% of the carbon-14 to decay. [3]

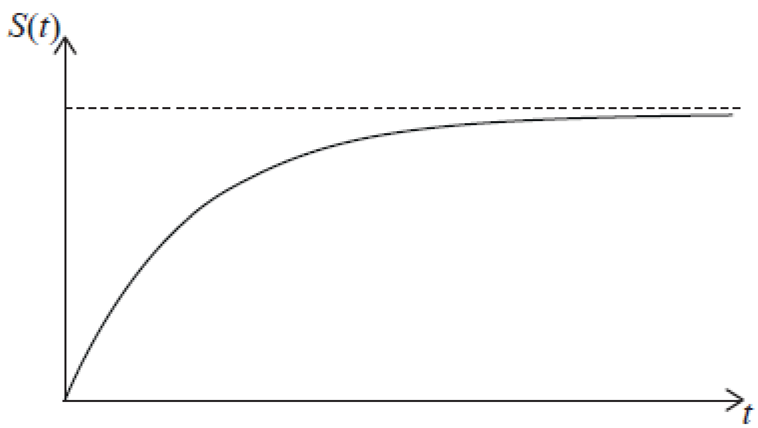
16. [Maximum mark: 6] 20N.1.SL.TZ0.T_12

Jean-Pierre jumps out of an airplane that is flying at constant altitude. Before opening his parachute, he goes through a period of freefall.

Jean-Pierre's vertical speed during the time of freefall, S, in $\mathrm{m\,s^{-1}}$, is modelled by the following function.

$$S(t) = K - 60\left(1.2^{-t}\right)\ ,\ t \geq 0$$

where t, is the number of seconds after he jumps out of the airplane, and K is a constant. A sketch of Jean-Pierre's vertical speed against time is shown below.



Jean-Pierre's initial vertical speed is $0\,\mathrm{m\,s^{-1}}$.

(a) Find the value of K. [2]

(b) In the context of the model, state what the horizontal asymptote represents. [1]

(c) Find Jean-Pierre's vertical speed after 10 seconds. Give your answer in $\mathrm{km\,h^{-1}}$. [3]

17. [Maximum mark: 6] 20N.1.SL.TZ0.S_4

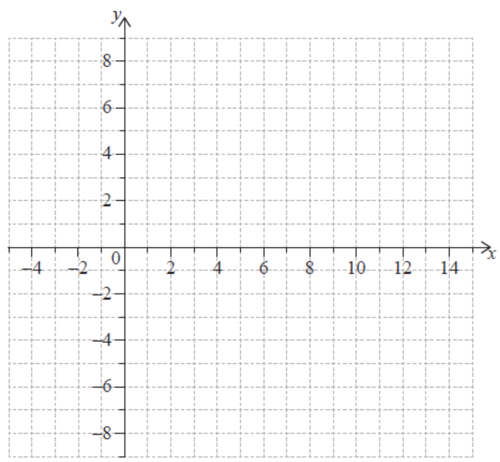
Let $f(x) = a\ \log_3(x-4)$, for $x > 4$, where $a > 0$.

Point $\mathrm{A}(13,\ 7)$ lies on the graph of f.

(a) Find the value of a. [3]

(b) The x-intercept of the graph of f is $(5,\ 0)$.

On the following grid, sketch the graph of f.



[3]

18. [Maximum mark: 15] 20N.1.AHL.TZ0.H_12

Consider the function defined by $f(x) = \frac{kx-5}{x-k}$, where $x \in \mathbb{R} \setminus \{k\}$ and $k^2 \neq 5$.

(a) State the equation of the vertical asymptote on the graph of $y = f(x)$. [1]

(b) State the equation of the horizontal asymptote on the graph of $y = f(x)$. [1]

(c) Use an algebraic method to determine whether f is a self-inverse function. [4]

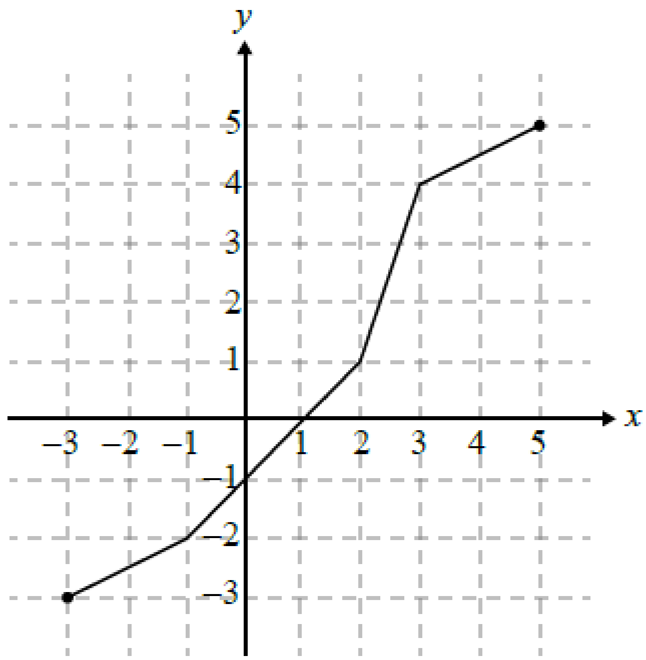
Consider the case where $k = 3$.

(d) Sketch the graph of $y = f(x)$, stating clearly the equations of any asymptotes and the coordinates of any points of intersections with the coordinate axes. [3]

(e) The region bounded by the x-axis, the curve $y = f(x)$, and the lines $x = 5$ and $x = 7$ is rotated through 2π about the x-axis. Find the volume of the solid generated, giving your answer in the form $\pi(a + b \ln 2)$, where $a, b \in \mathbb{Z}$. [6]

19. [Maximum mark: 6] 19N.2.AHL.TZ0.H_3

The following diagram shows the graph of $y = f(x)$, $-3 \leqslant x \leqslant 5$.



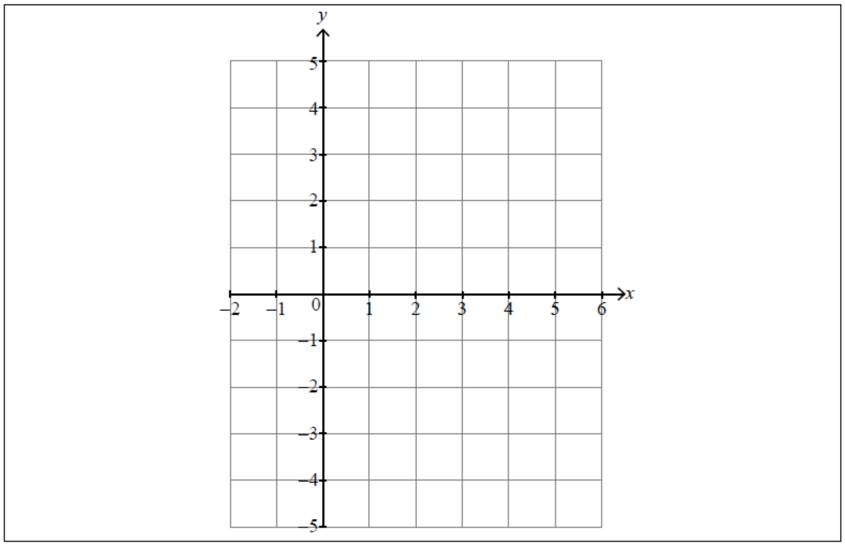
(a) Find the value of $(f \circ f)(1)$. [2]

(b) Given that $f^{-1}(a) = 3$, determine the value of a. [2]

(c) Given that $g(x) = 2f(x-1)$, find the domain and range of g. [2]

20. [Maximum mark: 5] 19M.1.AHL.TZ2.H_5

(a) Sketch the graph of $y = \frac{x-4}{2x-5}$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes.



[5]

21. [Maximum mark: 10] 19M.1.AHL.TZ2.H_11

Consider the functions f and g defined by $f(x) = \ln|x|, x \in \mathbb{R} \setminus \{0\}$, and $g(x) = \ln|x + k|, x \in \mathbb{R} \setminus \{-k\}$, where $k \in \mathbb{R}, k > 2$.

(a) Describe the transformation by which $f(x)$ is transformed to $g(x)$. [1]

(b) State the range of g. [1]

(c) Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same axes, clearly stating the points of intersection with any axes. [6]

The graphs of f and g intersect at the point P .

(d) Find the coordinates of P. [2]

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