Mock review [495 marks]

1. [Maximum mark: 6] SPM.1.SL.TZ0.3

At the end of a school day, the Headmaster conducted a survey asking students in how many classes they had used the internet.

The data is shown in the following table.

Number of classes in which the students used the internet	0	1	2	3	4	5	6
Number of students	20	24	30	k	10	3	1

(a) State whether the data is discrete or continuous. [1]

Markscheme

discrete *A1*

[1 mark]

The mean number of classes in which a student used the internet is 2.

(b) Find the value of k. [4]

Markscheme

$\frac{24+60+3k+40+15+6}{88+k} = 2$ *M1A1*

Note: Award *M1* for substitution into the formula for the mean, award **A1** for a correct equation.

attempt to solve their equation *(M1)*

k = 31 *A1*

[4 marks]

(c) It was not possible to ask every person in the school, so the Headmaster arranged the student names in alphabetical order and then asked every 10th person on the list.

Identify the sampling technique used in the survey. [1]

Markscheme

systematic *A1*

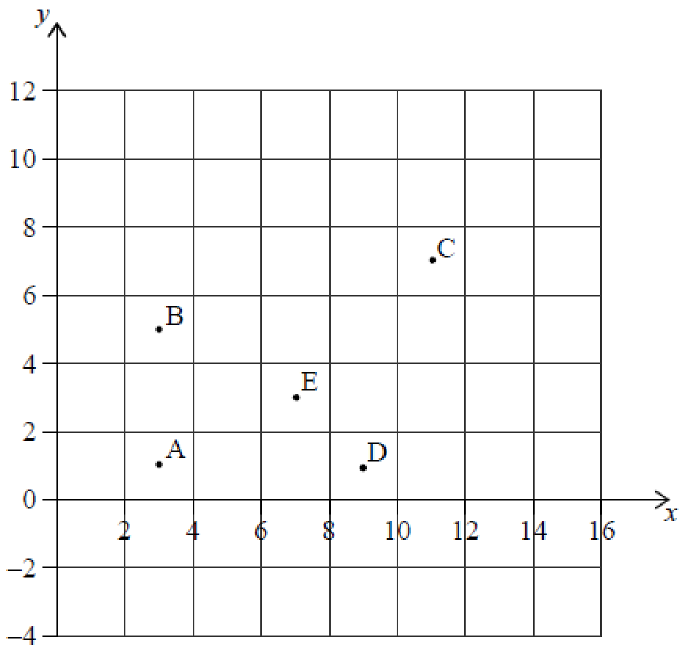
[1 mark]

2. [Maximum mark: 6] SPM.1.SL.TZ0.7

Points A(3, 1), B(3, 5), C(11, 7), D(9, 1) and E(7, 3) represent snow shelters in the Blackburn National Forest. These snow shelters are illustrated in the following coordinate axes.

Horizontal scale: 1 unit represents 1 km.

Vertical scale: 1 unit represents 1 km.



(a) Calculate the gradient of the line segment AE. [2]

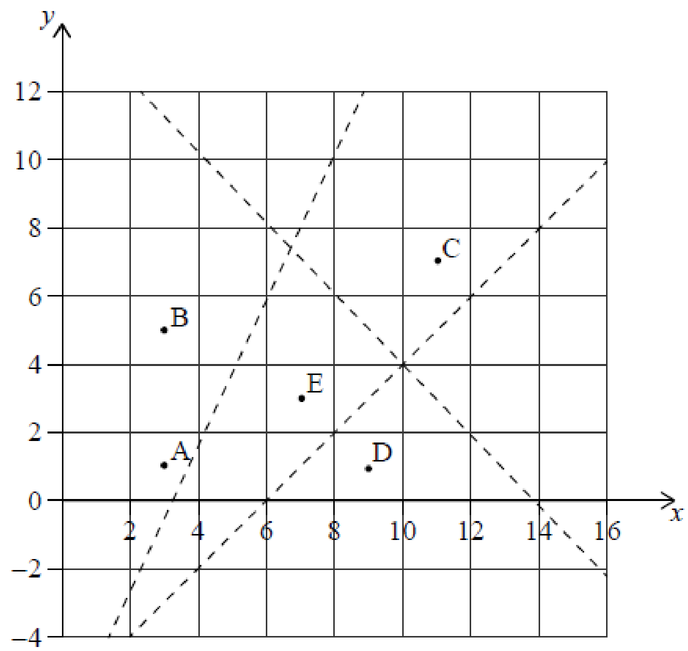
Markscheme

$\frac{3-1}{7-3}$ *(M1)*

$= 0.5$ ***A1***

[2 marks]

The Park Ranger draws three straight lines to form an incomplete Voronoi diagram.



(b) Find the equation of the line which would complete the Voronoi cell containing site E.

Give your answer in the form $ax + by + d = 0$ where $a, b, d \in \mathbb{Z}$. [3]

Markscheme

$y - 2 = -2(x - 5)$ ***(A1) (M1)***

Note: Award ***(A1)*** for their −2 seen, award ***(M1)*** for the correct substitution of (5, 2) and their normal gradient in equation of a line.

$2x + y - 12 = 0$ ***A1***

[3 marks]

(c) In the context of the question, explain the significance of the Voronoi cell containing site E. [1]

Markscheme

every point in the cell is closer to E than any other snow shelter ***A1***

[1 mark]

3. [Maximum mark: 6] SPM.1.SL.TZ0.12

Jae Hee plays a game involving a biased six-sided die.

The faces of the die are labelled −3, −1, 0, 1, 2 and 5.

The score for the game, X, is the number which lands face up after the die is rolled.

The following table shows the probability distribution for X.

Score x	-3	-1	0	1	2	5
$P(X=x)$	$\frac{1}{18}$	p	$\frac{3}{18}$	$\frac{1}{18}$	$\frac{2}{18}$	$\frac{7}{18}$

(a) Find the exact value of p. [1]

Markscheme

$\frac{4}{18}\left(\frac{2}{9}\right)$ ***A1***

[1 mark]

Jae Hee plays the game once.

(b) Calculate the expected score. [2]

Markscheme

$-3 \times \frac{1}{18} + (-1) \times \frac{4}{18} + 0 \times \frac{3}{18} + \ldots + 5 \times \frac{7}{18}$ ***(M1)***

Note: Award ***(M1)*** for their correct substitution into the formula for expected value.

$= 1.83 \left(\frac{33}{18}, 1.83333\ldots\right)$ ***A1***

[2 marks]

(c) Jae Hee plays the game twice and adds the two scores together.

Find the probability Jae Hee has a **total** score of −3. [3]

Markscheme

$2 \times \frac{1}{18} \times \frac{3}{18}$ ***(M1)(M1)***

Note: Award ***(M1)*** for $\frac{1}{18} \times \frac{3}{18}$, award ***(M1)*** for multiplying their product by 2.

$= \frac{1}{54}\left(\frac{6}{324}, 0.0185185\ldots, 1.85\%\right)$ ***A1***

[3 marks]

4. [Maximum mark: 5] SPM.1.AHL.TZ0.7

A particle, A, moves so that its velocity (v ms^{-1}) at time t is given by $v = 2 \sin t, t \geq 0$.

The kinetic energy (E) of the particle A is measured in joules (J) and is given by $E = 5v^2$.

(a) Write down an expression for E as a function of time. [1]

Markscheme

$E = 5(2 \sin t)^2 \ (= 20 \sin^2 t)$ ***A1***

[1 mark]

(b) Hence find $\frac{dE}{dt}$. [2]

Markscheme

$\frac{dE}{dt} = 40 \sin t \cos t$ ***(M1)A1***

[2 marks]

(c) Hence or otherwise find the first time at which the kinetic energy is changing at a rate of 5 J s^{-1}. [2]

Markscheme

$t = 0.126$ ***(M1)A1***

[2 marks]

5. [Maximum mark: 6] SPM.1.AHL.TZ0.11

A particle P moves with velocity $\boldsymbol{v} = \begin{pmatrix} -15 \\ 2 \\ 4 \end{pmatrix}$ in a magnetic field, $\boldsymbol{B} = \begin{pmatrix} 0 \\ d \\ 1 \end{pmatrix}$, $d \in \mathbb{R}$.

(a) Given that $\boldsymbol{v}$ is perpendicular to $\boldsymbol{B}$, find the value of d. [2]

Markscheme

$15 \times 0 + 2d + 4 = 0$ ***(M1)***

$d = -2$ ***A1***

[2 marks]

(b) The force, $\boldsymbol{F}$, produced by P moving in the magnetic field is given by the vector equation $\boldsymbol{F} = a$ $\boldsymbol{v} \times \boldsymbol{B}$, $a \in \mathbb{R}^+$.

Given that $|\boldsymbol{F}| = 14$, find the value of a. [4]

Markscheme

$a \begin{pmatrix} -15 \\ 2 \\ 4 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ ***(M1)***

$= a \begin{pmatrix} 10 \\ 15 \\ 30 \end{pmatrix} \left(= 5a \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \right)$ ***A1***

magnitude is $5a\sqrt{2^2 + 3^2 + 6^2} = 14$ ***M1***

$a = \frac{14}{35}$ $(= 0.4)$ ***A1***

[4 marks]

6. [Maximum mark: 7] SPM.1.AHL.TZ0.15

Let $w = a\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}$, where $a \in \mathbb{R}^+$.

for $a = 2$,

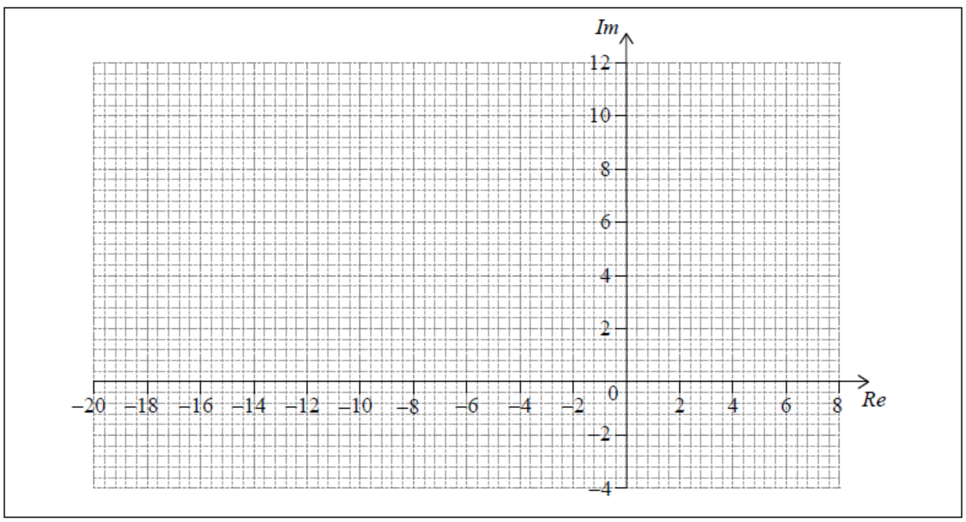
(a.i) find the values of w^2, w^3, and w^4. [2]

Markscheme

$4\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}, 8\mathrm{e}^{\frac{3\pi}{4}\mathrm{i}}, 16\mathrm{e}^{\pi\mathrm{i}}\ (= 4\mathrm{i},\ -4\sqrt{2} + 4\sqrt{2}\mathrm{i},\ -16)$ ***(M1)A1***

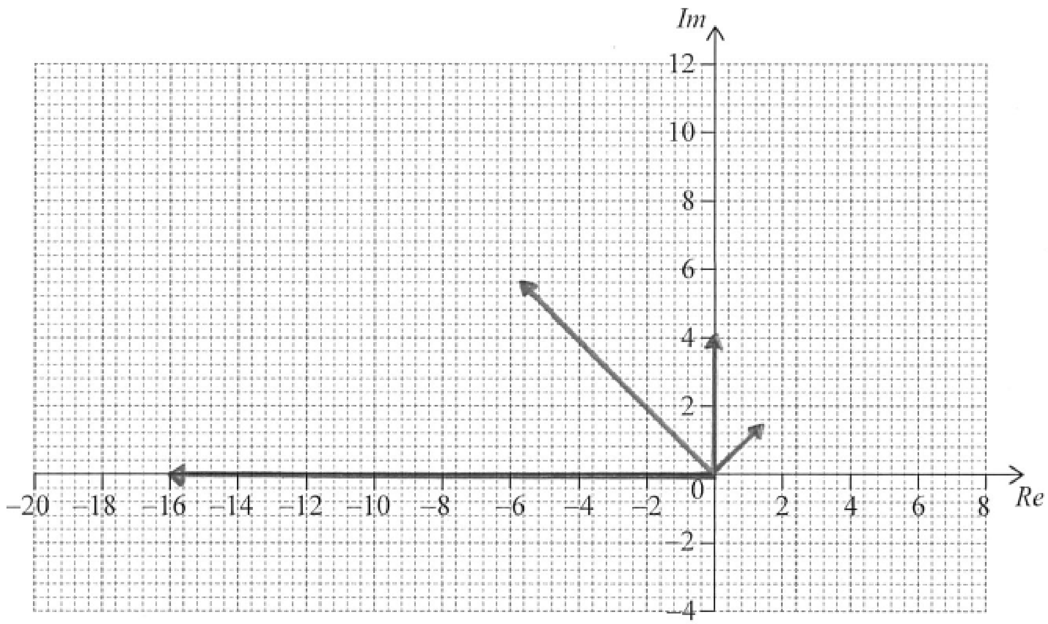
[2 marks]

(a.ii) draw w, w^2, w^3, and w^4 on the following Argand diagram.



[3]

Markscheme



A3

Note: Award ***A1*** for correct arguments, award ***A1*** for $4\mathrm{i}$ and −16 clearly indicated, award ***A1*** for $|w| < 4$ and $4 < |w^3| < 16$.

[3 marks]

(b) Let $z = \frac{w}{2-\mathrm{i}}$.

Find the value of a for which successive powers of z lie on a circle. [2]

Markscheme

$2^2 + 1^2 = a^2$ ***M1***

$a = \sqrt{5}\ (= 2.24)$ ***A1***

[2 marks]

7. [Maximum mark: 6] SPM.1.AHL.TZ0.17

Mr Burke teaches a mathematics class with 15 students. In this class there are 6 female students and 9 male students.

Each day Mr Burke randomly chooses one student to answer a homework question.

In the first month, Mr Burke will teach his class 20 times.

(a) Find the probability he will choose a female student 8 times. [2]

Markscheme

$\mathrm{P}(X = 8)$ ***(M1)***

Note: Award ***(M1)*** for evidence of recognizing binomial probability. *eg,* $\mathrm{P}(X = 8)$, $X \sim \mathrm{B}\left(20,\ \frac{6}{15}\right)$.

$= 0.180\ (0.179705\ldots)$ ***A1***

[2 marks]

(b) The Head of Year, Mrs Smith, decides to select a student at random from the year group to read the notices in assembly. There are 80 students in total in the year group. Mrs Smith calculates the probability of picking a male student 8 times in the first 20 assemblies is 0.153357 correct to 6 decimal places.

Find the number of male students in the year group. [4]

Markscheme

let x be the number of male students

recognize that probability of selecting a male is equal to $\frac{x}{80}$ ***(A1)***

$\left(\text{set up equation } {}^{20}\mathrm{C}_8\left(\frac{x}{80}\right)^8\left(\frac{80-x}{80}\right)^{12} =\right) 0.153357$ ***(M1)***

number of male students $= 37$ ***(M1)A1***

Note: Award ***(M1)A0*** for 27.

[4 marks]

8. [Maximum mark: 17] SPM.2.SL.TZ0.3

The Malvern Aquatic Center hosted a 3 metre spring board diving event. The judges, Stan and Minsun awarded 8 competitors a score out of 10. The raw data is collated in the following table.

Competitors	A	B	C	D	E	F	G	H
Stan's score (x)	4.1	3	4.3	6	7.1	6	7.5	6
Minsun's score (y)	4.7	4.6	4.8	7.2	7.8	9	9.5	7.2

(a.i) Write down the value of the Pearson's product–moment correlation coefficient, r. [2]

Markscheme

0.909 (0.909181…) ***A2***

[2 marks]

(a.ii) Using the value of r, interpret the relationship between Stan's score and Minsun's score. [2]

Markscheme

(very) strong and positive ***A1A1***

Note: Award ***A1*** for (very) strong ***A1*** for positive.

[2 marks]

(b) Write down the equation of the regression line y on x. [2]

Markscheme

$y = 1.14x + 0.578 \ \ (y = 1.14033\ldots x + 0.578183\ldots)$ ***A1A1***

Note: Award ***A1*** for $1.14x$, ***A1*** for 0.578. Award a maximum of ***A1A0*** if the answer is not an equation in the form $y = mx + c$.

[2 marks]

(c.i) Use your regression equation from part (b) to estimate Minsun's score when Stan awards a perfect 10. [2]

Markscheme

1.14 × 10 + 0.578 ***M1***

12.0 (11.9814…) ***A1***

[2 marks]

(c.ii) State whether this estimate is reliable. Justify your answer. [2]

Markscheme

no the estimate is not reliable ***A1***

outside the known data range ***R1***
OR
a score greater than 10 is not possible ***R1***

Note: Do not award ***A1R0***.

[2 marks]

The Commissioner for the event would like to find the Spearman's rank correlation coefficient.

(d) **Copy** and complete the information in the following table.

Competitors	A	B	C	D	E	F	G	H
Stan's Rank		8					1	4
Minsun's Rank		8					1	4.5

[2]

Markscheme

Competitors	A	B	C	D	E	F	G	H
Stan's rank	7	8	6	4	2	4	1	4
Minsun's rank	7	8	6	4.5	3	2	1	4.5

A1A1

Note: Award ***A1*** for correct ranks for Stan. Award ***A1*** for correct ranks for Minsun.

[2 marks]

(e.i) Find the value of the Spearman's rank correlation coefficient, r_s. [2]

Markscheme

0.933 (0.932673…) ***A2***

[2 marks]

(e.ii) Comment on the result obtained for r_s. [2]

Markscheme

Stan and Minsun strongly agree on the ranking of competitors. ***A1A1***

Note: Award ***A1*** for "strongly agree", ***A1*** for reference to a rank order.

[2 marks]

(f) The Commissioner believes Minsun's score for competitor G is too high and so decreases the score from 9.5 to 9.1.

Explain why the value of the Spearman's rank correlation coefficient r_s does not change. [1]

Markscheme

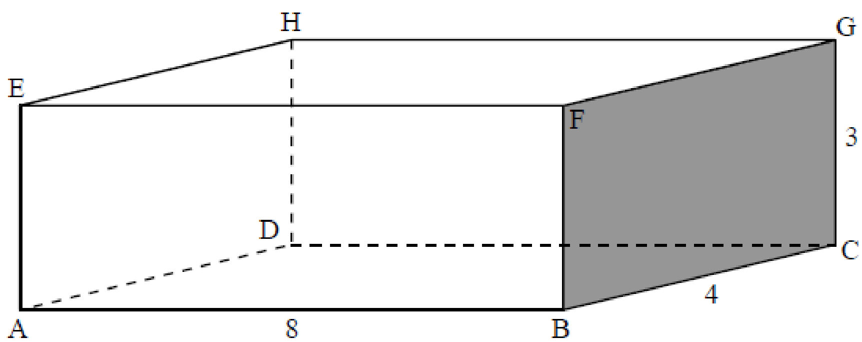
decreasing the score to 9.1, does not change the rank of competitor G ***A1***

[1 mark]

9. [Maximum mark: 15] SPM.2.SL.TZ0.4

The Happy Straw Company manufactures drinking straws.

The straws are packaged in small closed rectangular boxes, each with length 8 cm, width 4 cm and height 3 cm. The information is shown in the diagram.



(a) Calculate the surface area of the box in cm^2. [2]

Markscheme

$2(8 \times 4 + 3 \times 4 + 3 \times 8)$ ***M1***

$= 136$ (cm^2) ***A1***

[2 marks]

(b) Calculate the length AG. [2]

Markscheme

$\sqrt{8^2 + 4^2 + 3^2}$ ***M1***

(AG =) 9.43 (cm) (9.4339…, $\sqrt{89}$) ***A1***

[2 marks]

Each week, the Happy Straw Company sells x boxes of straws. It is known that $\frac{dP}{dx} = -2x + 220$, $x \geq 0$, where P is the weekly profit, in dollars, from the sale of x thousand boxes.

(c) Find the number of boxes that should be sold each week to maximize the profit. [3]

Markscheme

$-2x + 220 = 0$ ***M1***

$x = 110$ ***A1***

110 000 (boxes) ***A1***

[3 marks]

(d) Find $P(x)$. [5]

Markscheme

$P(x) = \int -2x + 220\,\mathrm{d}x$ ***M1***

Note: Award ***M1*** for evidence of integration.

$P(x) = -x^2 + 220x + c$ ***A1A1***

Note: Award ***A1*** for either $-x^2$ or $220x$ award ***A1*** for both correct terms and constant of integration.

$1700 = -(20)^2 + 220(20) + c$ ***M1***

$c = -2300$

$P(x) = -x^2 + 220x - 2300$ ***A1***

[5 marks]

(e) Find the least number of boxes which must be sold each week in order to make a profit. [3]

Markscheme

$-x^2 + 220x - 2300 = 0$ ***M1***

$x = 11.005$ ***A1***

11 006 (boxes) ***A1***

Note: Award ***M1*** for their $P(x) = 0$, award ***A1*** for their correct solution to x.
Award the final ***A1*** for expressing their solution to the minimum number of boxes. Do not accept 11 005, the nearest integer, nor 11 000, the answer expressed to 3 significant figures, as these will not satisfy the demand of the question.

[3 marks]

10. [Maximum mark: 18] SPM.2.AHL.TZ0.3

In this question, give all answers to two decimal places.

Bryan decides to purchase a new car with a price of €14 000, but cannot afford the full amount. The car dealership offers two options to finance a loan.

Finance option A:

A 6 year loan at a nominal annual interest rate of 14 % **compounded quarterly**. No deposit required and repayments are made each quarter.

(a.i) Find the repayment made each quarter. [3]

Markscheme

N = 24
I % = 14
PV = −14000
FV = 0
P/Y = 4
C/Y = 4 ***(M1)(A1)***

Note: Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct. Accept PV = 14000.

(€)871.82 ***A1***

[3 marks]

(a.ii) Find the total amount paid for the car. [2]

Markscheme

$4 \times 6 \times 871.82$ ***(M1)***

(€) 20923.68 ***A1***

[2 marks]

(a.iii) Find the interest paid on the loan. [2]

Markscheme

$20923.68 - 14000$ ***(M1)***

(€) 6923.68 ***A1***

[2 marks]

Finance option B:

A 6 year loan at a nominal annual interest rate of r % **compounded monthly**. Terms of the loan require a 10 % deposit and monthly repayments of €250.

(b.i) Find the amount to be borrowed for this option. [2]

Markscheme

0.9×14000 $(= 14000 - 0.10 \times 14000)$ ***M1***

(€) 12600.00 ***A1***

[2 marks]

(b.ii) Find the annual interest rate, r. [3]

Markscheme

N = 72

PV = 12600

PMT = −250

FV = 0

P/Y = 12

C/Y = 12 ***(M1)(A1)***

Note: Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for all entries correct. Accept PV = −12600 provided PMT = 250.

12.56(%) ***A1***

[3 marks]

(c) State which option Bryan should choose. Justify your answer. [2]

Markscheme

EITHER

Bryan should choose Option A ***A1***

no deposit is required ***R1***

Note: Award ***R1*** for stating that no deposit is required. Award ***A1*** for the correct choice from that fact. Do not award ***R0A1***.

OR

Bryan should choose Option B ***A1***

cost of Option A (6923.69) > cost of Option B ($72 \times 250 - 12600 = 5400$) ***R1***

Note: Award ***R1*** for a correct comparison of costs. Award ***A1*** for the correct choice from that comparison. Do not award ***R0A1***.

[2 marks]

(d) Bryan chooses option B. The car dealership invests the money Bryan pays as soon as they receive it.

If they invest it in an account paying 0.4 % interest per month and inflation is 0.1 % per month, calculate the real amount of money the car dealership has received by the end of the 6 year period. [4]

Markscheme

real interest rate is $0.4 - 0.1 = 0.3\%$ ***(M1)***

value of other payments $250 + 250 \times 1.003 + \ldots + 250 \times 1.003^{71}$

use of sum of geometric sequence formula or financial app on a GDC ***(M1)***

= 20 058.43

value of deposit at the end of 6 years

$1400 \times (1.003)^{72} = 1736.98$ ***(A1)***

Total value is (€) 21 795.41 ***A1***

Note: Both ***M*** marks can awarded for a correct use of the GDC's financial app:

N = 72 (6 × 12)
I % = 3.6 (0.3 × 12)
PV = 0
PMT = −250
FV =
P/Y = 12
C/Y = 12

OR

N = 72 (6 × 12)

I % = 0.3

PV = 0

PMT = −250

FV =

P/Y = 1

C/Y = 1

[4 marks]

11. [Maximum mark: 14] SPM.2.AHL.TZ0.4

An aircraft's position is given by the coordinates (x, y, z), where x and y are the aircraft's displacement east and north of an airport, and z is the height of the aircraft above the ground. All displacements are given in kilometres.

The velocity of the aircraft is given as $\begin{pmatrix} -150 \\ -50 \\ -20 \end{pmatrix} \text{km h}^{-1}$.

At 13:00 it is detected at a position 30 km east and 10 km north of the airport, and at a height of 5 km. Let t be the length of time in hours from 13:00.

(a) Write down a vector equation for the displacement, ***r*** of the aircraft in terms of t. [2]

Markscheme

r $= \begin{pmatrix} 30 \\ 10 \\ 5 \end{pmatrix} + t\begin{pmatrix} -150 \\ -50 \\ -20 \end{pmatrix}$ ***A1A1***

[2 marks]

If the aircraft continued to fly with the velocity given

(b.i) verify that it would pass directly over the airport. [2]

Markscheme

when $x = 0, t = \frac{30}{150} = 0.2$ ***M1***

EITHER

when $y = 0, t = \frac{10}{150} = 0.2$ ***A1***

since the two values of t are equal the aircraft passes directly over the airport

OR

$t = 0.2, y = 0$ ***A1***

[2 marks]

(b.ii) state the height of the aircraft at this point. [1]

Markscheme

height = 5 – 0.2 × 20 = 1 km ***A1***

[1 mark]

(b.iii) find the time at which it would fly directly over the airport. [1]

Markscheme

time 13:12 ***A1***

[1 mark]

When the aircraft is 4 km above the ground it continues to fly on the same bearing but adjusts the angle of its descent so that it will land at the point (0, 0, 0).

(c.i) Find the time at which the aircraft is 4 km above the ground. [2]

Markscheme

$5 - 20t = 4 \Rightarrow t = \frac{1}{20}$ (3 minutes) ***(M1)***

time 13:03 ***A1***

[2 marks]

(c.ii) Find the direct distance of the aircraft from the airport at this point. [3]

Markscheme

displacement is $\begin{pmatrix} 22.5 \\ 7.5 \\ 4 \end{pmatrix}$ ***A1***

distance is $\sqrt{22.5^2 + 7.5^2 + 4^2}$ ***(M1)***

= 24.1 km ***A1***

[3 marks]

(d) Given that the velocity of the aircraft, after the adjustment of the angle of descent, is $\begin{pmatrix} -150 \\ -50 \\ a \end{pmatrix}$ $\mathrm{km\,h^{-1}}$, find the value of a. [3]

Markscheme

METHOD 1

time until landing is $12-3=9$ minutes *M1*

height to descend = $4\,\mathrm{km}$

$a=\frac{-4}{\frac{9}{60}}$ *M1*

$=-26.7$ *A1*

METHOD 2

$$\begin{pmatrix}-150\\-50\\a\end{pmatrix}=s\begin{pmatrix}22.5\\7.5\\4\end{pmatrix}$$ *M1*

$-150=22.5\,s\Rightarrow s=-\frac{20}{3}$ *M1*

$a=-\frac{20}{3}\times 4$

$=-26.7$ *A1*

[3 marks]

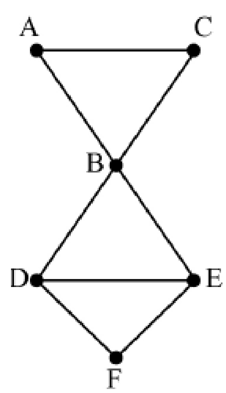
12. [Maximum mark: 17] SPM.2.AHL.TZ0.5

The following table shows the costs in US dollars (US$) of direct flights between six cities. Blank cells indicate no direct flights. The rows represent the departure cities. The columns represent the destination cities.

		Destination city					
		A	**B**	**C**	**D**	**E**	**F**
Departure city	**A**		90	150			
	B	90		80	70	140	
	C	150	80				
	D		70			100	180
	E		140		100		210
	F				180	210	

(a) Show the direct flights between the cities as a graph. [2]

Markscheme



A2

[2 marks]

(b) Write down the adjacency matrix for this graph. [2]

Markscheme

attempt to form an adjacency matrix *M1*

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$ ***A1***

[2 marks]

(c) Using your answer to part (b), find the number of different ways to travel from and return to city A in exactly 6 flights. [2]

Markscheme

raising the matrix to the power six ***(M1)***

50 ***A1***

[2 marks]

(d) State whether or not it is possible to travel from and return to city A in exactly 6 flights, having visited each of the other 5 cities exactly once. Justify your answer. [2]

Markscheme

not possible ***A1***

because you must pass through B twice ***R1***

Note: Do not award ***A1R0***.

[2 marks]

The following table shows the least cost to travel between the cities.

Departure city		Destination city					
		A	**B**	**C**	**D**	**E**	**F**
	A	0	90	150	160	a	b
	B	90	0	80	70	140	250
	C	150	80	0	150	220	330
	D	160	70	150	0	100	180
	E	a	140	220	100	0	210
	F	b	250	330	180	210	0

(e) Find the values of a and b. [2]

Markscheme

$a = 230, b = 340$ ***A1A1***

[2 marks]

A travelling salesman has to visit each of the cities, starting and finishing at city A.

(f) Use the nearest neighbour algorithm to find an upper bound for the cost of the trip. [3]

Markscheme

A → B → D → E → F → C → A ***(M1)***

90 + 70 + 100 + 210 + 330 + 150 ***(A1)***

(US$) 950 ***A1***

[3 marks]

(g) By deleting vertex A, use the deleted vertex algorithm to find a lower bound for the cost of the trip. [4]

Markscheme

finding weight of minimum spanning tree ***M1***

70 + 80 + 100 + 180 = (US$) 430 ***A1***

adding in two edges of minimum weight ***M1***

430 + 90 + 150 = (US$) 670 ***A1***

[4 marks]

13. [Maximum mark: 14] SPM.2.AHL.TZ0.6

A city has two cable companies, X and Y. Each year 20 % of the customers using company X move to company Y and 10 % of the customers using company Y move to company X. All additional losses and gains of customers by the companies may be ignored.

(a) Write down a transition matrix ***T*** representing the movements between the two companies in a particular year. [2]

Markscheme

$$\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$$ ***M1A1***

[2 marks]

(b) Find the eigenvalues and corresponding eigenvectors of ***T***. [4]

Markscheme

$$\begin{vmatrix} 0.8-\lambda & 0.1 \\ 0.2 & 0.9-\lambda \end{vmatrix} = 0$$ ***M1***

$\lambda = 1$ and 0.7 ***A1***

eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***(M1)A1***

Note: Accept any scalar multiple of the eigenvectors.

[4 marks]

(c) Hence write down matrices ***P*** and ***D*** such that $\boldsymbol{T} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$. [2]

Markscheme

EITHER

$\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix}$ ***A1A1***

OR

$\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ $\boldsymbol{D} = \begin{pmatrix} 0.7 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1A1***

[2 marks]

Initially company X and company Y both have 1200 customers.

(d) Find an expression for the number of customers company X has after n years, where $n \in \mathbb{N}$. [5]

Markscheme

$\boldsymbol{P}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ ***A1***

$\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.7^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1200 \\ 1200 \end{pmatrix}$ ***M1A1***

attempt to multiply matrices ***M1***

so in company A, after n years, $400\left(2 + 0.7^n\right)$ ***A1***

[5 marks]

(e) Hence write down the number of customers that company X can expect to have in the long term. [1]

Markscheme

$400 \times 2 = 800$ ***A1***

[1 mark]

14. [Maximum mark: 5] EXN.1.SL.TZ0.1

A school consists of 740 students divided into 5 grade levels. The numbers of students in each grade are shown in the table below.

Grade	8	9	10	11	12
Number of students	120	125	119	195	181

The Principal of the school wishes to select a sample of 25 students. She wishes to ensure that, as closely as possible, the proportion of the students from each grade in the sample is the same as the proportions in the school.

(a) Calculate the number of grade 12 students who should be in the sample. [3]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$\frac{181}{740} \times 25 = 6.11486\ldots$ **M1(A1)**

6 (students) **A1**

[3 marks]

(b) The Principal selects the students for the sample by asking those who took part in a previous survey if they would like to take part in another. She takes the first of those who reply positively, up to the maximum needed for the sample.

State which two of the sampling methods listed below best describe the method used.

Stratified Quota Convenience Systematic Simple random [2]

Markscheme

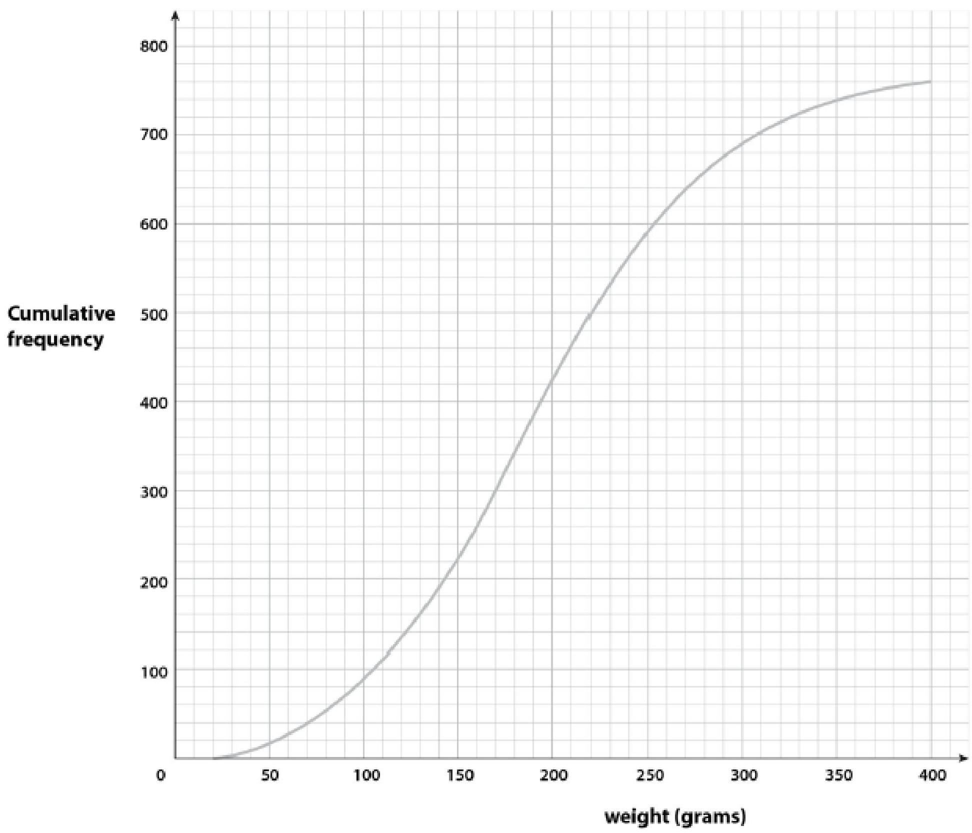
Quota and convenience **A1A1**

Note: Award **A1A0** for one correct and one incorrect answer.

[2 marks]

15. [Maximum mark: 7] EXN.1.SL.TZ0.4

A food scientist measures the weights of 760 potatoes taken from a single field and the distribution of the weights is shown by the cumulative frequency curve below.



(a) Find the number of potatoes in the sample with a weight of more than 200 grams. [2]

Markscheme

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$760 - 420 = 340\,(\mathrm{g})$ **(M1)A1**

[2 marks]

(b.i) Find the median weight. [1]

Markscheme

Median $= 190\,(\text{g})$ **A1**

[1 mark]

(b.ii) Find the lower quartile. [1]

Markscheme

Lower quartile $= 135 - 140\,(\text{g})$ **A1**

[1 mark]

(b.iii) Find the upper quartile. [1]

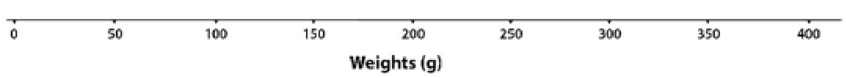
Markscheme

Upper quartile $= 242 - 247\,(\text{g})$ **A1**

[1 mark]

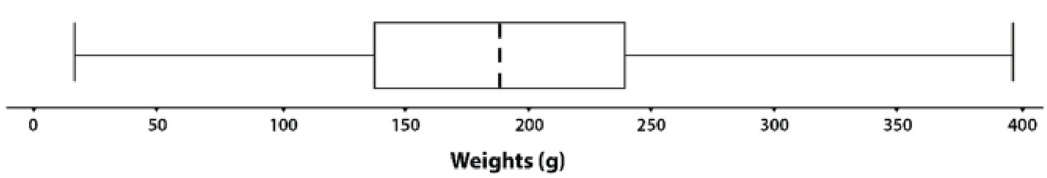
(c) The weight of the smallest potato in the sample is 20 grams and the weight of the largest is 400 grams.

Use the scale shown below to draw a box and whisker diagram showing the distribution of the weights of the potatoes. You may assume there are no outliers.



[2]

Markscheme



M1A1

Note: The **M1** is for a box and whisker plot and the **A1** for all 5 statistics in the right places.

[2 marks]

16. [Maximum mark: 7] EXN.1.SL.TZ0.7

Consider the curve $y = x^2 - 4x + 2$.

(a) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. [1]

Markscheme

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$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 4$ **A1**

[1 mark]

(b) Show that the normal to the curve at the point where $x = 1$ is $2y - x + 3 = 0$. [6]

Markscheme

Gradient at $x = 1$ is -2 **M1**

Gradient of normal is $\frac{1}{2}$ **A1**

When $x = 1$ $y = 1 - 4 + 2 = -1$ **(M1)A1**

EITHER

$y + 1 = \frac{1}{2}(x - 1)$ **M1**

$2y + 2 = x - 1$ or $y + 1 = \frac{1}{2}x - \frac{1}{2}$ **A1**

OR

$-1 = \frac{1}{2} \times 1 + c$ **M1**

$y = \frac{1}{2}x - \frac{3}{2}$ **A1**

THEN

$2y - x + 3 = 0$ **AG**

[6 marks]

17. [Maximum mark: 7] EXN.1.SL.TZ0.8

The water temperature (T) in Lake Windermere is measured on the first day of eight consecutive months (m) from January to August (months 1 to 8) and the results are shown below. The value for May (month 5) has been accidently deleted.

	Jan	Feb	March	April	May	June	July	August
Month (*m*)	1	2	3	4	5	6	7	8
Temperature (*T*)(°C)	5.2	8.0	7.2	8.9		12.6	15.5	15.4

(a) Assuming the data follows a linear model for this period, find the regression line of T on m for the remaining data. [2]

Markscheme

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$(T = 1.517\ldots m + 3.679\ldots)$ $T = 1.52m + 3.68$ **A1A1**

[2 marks]

(b) Use your line to find an estimate for the water temperature on the first day of May. [2]

Markscheme

$11.3\ (11.2671\ldots)\ (^\circ\mathrm{C})$ **(M1)A1**

[2 marks]

(c.i) Explain why your line should not be used to estimate the value of m at which the temperature is $10.0\ ^\circ\mathrm{C}$. [1]

Markscheme

Because the line should only be used to estimate T from m and not m from T. **R1**

[1 mark]

(c.ii) Explain in context why your line should not be used to predict the value for December (month 12). [1]

Markscheme

Because the temperatures are no longer going up at a steady rate, ***or*** because we know that winter is approaching so the temperature will go down, ***or*** temperatures are not likely to continue increasing. **R1**

[1 mark]

(d) State a more appropriate model for the water temperature in the lake over an extended period of time. You are not expected to calculate any parameters. [1]

Markscheme

Trigonometric or sinusoidal **A1**

[1 mark]

18. [Maximum mark: 7] EXN.1.SL.TZ0.9

Sophia pays $\$200$ into a bank account at the end of each month. The annual interest paid on money in the account is 3.1% which is compounded monthly.

(a) Find the value of her investment after a period of 5 years. [3]

Markscheme

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Number of time periods $12 \times 5 = 60$ **(A1)**

N = 60
I% = 3.1
PV = 0
PMT = 200
P/Y = 12
C/Y = 12
Value $(\$)12,961.91$ **(M1)A1**

[3 marks]

The average rate of inflation per year over the 5 years was 2%.

(b) Find an approximation for the real interest rate for the money invested in the account. [2]

Markscheme

METHOD 1

Real interest rate $= 3.1 - 2.0 = 1.1\%$ **(M1)A1**

METHOD 2

$\frac{1+0.031}{1+0.02} = 1.01078\ldots$ **(M1)**

1.08% (accept 1.1%) **A1**

[2 marks]

(c) Hence find the real value of Sophia's investment at the end of 5 years. [2]

Markscheme

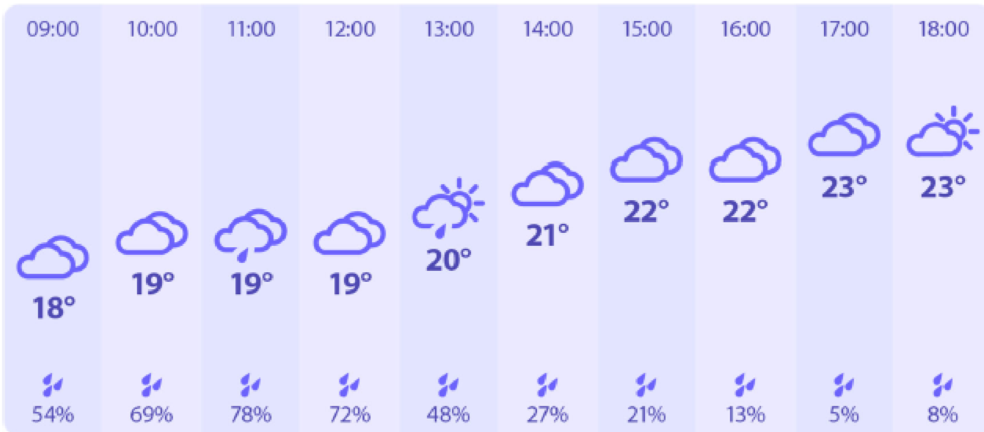
N = 60
I% = 1.1
PV = 0
PMT = 200
P/Y = 12
C/Y = 12
$(\$)12,300\ (12,330.33\ldots)$ **(M1)A1**

Note: Award **A1** for $\$12,300$ only.

[2 marks]

19. [Maximum mark: 7] EXN.1.SL.TZ0.10

The diagram below shows part of the screen from a weather forecasting website showing the data for town A. The percentages on the bottom row represent the likelihood of some rain during the hour leading up to the time given. For example there is a 69% chance (a probability of 0.69) of rain falling on any point in town A between 0900 and 1000.



Paula works at a building site in the area covered by this page of the website from 0900 to 1700. She has lunch from 1300 to 1400.

(a) Write down the probability it rains during Paula's lunch break. [1]

Markscheme

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Note: Accept probabilities written as percentages throughout.

0.27 **A1**

[1 mark]

In the following parts you may assume all probabilities are independent.

Paula needs to work outside between 1000 and 1300.

(b) Find the probability it will rain in each of the three hours Paula is working outside. [2]

Markscheme

Note: Accept probabilities written as percentages throughout.

$0.78 \times 0.72 \times 0.48$ **(M1)**

$= 0.270\ (0.269568)$ **A1**

[2 marks]

Paula will also spend her lunchtime outside.

(c) Find the probability it will not rain while Paula is outside. [2]

Markscheme

Note: Accept probabilities written as percentages throughout.

$0.22 \times 0.28 \times 0.52 \times 0.73$ **(M1)**

$= 0.0234\ (0.02338336)$ **A1**

[2 marks]

(d) Find the probability it will rain at least once while Paula is outside. [2]

Markscheme

Note: Accept probabilities written as percentages throughout.

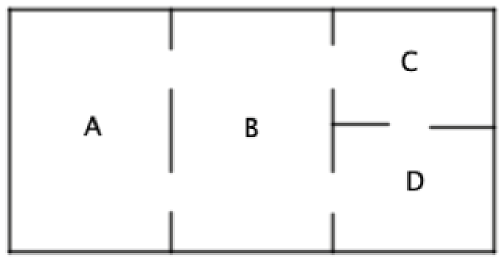
$1 - 0.02338336$ **(M1)**

$= 0.977\ (0.976616\ldots)$ **A1**

[2 marks]

20. [Maximum mark: 5] EXN.1.AHL.TZ0.5

A robot moves around the maze shown below.



Whenever it leaves a room it is equally likely to take any of the exits.

The time interval between the robot entering and leaving a room is the same for all transitions.

(a) Find the transition matrix for the maze. [3]

Markscheme

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$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 0.5 & 0 & 0 \\ 1 & 0 & 0.5 & 0.5 \\ 0 & 0.25 & 0 & 0.5 \\ 0 & 0.25 & 0.5 & 0 \end{pmatrix} \end{array}$$

(M1)A1A1

Note: Award **A1A0** if there is one error in the matrix. **A0A0** for more than one error.

[3 marks]

(b) A scientist sets up the robot and then leaves it moving around the maze for a long period of time.

Find the probability that the robot is in room B when the scientist returns. [2]

Markscheme

Steady state column matrix is $\begin{pmatrix} 0.2 \\ 0.4 \\ 0.2 \\ 0.2 \end{pmatrix}$ **(M1)**

Probability it is in room B is 0.4 **A1**

[2 marks]

21. [Maximum mark: 6] EXN.1.AHL.TZ0.7

The position of a helicopter relative to a communications tower at the top of a mountain at time t (hours) can be described by the vector equation below.

$$\boldsymbol{r} = \begin{pmatrix} 20 \\ -25 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4.2 \\ 5.8 \\ -0.5 \end{pmatrix}$$

The entries in the column vector give the displacements east and north from the communications tower and above/below the top of the mountain respectively, all measured in kilometres.

(a) Find the speed of the helicopter. [2]

Markscheme

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$|v| = \sqrt{4.2^2 + 5.8^2 + 0.5^2}$ **(M1)**

$7.18\ (7.1784\ldots)\ \left(\text{kmh}^{-1}\right)$ **A1**

[2 marks]

(b) Find the distance of the helicopter from the communications tower at $t = 0$. [2]

Markscheme

$$\boldsymbol{r} = \begin{pmatrix} 20 \\ -25 \\ 0 \end{pmatrix}$$

$|\boldsymbol{r}| = \sqrt{20^2 + 25^2}$ **(M1)**

$= \sqrt{1025} = 32.0\ (32.0156\ldots)\ (\text{km})$ **A1**

[2 marks]

(c) Find the bearing on which the helicopter is travelling. [2]

Markscheme

Bearing is $\arctan\left(\frac{4.2}{5.8}\right)$ or 90°-$\arctan\left(\frac{5.8}{4.2}\right)$ **(M1)**

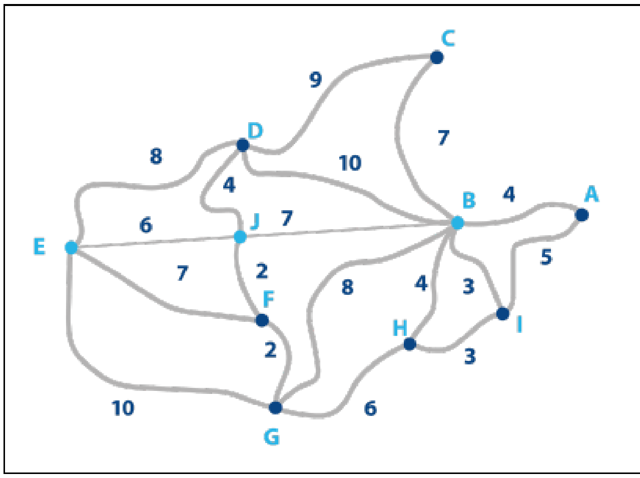
035.9° $(35.909\ldots)$ **A1**

[2 marks]

22. [Maximum mark: 7] EXN.1.AHL.TZ0.11

Nymphenburg Palace in Munich has extensive grounds with 9 points of interest (stations) within them.

These nine points, along with the palace, are shown as the vertices in the graph below. The weights on the edges are the walking times in minutes between each of the stations and the total of all the weights is 105 minutes.



A	Nymphenburg Palace
B	Fountain
C	Magdalenenklause
D	Pagodenburg
E	Large Cascade
F	Pan with Spring
G	Badenburg
H	Amalienburg
I	Summerhouse
J	Bridge

Anders decides he would like to walk along all the paths shown beginning and ending at the Palace (vertex A).

Use the Chinese Postman algorithm, clearly showing all the stages, to find the shortest time to walk along all the paths. [7]

Markscheme

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Odd vertices are B, F, H and I **(M1)A1**

Pairing the vertices **M1**

BF and HI $9 + 3 = 12$
BH and FI $4 + 11 = 15$
BI and FH $3 + 8 = 11$ **A2**

Note: award **A1** for two correct totals.

Shortest time is $105 + 11 = 116$ (minutes) **M1A1**

[7 marks]

23. [Maximum mark: 5] EXN.1.AHL.TZ0.14

(a) Write down $2+5\mathrm{i}$ in exponential form. [2]

Markscheme

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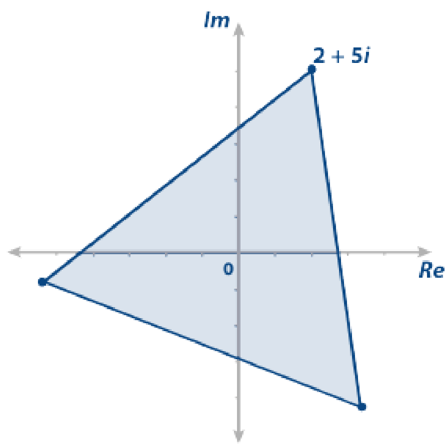
$5.385\ldots \mathrm{e}^{1.1902\ldots \mathrm{i}} \approx 5.39\mathrm{e}^{1.19\mathrm{i}}$ **A1A1**

Note: Accept equivalent answers: $5.39\mathrm{e}^{-5.09\mathrm{i}}$

[2 marks]

(b)



An equilateral triangle is to be drawn on the Argand plane with one of the vertices at the point corresponding to $2+5\mathrm{i}$ and all the vertices equidistant from 0.

Find the points that correspond to the other two vertices. Give your answers in Cartesian form. [3]

Markscheme

multiply by $\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$ **(M1)**

$-5.33-0.77\mathrm{i},\ 3.33-4.23\mathrm{i}$ **A1A1**

[3 marks]

24. [Maximum mark: 8] EXN.1.AHL.TZ0.15

Consider the function $f(x) = \sqrt{-ax^2 + x + a},\ a \in \mathbb{R}^+$.

(a) Find $f\prime(x)$. [2]

Markscheme

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$f\prime(x) = (-2ax + 1) \times \frac{1}{2} \times \left(-ax^2 + x + a\right)^{-\frac{1}{2}}$

Note: M1 is for use of the chain rule.

$= \frac{-2ax+1}{2\sqrt{-ax^2+x+a}}$ **M1A1**

[2 marks]

For $a > 0$ the curve $y = f(x)$ has a single local maximum.

(b) Find in terms of a the value of x at which the maximum occurs. [2]

Markscheme

$-2ax + 1 = 0$ **(M1)**

$x = \frac{1}{2a}$ **A1**

[2 marks]

(c) Hence find the value of a for which y has the smallest possible maximum value. [4]

Markscheme

Value of local maximum $= \sqrt{-a \times \frac{1}{4a^2} + \frac{1}{2a} + a}$ **M1A1**

$= \sqrt{\frac{1}{4a} + a}$

This has a minimum value when $a = 0.5$ **(M1)A1**

[4 marks]

25. [Maximum mark: 8] EXN.1.AHL.TZ0.16

The cars for a fairground ride hold four people. They arrive at the platform for loading and unloading every 30 seconds.

During the hour from 9 am the arrival of people at the ride in any interval of t minutes can be modelled by a Poisson distribution with a mean of $9t\,(0 < t < 60)$.

When the 9 am car leaves there is no one in the queue to get on the ride.

Shunsuke arrives at 9.01 am.

(a) Find the probability that more than 7 people arrive at the ride before Shunsuke. [2]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

Let X be the number of people who arrive between 9.00 am and 9.01 am

$X \sim \text{Po}(9)$

$\text{P}(X > 7) = \text{P}(X \geq 8)$ **(M1)**

$0.676\ (0.67610\ldots)$ **A1**

[2 marks]

(b) Find the probability there will be space for him on the 9.01 car. [6]

Markscheme

Mean number of people arriving each 30 seconds is 4.5 **(M1)**

Let X_1 be the number who arrive in the first 30 seconds and X_2 the number who arrive in the second 30 seconds.

P(Shunsuke will be able to get on the ride)

$= \text{P}(X_1 \leq 4) \times \text{P}(X_2 \leq 3) + \text{P}(X_1 = 5) \times \text{P}(X_2 \leq 2) + \text{P}(X_1 = 6) \times \text{P}(X_2 \leq 1) + \text{P}(X_1 = 7) \times \text{P}(X_2 = 0)$ **M1M1**

Note: M1 for first term, **M1** for any of the other terms.

null **(A1)(A1)**

Note: (A1) for one correct value, **(A1)(A1)** for four correct values.

$= 0.221\ (0.220531\ldots)$ **A1**

[6 marks]

26. [Maximum mark: 12] EXN.2.SL.TZ0.2

A box of chocolates is to have a ribbon tied around it as shown in the diagram below.

The box is in the shape of a cuboid with a height of 3 cm. The length and width of the box are x and y cm.

After going around the box an extra 10 cm of ribbon is needed to form the bow.

(a) Find an expression for the total length of the ribbon L in terms of x and y. [2]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$L = 2x + 2y + 12 + 10 = 2x + 2y + 22$ **A1A1**

Note: A1 for $2x + 2y$ and **A1** for $12 + 10$ or 22.

[2 marks]

The volume of the box is $450\,\text{cm}^3$.

(b) Show that $L = 2x + \frac{300}{x} + 22$ [3]

Markscheme

$V = 3xy = 450$ **A1**

$y = \frac{150}{x}$ **A1**

$L = 2x + 2\left(\frac{150}{x}\right) + 22$ **M1**

$L = 2x + \frac{300}{x} + 22$ **AG**

[3 marks]

(c) Find $\frac{\mathrm{d}L}{\mathrm{d}x}$ [3]

Markscheme

$L = 2x + 300x^{-1} + 22$ **(M1)**

$\frac{\mathrm{d}L}{\mathrm{d}x} = 2 - \frac{300}{x^2}$ **A1A1**

Note: A1 for 2 (and 0), **A1** for $\frac{300}{x^2}$.

[3 marks]

(d) Solve $\frac{\mathrm{d}L}{\mathrm{d}x} = 0$ [2]

Markscheme

$\frac{300}{x^2} = 2$ **(M1)**

$x = \sqrt{150} = 12.2\ \ (12.2474\ldots)$ **A1**

[2 marks]

(e) Hence or otherwise find the minimum length of ribbon required. [2]

Markscheme

$L = 2\sqrt{150} + \frac{300}{\sqrt{150}} + 22 = 71.0\ \ (70.9897\ldots)\,\mathrm{cm}$ **(M1)A1**

[2 marks]

27. [Maximum mark: 9] EXM.1.SL.TZ0.4

The diagram below is part of a Voronoi diagram.

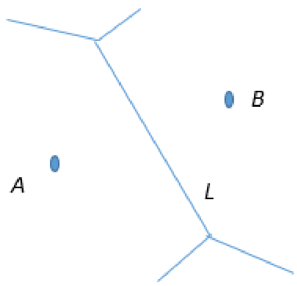


Diagram not to scale

A and *B* are sites with *B* having the co-ordinates of (4, 6). *L* is an edge; the equation of this perpendicular bisector of the line segment from *A* to *B* is $y = -2x + 9$

Find the co-ordinates of the point *A*. [9]

Markscheme

Line from *A* to *B* will have the form $y = \frac{1}{2}x + c$ ***M1A1***

Through $(4,\ 6) \Rightarrow c = 4$ so line is $y = \frac{1}{2}x + 4$ ***M1A1***

Intersection of $y = -2x + 9$ and $y = \frac{1}{2}x + 4$ is (2, 5) ***M1A1***

Let $A = (p,\ q)$ then $(2,\ 5) = \left(\frac{p+4}{2},\ \frac{q+6}{2}\right) \Rightarrow p = 0,\ q = 4$ ***M1A1A1***

$A = (0, 4)$

[9 marks]

28. [Maximum mark: 9] EXM.1.SL.TZ0.5

A set of data comprises of five numbers x_1, x_2, x_3, x_4, x_5 which have been placed in ascending order.

(a) Recalling definitions, such as the Lower Quartile is the $\frac{n+1}{4}th$ piece of data with the data placed in order, find an expression for the Interquartile Range. [2]

Markscheme

$LQ = \frac{x_1+x_2}{2}$, $UQ = \frac{x_4+x_5}{2}$, $IQR = \frac{x_4+x_5-x_1-x_2}{2}$ ***M1A1***

[2 marks]

(b) Hence, show that a data set with only 5 numbers in it cannot have any outliers. [5]

Markscheme

$UQ + 1.5IQR = 1.25x_4 + 1.25x_5 - 0.75x_1 - 0.75x_2 \geqslant x_5$ ***M1A1***

Since $1.25x_4 + 0.25x_5 \geqslant 0.75x_1 + 0.75x_2$ due to the ascending order. ***R1***

Similarly $LQ - 1.5IQR = 1.25x_1 + 1.25x_2 - 0.75x_4 - 0.75x_5 \leqslant x_1$ ***M1A1***

Since $0.25x_1 + 1.25x_2 \leqslant 0.75x_3 + 0.75x_4$ due to the ascending order.

So there are no outliers for a data set of 5 numbers. ***AG***

[5 marks]

(c) Give an example of a set of data with 7 numbers in it that does have an outlier, justify this fact by stating the Interquartile Range. [2]

Markscheme

For example 1, 2, 3, 4, 5, 6, 100 where $IQR = 4$ ***A1A1***

[2 marks]

29. [Maximum mark: 5] EXM.1.SL.TZ0.1

Give your answers to this question correct to two decimal places.

Gen invests $2400 in a savings account that pays interest at a rate of 4% per year, compounded annually. She leaves the money in her account for 10 years, and she does not invest or withdraw any money during this time.

(a) Calculate the value of her savings after 10 years. [2]

Markscheme

$2400(1.04)^{10} = \$3552.59$ ***M1A1***

[2 marks]

(b) The rate of inflation during this 10 year period is 1.5% per year.

Calculate the real value of her savings after 10 years. [3]

Markscheme

real interest rate = $4 - 1.5 = 2.5\%$ ***A1***

$2400(1.025)^{10} = \$3072.20$ ***M1A1***

[3 marks]

30. [Maximum mark: 6] EXM.1.SL.TZ0.2

Kayla wants to measure the extent to which two judges in a gymnastics competition are in agreement. Each judge has ranked the seven competitors, as shown in the table, where 1 is the highest ranking and 7 is the lowest.

Competitor	A	B	C	D	E	F	G
Judge 1	1	2	3	3	5	6	6
Judge 2	2	3	1	4	5	5	7

(a) Calculate Spearman's rank correlation coefficient for this data. [5]

Markscheme

average equal ranks ***M1***

Competitor	A	B	C	D	E	F	G
Judge 1	1	2	3.5	3.5	5	6.5	6.5
Judge 2	2	3	1	4	5.5	5.5	7

A1A1

$r_s = 0.817$ ***A2***

[5 marks]

(b) State what conclusion Kayla can make from the answer in part (a). [1]

Markscheme

There is strong agreement between the two judges. ***R1***

[1 mark]

31. [Maximum mark: 6] EXM.1.SL.TZ0.6

Yejin plans to retire at age 60. She wants to create an annuity fund, which will pay her a monthly allowance of $4000 during her retirement. She wants to save enough money so that the payments last for 30 years. A financial advisor has told her that she can expect to earn 5% interest on her funds, compounded annually.

(a) Calculate the amount Yejin needs to have saved into her annuity fund, in order to meet her retirement goal. [3]

Markscheme

Use of finance solver ***M1***

$N = 360$, $I = 5\%$, Pmt = 4000, FV = 0, PpY = 12, CpY = 1 ***A1***

$755000 (correct to 3 s.f.) ***A1***

[3 marks]

(b) Yejin has just turned 28 years old. She currently has no retirement savings. She wants to save part of her salary each month into her annuity fund.

Calculate the amount Yejin needs to save each month, to meet her retirement goal. [3]

Markscheme

$N = 384$, $I = 5\%$, PV = 0, FV = 754638, PpY = 12, CpY = 1 ***M1A1***

$817 per month (correct to 3 s.f.) ***A1***

[3 marks]

32. [Maximum mark: 4] EXM.1.AHL.TZ0.2

If $A = \begin{pmatrix} 2p & 3 \\ -4p & p \end{pmatrix}$ and $\det A = 14$, find the possible values of p. [4]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$2p^2 + 12p = 14$ ***(M1)(A1)***

$p^2 + 6p - 7 = 0$

$(p + 7)(p - 1) = 0$ ***(A1)***

$p = -7$ or $p = 1$ ***(A1) (C4)***

Note: Both answers are required for the final ***(A1)***.

[4 marks]

33. [Maximum mark: 4] EXM.1.AHL.TZ0.3

A and B are 2 × 2 matrices, where $A = \begin{bmatrix} 5 & 2 \\ 2 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 11 & 2 \\ 44 & 8 \end{bmatrix}$. Find B [4]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$B = (BA)A^{-1}$ ***(M1)***

$= -\frac{1}{4}\begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -2 & 5 \end{pmatrix}$ ***(M1)***

$= -\frac{1}{4}\begin{pmatrix} -4 & -12 \\ -16 & -48 \end{pmatrix}$ ***(A1)***

$= \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$ ***(A1)***

OR

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}$ ***(M1)***

$\Rightarrow \left.\begin{matrix} 5a + 2b = 11 \\ 2a \qquad = 2 \end{matrix}\right\}$

$\Rightarrow a = 1,\ b = 3$ ***(A1)***

$\left.\begin{matrix} 5c + 2d = 44 \\ 2c \ = 8 \end{matrix}\right\}$

$\Rightarrow c = 4,\ d = 12$ ***(A1)***

$B = \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$ ***(A1) (C4)***

Note: Correct solution with inversion (ie *AB* instead of *BA*) earns FT marks, (maximum ***[3 marks]***).

[4 marks]

34. [Maximum mark: 6] EXM.1.AHL.TZ0.28

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} e^x & e^{-x} \\ 2+e^x & 1 \end{pmatrix}$, where $x \in \mathbb{R}$.

Find the value of x for which $\boldsymbol{A}$ is singular. [6]

Markscheme

finding $\det \boldsymbol{A} = e^x - e^{-x}(2+e^x)$ or equivalent ***A1***

$\boldsymbol{A}$ is singular $\Rightarrow \det \boldsymbol{A} = 0$ ***(R1)***

$e^x - e^{-x}(2+e^x) = 0$

$e^{2x} - e^x - 2 = 0$ ***A1***

solving for e^x ***(M1)***

$e^x > 0$ (or equivalent explanation) ***(R1)***

$e^x = 2$

$x = \ln 2$ (only) ***A1 N0***

[6 marks]

35. [Maximum mark: 2] EXM.1.AHL.TZ0.31

Find the determinant of $\boldsymbol{A}$, where $\boldsymbol{A} = \begin{pmatrix} 3 & 1 & 2 \\ 9 & 5 & 8 \\ 7 & 4 & 6 \end{pmatrix}$. [2]

Markscheme

$\det \boldsymbol{A} = -2$ ***A2***

[2 marks]

36. [Maximum mark: 5] EXM.1.AHL.TZ0.32

If $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix}$ and $\boldsymbol{A}^2$ is a matrix whose entries are all 0, find k. [5]

Markscheme

$\boldsymbol{A}^2 = \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix}$ ***M1***

$= \begin{pmatrix} 1+2k & 0 \\ 0 & 2k+1 \end{pmatrix}$ ***A2***

Note: Award ***A2*** for 4 correct, ***A1*** for 2 or 3 correct.

$1 + 2k = 0$ ***M1***

$k = -\frac{1}{2}$ ***A1***

[5 marks]

37. [Maximum mark: 5] EXM.1.AHL.TZ0.33

Given that $\boldsymbol{M} = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$ and that $\boldsymbol{M}^2 - 6\boldsymbol{M} + k\boldsymbol{I} = 0$ find k. [5]

Markscheme

$\boldsymbol{M}^2 = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -18 & 19 \end{pmatrix}$ ***M1A1***

$\Rightarrow \begin{pmatrix} 7 & -6 \\ -18 & 19 \end{pmatrix} - \begin{pmatrix} 12 & -6 \\ -18 & 24 \end{pmatrix} + k\boldsymbol{I} = 0$ ***(M1)***

$\Rightarrow \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix} + k\boldsymbol{I} = 0$ ***(A1)***

$\Rightarrow k = 5$ ***A1***

[5 marks]

38. [Maximum mark: 6] EXM.1.AHL.TZ0.34

The square matrix $\boldsymbol{X}$ is such that $\boldsymbol{X}^3 = 0$. Show that the inverse of the matrix $(\boldsymbol{I} - \boldsymbol{X})$ is $\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2$. [6]

Markscheme

For multiplying $(\boldsymbol{I} - \boldsymbol{X})(\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2)$ ***M1***

$= \boldsymbol{I}^2 + \boldsymbol{IX} + \boldsymbol{IX}^2 - \boldsymbol{XI} - \boldsymbol{X}^2 - \boldsymbol{X}^3 = \boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2 - \boldsymbol{X} - \boldsymbol{X}^2 - \boldsymbol{X}^3$ ***(A1)(A1)***

$= \boldsymbol{I} - \boldsymbol{X}^3$ ***A1***

$= \boldsymbol{I}$ ***A1***

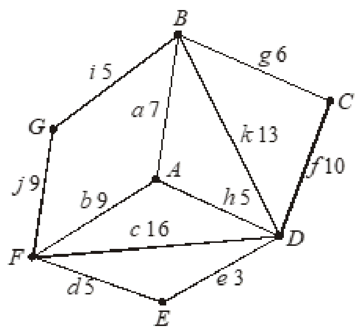
$\boldsymbol{AB} = \boldsymbol{I} \Rightarrow \boldsymbol{A}^{-1} = \boldsymbol{B}$ ***(R1)***

$(\boldsymbol{I} - \boldsymbol{X})\,(\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2) = \boldsymbol{I} \Rightarrow (\boldsymbol{I} - \boldsymbol{X})^{-1} = \boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2$ ***AG N0***

[5 marks]

39. [Maximum mark: 8] EXM.1.AHL.TZ0.36

Apply Prim's algorithm to the weighted graph given below to obtain the minimal spanning tree starting with the vertex A.



Find the weight of the minimal spanning tree. [8]

Markscheme

We start with point A and write S as the set of vertices and T as the set of edges.
The weights on each edge will be used in applying Prim's algorithm.
Initially, $S = \{A\}$, $T = \emptyset$. In each subsequent stage, we shall update S and T.

Step 1: Add edge h: So $S = \{A, D\}$, $T = \{h\}$
Step 2: Add edge e: So $S = \{A, D, E\}$ $T = \{h, e\}$
Step 3: Add edge d: Then $S = \{A, D, E, F\}$ $T = \{h, e, d\}$
Step 4: Add edge a: Then $S = \{A, D, E, F, B\}$ $T = \{h, e, d, a\}$
Step 5: Add edge i: Then $S = \{A, D, E, F, B, G\}$ $T = \{h, e, d, a, i\}$
Step 6: Add edge g: Then $S = (A, D, E, F, B, G, C\}$ $T = \{h, e, d, a, I, g\}$ ***(M4)(A3)***

Notes: Award ***(M4)(A3)*** for all 6 correct,
(M4)(A2) for 5 correct;
(M3)(A2) for 4 correct,
(M3)(A1) for 3 correct;
(M1)(A1) for 2 correct,
(M1)(A0) for 1 correct

OR

(M2) for the correct definition of Prim's algorithm,
(M2) for the correct application of Prim's algorithm,
(A3) for the correct answers at the last three stages.

Now S has all the vertices and the minimal spanning tree is obtained.

The weight of the edges in T is $5 + 3 + 5 + 7 + 5 + 6$

$= 31$ ***(A1)***

[8 marks]

40. [Maximum mark: 6] EXM.1.AHL.TZ0.1

The cost adjacency matrix below represents the distance in kilometres, along routes between bus stations.

	A	B	C	D	E
A	-	x	2	6	p
B	x	-	5	7	q
C	2	5	-	3	r
D	6	7	3	-	s
E	p	q	r	s	-

All the values in the matrix are positive, distinct integers.

It is decided to electrify some of the routes, so that it will be possible to travel from any station to any other station solely on electrified routes. In order to achieve this with a minimal total length of electrified routes, Prim's algorithm for a minimal spanning tree is used, starting at vertex A.

The algorithm adds the edges in the following order:

AB AC CD DE.

There is only one minimal spanning tree.

(a) Find with a reason, the value of x. [2]

Markscheme

AB must be the length of the smallest edge from A so $x = 1$. ***R1A1***

[2 marks]

(b) If the total length of the minimal spanning tree is 14, find the value of s. [2]

Markscheme

$1 + 2 + 3 + s = 14 \Rightarrow s = 8$ ***M1A1***

[2 marks]

(c) Hence, state, with a reason, what can be deduced about the values of p, q, r. [2]

Markscheme

The last minimal edge chosen must connect to E , so since $s = 8$ each of p, q, r must be ≥ 9. ***R1A1***

[2 marks]

41. [Maximum mark: 9] EXM.1.AHL.TZ0.6

Let $\boldsymbol{C} = \begin{pmatrix} -2 & 4 \\ 1 & 7 \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}$.

The 2×2 matrix $\boldsymbol{Q}$ is such that $3\boldsymbol{Q} = 2\boldsymbol{C} - \boldsymbol{D}$

(a) Find $\boldsymbol{Q}$. [3]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$3\boldsymbol{Q} = \begin{pmatrix} -4 & 8 \\ 2 & 14 \end{pmatrix} - \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}$ ***(A1)***

$3\boldsymbol{Q} = \begin{pmatrix} -9 & 6 \\ 3 & 14 - a \end{pmatrix}$ ***(A1)***

$\boldsymbol{Q} = \begin{pmatrix} -3 & 2 \\ 1 & \frac{14-a}{3} \end{pmatrix}$ ***(A1) (N3)***

[3 marks]

(b) Find $\boldsymbol{CD}$. [4]

Markscheme

$\boldsymbol{CD} = \begin{pmatrix} -2 & 4 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}$

$= \begin{pmatrix} -14 & -4 + 4a \\ -2 & 2 + 7a \end{pmatrix}$ ***(A1)(A1)(A1)(A1) (N4)***

[4 marks]

(c) Find $\boldsymbol{D}^{-1}$. [2]

Markscheme

$\det \boldsymbol{D} = 5a + 2$ (may be implied) ***(A1)***

$\boldsymbol{D}^{-1} = \frac{1}{5a+2} \begin{pmatrix} a & -2 \\ 1 & 5 \end{pmatrix}$ ***(A1) (N2)***

[2 marks]

42. [Maximum mark: 6] EXM.1.AHL.TZ0.10

Let $\boldsymbol{A} = \begin{pmatrix} 3 & 2 \\ k & 4 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Find, in terms of k,

(a) $2\boldsymbol{A} - \boldsymbol{B}$. [3]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$2\boldsymbol{A} = \begin{pmatrix} 6 & 4 \\ 2k & 8 \end{pmatrix}$ ***(A1)***

$2\boldsymbol{A} - \boldsymbol{B} = \begin{pmatrix} 4 & 2 \\ 2k - 1 & 5 \end{pmatrix}$ ***A2 N3***

[3 marks]

(b) det $(2\boldsymbol{A} - \boldsymbol{B})$. [3]

Markscheme

Evidence of using the definition of determinant ***(M1)***

Correct substitution ***(A1)***

eg $4(5) - 2(2k - 1)$, $20 - 2(2k - 1)$, $20 - 4k + 2$

det $(2\boldsymbol{A} - \boldsymbol{B}) = 22 - 4k$ ***A1 N3***

[3 marks]

43. [Maximum mark: 7] EXM.1.AHL.TZ0.17

Sue sometimes goes out for lunch. If she goes out for lunch on a particular day then the probability that she will go out for lunch on the following day is 0.4. If she does not go out for lunch on a particular day then the probability she will go out for lunch on the following day is 0.3.

(a) Write down the transition matrix for this Markov chain. [2]

Markscheme

$$\begin{pmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{pmatrix}$$ ***M1A1***

[2 marks]

(b) We know that she went out for lunch on a particular Sunday, find the probability that she went out for lunch on the following Tuesday. [2]

Markscheme

$$\begin{pmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.34 \\ 0.66 \end{pmatrix}$$ ***M1***

So probability is 0.34 ***A1***

[2 marks]

(c) Find the steady state probability vector for this Markov chain. [3]

Markscheme

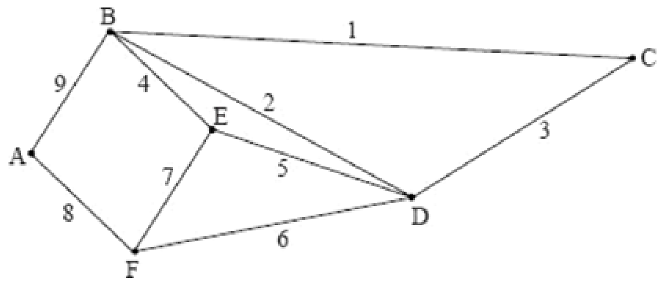
$$\begin{pmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} p \\ 1-p \end{pmatrix} \Rightarrow 0.4p + 0.3\left(1-p\right) = p \Rightarrow p = \frac{1}{3}$$ ***M1A1***

So vector is $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$ ***A1***

[or by investigating high powers of the transition matrix]

[3 marks]

44. [Maximum mark: 9] EXM.1.AHL.TZ0.24



The above diagram shows the weighted graph G.

(a.i) Write down the adjacency matrix for G. [1]

Markscheme

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$ ***A1***

[1 mark]

(a.ii) Find the number of distinct walks of length 4 beginning and ending at A. [3]

Markscheme

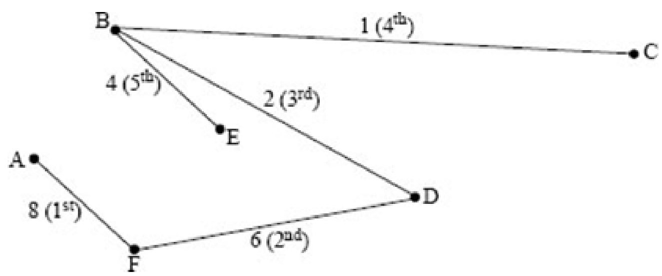
We require the (A, A) element of M^4 which is 13. ***M1A2***

[3 marks]

(b) Starting at A, use Prim's algorithm to find and draw the minimum spanning tree for G.

Your solution should indicate clearly the way in which the tree is constructed. [5]

Markscheme



A1A1A1A1A1

[5 marks]

45. [Maximum mark: 8] EXM.1.AHL.TZ0.37

In this part, marks will only be awarded if you show the correct application of the required algorithms, and show all your working.

In an offshore drilling site for a large oil company, the distances between the planned wells are given below in metres.

	1	2	3	4	5	6	7	8	9	10
2	30									
3	40	60								
4	90	190	130							
5	80	200	10	160						
6	70	40	20	40	130					
7	60	120	50	90	30	60				
8	50	140	90	70	140	70	40			
9	40	170	140	60	50	90	50	70		
10	200	80	150	110	90	30	190	90	100	
11	150	30	200	120	190	120	60	190	150	200

It is intended to construct a network of paths to connect the different wells in a way that minimises the sum of the distances between them.

Use Prim's algorithm, starting at vertex 3, to find a network of paths of minimum total length that can span the whole site. [8]

Markscheme

Vertices added to the Tree	Edge added	Weight
3	$\varnothing$	0
5	3, 5	10
6	3, 6	20
7	5, 7	30
10	6, 10	30
1	3, 1	40
2	1, 2	30
11	2, 11	30
9	1, 9	40
4	6, 4	40
8	7, 8	40
		310

(R2)(A4)(M1)

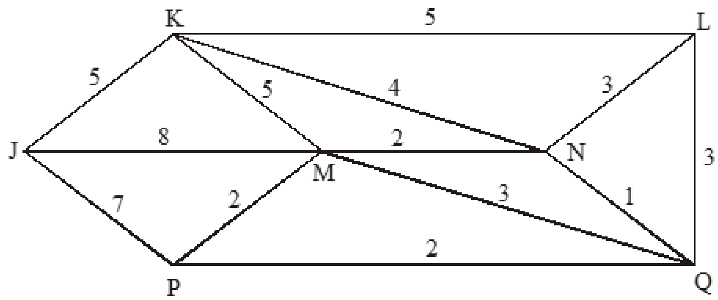
(A1)

Note: Award ***(R2)*** for correct algorithms, ***(R1)*** for 1 error, ***(R0)*** for 2 or more errors.
Award ***(A4)*** for correct calculations, ***(A3)*** for 1 error, ***(A2)*** for 2 errors, ***(A1)*** for 3 errors, ***(A0)*** for 4 or more errors.
Award ***(M1)*** for tree/table/method.
Award ***(A1)*** for minimum weight.

[8 marks]

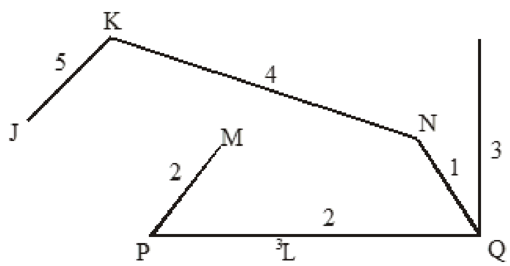
46. [Maximum mark: 6] EXM.1.AHL.TZ0.38

The diagram below shows a weighted graph.



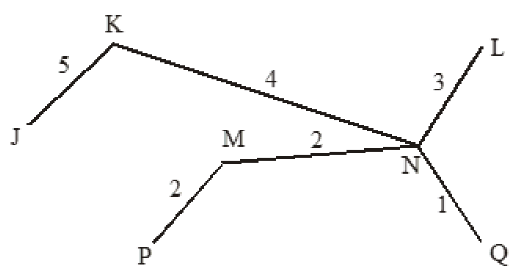
Use Prim's algorithms to find a minimal spanning tree, starting at J. Draw the tree, and find its total weight. [6]

Markscheme



(C4)

OR



(C4)

Total weight = 17 ***(A2)***

Note: There are other possible spanning trees.

[6 marks]

47. [Maximum mark: 10] EXM.1.AHL.TZ0.40

The weights of the edges of a complete graph G are shown in the following table.

	A	B	C	D	E	F
A	–	5	4	7	6	2
B	5	–	6	3	5	4
C	4	6	–	8	1	6
D	7	3	8	–	7	3
E	6	5	1	7	–	3
F	2	4	6	3	3	–

Starting at B, use Prim's algorithm to find and draw a minimum spanning tree for G. Your solution should indicate the order in which the vertices are added. State the total weight of your tree. [10]

Markscheme

Different notations may be used but the edges should be added in the following order.

Using Prim's Algorithm, ***(M1)***

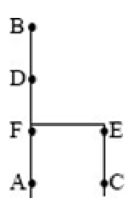
BD ***A1***

DF ***A1***

FA ***A1***

FE ***A1***

EC ***A1***



A2

Total weight = 12 ***A2***

[10 marks]

48. [Maximum mark: 12] EXM.1.AHL.TZ0.41

The weights of the edges in a simple graph G are given in the following table.

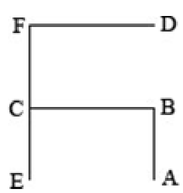
Vertices	A	B	C	D	E	F
A	–	4	6	16	15	17
B	4	–	5	17	9	16
C	6	5	–	15	8	14
D	16	17	15	–	15	7
E	15	9	8	15	–	18
F	17	16	14	7	18	–

Use Prim's Algorithm, starting with vertex F, to find and draw the minimum spanning tree for *G*. Your solution should indicate the order in which the edges are introduced. [12]

Markscheme

The edges are introduced in the following order:

FD, FC, CB, BA, CE ***A2A2A2A2A2***



A2

[12 marks]

49. [Maximum mark: 6] EXM.1.AHL.TZ0.54

Consider the system of equations $\boldsymbol{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ where $\boldsymbol{A} = \begin{pmatrix} k+1 & -k \\ 2 & k-1 \end{pmatrix}$ and $k \in \mathbb{R}$.

(a) Find det ***A***. [2]

Markscheme

Attempting to find det ***A*** ***(M1)***

det ***A*** $= k^2 + 2k - 1$ ***A1 N2***

[2 marks]

(b) Find the set of values of k for which the system has a unique solution. [4]

Markscheme

System has a unique solution provided det ***A*** ≠ 0 ***(R1)***

$k^2 + 2k - 1 \neq 0$ ***(A1)***

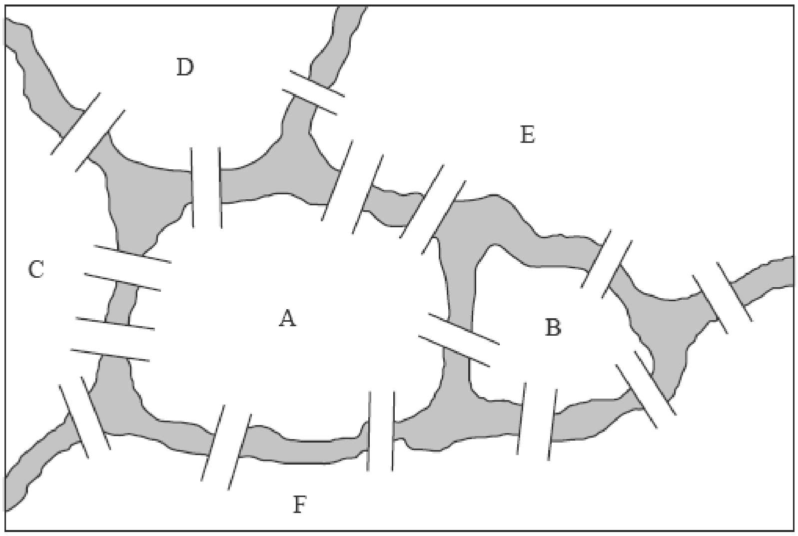
Solving $k^2 + 2k - 1 \neq 0$ or equivalent for k ***M1***

$k \in \mathbb{R} \setminus \left\{-1 \pm \sqrt{2}\right\}$ $\left(\text{accept } k \neq -1 \pm \sqrt{2},\ k \neq -2.41,\ 0.414\right)$ ***A1 N3***

[4 marks]

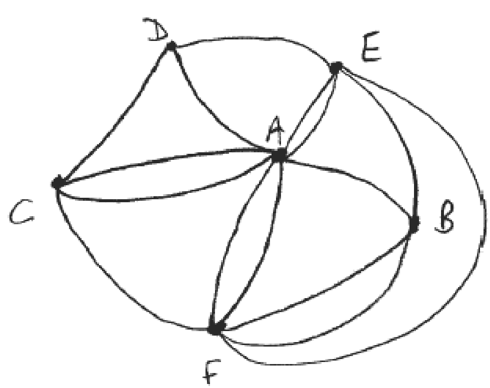
50. [Maximum mark: 12] EXM.2.AHL.TZ0.17

A canal system divides a city into six land masses connected by fifteen bridges, as shown in the diagram below.



(a) Draw a graph to represent this map. [2]

Markscheme



A2

[2 marks]

(b) Write down the adjacency matrix of the graph. [2]

Markscheme

$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & 2 \\ 1 & 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$ ***A2***

Note: Award A1 for one error or omission, A0 for more than one error or omission. Two symmetrical errors count as one error.

[2 marks]

(c) List the degrees of each of the vertices. [2]

Markscheme

A B C D E F

(8, 4 4, 3 5, 6) ***A2***

Note: Award no more than A1 for one error, A0 for more than one error.

[2 marks]

State with reasons whether or not this graph has

(d.i) an Eulerian circuit. [2]

Markscheme

no, because there are odd vertices ***M1A1***

[2 marks]

(d.ii) an Eulerian trail. [2]

Markscheme

yes, because there are exactly two odd vertices ***M1A1***

[2 marks]

(e) Find the number of walks of length 4 from E to F. [2]

Markscheme

$$M^4 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 309 & 174 & 140 & 118 & 170 & 214 \\ 174 & 117 & 106 & 70 & 122 & 132 \\ 140 & 106 & 117 & 66 & 134 & 138 \\ 118 & 70 & 66 & 53 & 80 & 102 \\ 170 & 122 & 134 & 80 & 157 & 170 \\ 214 & 132 & 138 & 102 & 170 & 213 \end{pmatrix} \end{array}$$

(M1)A1

number of walks of length 4 is 170

Note: The complete matrix need not be shown. Only one of the FE has to be shown.

[2 marks]

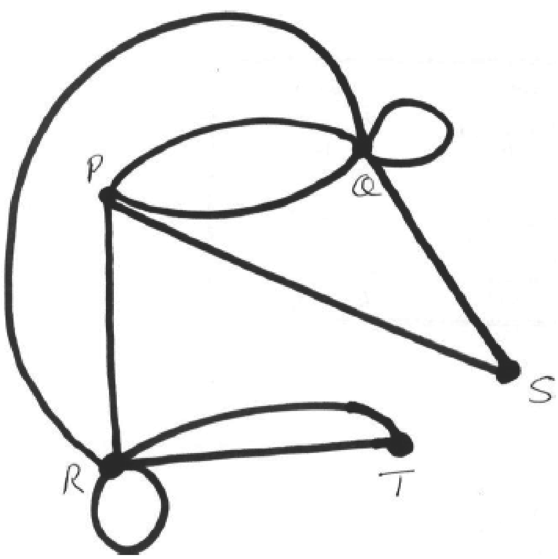
51. [Maximum mark: 19] EXM.2.AHL.TZ0.18

The adjacency matrix of the graph G, with vertices P, Q, R, S, T is given by:

$$\begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array}\begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix} \end{array}$$

(a) Draw the graph of G. [3]

Markscheme



A3

Note: Award A2 for one missing or misplaced edge,

A1 for two missing or misplaced edges.

[3 marks]

(b.i) Define an Eulerian circuit. [1]

Markscheme

an Eulerian circuit is one that contains every edge of the graph exactly once ***A1***

[1 mark]

(b.ii) Write down an Eulerian circuit in G starting at P. [2]

Markscheme

a possible Eulerian circuit is

P → Q → S → P → Q → Q → R → T → R → R → P ***A2***

[2 marks]

(c.i) Define a Hamiltonian cycle. [2]

Markscheme

a Hamiltonian cycle passes through each vertex of the graph ***A1***

exactly once ***A1***

[2 marks]

(c.ii) Explain why it is not possible to have a Hamiltonian cycle in G. [3]

Markscheme

to pass through T, you must have come from R and must return to R. ***R3***

hence there is no Hamiltonian cycle

[3 marks]

(d.i) Find the number of walks of length 5 from P to Q. [4]

Markscheme

using the adjacency matrix $\boldsymbol{A} = \begin{pmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}$, ***(M1)***

we need the entry in the first row second column of the matrix $\boldsymbol{A}^5$ ***(M1)***

$$\boldsymbol{A}^5 = \begin{pmatrix} 245 & 309 & 274 & 143 & 126 \\ 309 & 363 & 322 & 168 & 156 \\ 274 & 322 & 295 & 141 & 164 \\ 143 & 168 & 141 & 77 & 72 \\ 126 & 156 & 164 & 72 & 72 \end{pmatrix}$$ ***(A1)***

hence there are 309 ways ***A1***

[4 marks]

(d.ii) Which pairs of distinct vertices have more than 15 walks of length 3 between them? [4]

Markscheme

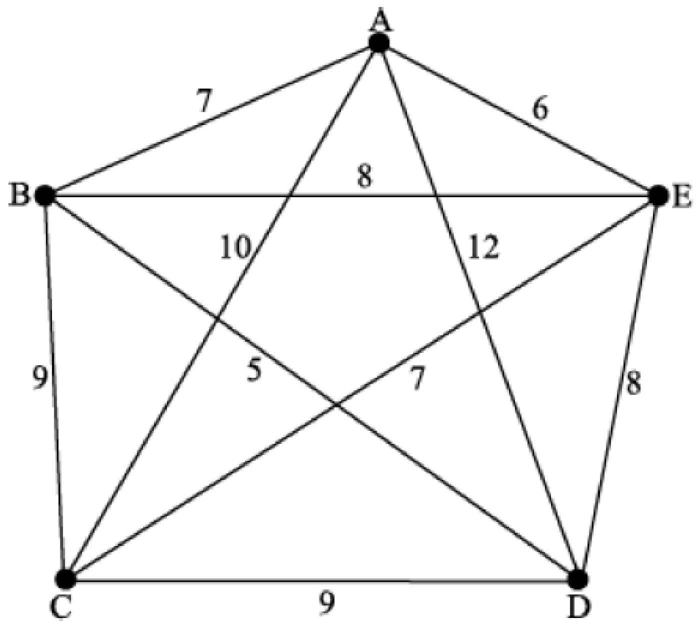
$$\boldsymbol{A}^3 = \begin{pmatrix} 13 & 21 & 17 & 10 & 6 \\ 21 & 22 & 19 & 11 & 8 \\ 17 & 19 & 18 & 7 & 14 \\ 10 & 11 & 7 & 5 & 4 \\ 6 & 8 & 14 & 4 & 4 \end{pmatrix}$$ ***(M1)***

hence the pairs of vertices are PQ, PR and QR ***A1A1A1***

[4 marks]

52. [Maximum mark: 14] EXM.2.AHL.TZ0.19

Let G be the graph below.



(a) Find the total number of Hamiltonian cycles in G, starting at vertex A. Explain your answer. [3]

Markscheme

Starting from vertex A there are 4 choices. From the next vertex there are three choices, etc… ***M1R1***

So the number of Hamiltonian cycles is 4! = 24. ***A1 N1***

[3 marks]

(b.i) Find a minimum spanning tree for the subgraph obtained by deleting A from G. [3]

Markscheme

Start (for instance) at B, using Prim´s algorithm Then D is the nearest vertex ***M1***

Next E is the nearest vertex ***A1***

Finally C is the nearest vertex So a minimum spanning tree is B → D → E → C ***A1 N1***

[3 marks]

(b.ii) Hence, find a lower bound for the travelling salesman problem for G. [3]

Markscheme

A lower bound for the travelling salesman problem is then obtained by adding the weights of AB and AE to the weight of the minimum ***M1***

spanning tree (ie 20) ***A1***

A lower bound is then $20 + 7 + 6 = 33$ ***A1 N1***

[3 marks]

(c) Give an upper bound for the travelling salesman problem for the graph above. [2]

Markscheme

ABCDE is an Hamiltonian cycle ***A1***

Thus an upper bound is given by $7 + 9 + 9 + 8 + 6 = 39$ ***A1***

[2 marks]

(d) Show that the lower bound you have obtained is not the best possible for the solution to the travelling salesman problem for G. [3]

Markscheme

Eliminating C from G a minimum spanning tree is $E \to A \to B \to D$ ***M1***

of weight 18 ***A1***

Adding BC to CE$(18 + 9 + 7)$ gives a lower bound of $34 > 33$ ***A1***

So 33 not the best lower bound. ***AG N0***

[3 marks]

53. [Maximum mark: 12] EXM.2.AHL.TZ0.2

(a) Briefly explain the two differences in the application of Prim's and Kruskal's algorithms for finding a minimum spanning tree in a weighted connected graph. [2]

Markscheme

In Prim's algorithm you start at a particular (given) vertex, whereas in Kruskal's you start with the smallest edge. ***A1***

In Prim's as smallest edges are added (never creating a circuit) the created graph always remains connected, whereas in Kruskal's this requirement to always be connected is not necessary. ***A1***

[2 marks]

The cost adjacency matrix for the complete graph K_6 is given below.

	A	B	C	D	E	F
A	-	1	3	4	5	6
B	1	-	5	6	7	8
C	3	5	-	9	10	11
D	4	6	9	-	12	13
E	5	7	10	12	-	2
F	6	8	11	13	2	-

It represents the distances in kilometres along dusty tracks connecting villages on an island. Find the minimum spanning tree for this graph; in all 3 cases state the order in which the edges are added.

(b.i) Using Kruskal's algorithm. [2]

Markscheme

Edges added in the order

AB EF AC AD AE ***A1A1***

[**note** ***A1*** for the first 2 edges ***A1*** for other 3]

[2 marks]

(b.ii) Using Prim's algorithm starting at vertex A. [2]

Markscheme

Edges added in the order

AB AC AD AE EF ***A1A1***

[**note** ***A1*** for the first 2 edges ***A1*** for other 3]

[2 marks]

(b.iii) Using Prim's algorithm starting at vertex F. [2]

Markscheme

Edges added in the order

FE AE AB AC AD ***A1A1***

[**note** ***A1*** for the first 2 edges ***A1*** for other 3]

[2 marks]

It is desired to tarmac some of these tracks so that it is possible to walk from any village to any other village walking entirely on tarmac.

(c.i) State the total minimum length of the tracks that have to be tarmacked. [2]

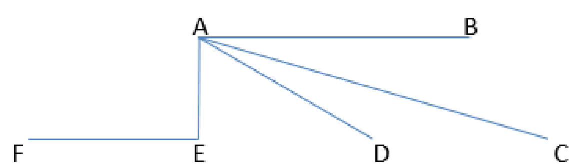
Markscheme

$1 + 2 + 3 + 4 + 5 = 15$ ***M1A1***

[2 marks]

(c.ii) Sketch the tracks that are to be tarmacked. [2]

Markscheme



A2

[2 marks]

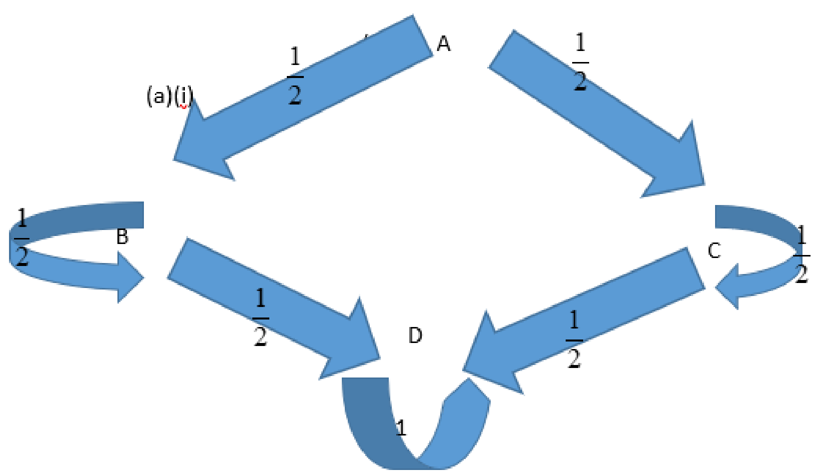
54. [Maximum mark: 26] EXM.3.AHL.TZ0.4

This question will connect Markov chains and directed graphs.

Abi is playing a game that involves a fair coin with heads on one side and tails on the other, together with two tokens, one with a fish's head on it and one with a fish's tail on it. She starts off with no tokens and wishes to win them both. On each turn she tosses the coin, if she gets a head she can claim the fish's head token, provided that she does not have it already and if she gets a tail she can claim the fish's tail token, provided she does not have it already. There are 4 states to describe the tokens in her possession; A: no tokens, B: only a fish's head token, C: only a fish's tail token, D: both tokens. So for example if she is in state B and tosses a tail she moves to state D, whereas if she tosses a head she remains in state B.

(a.i) Draw a transition state diagram for this Markov chain problem. [3]

Markscheme



M1A2

[3 marks]

(a.ii) Explain why for any transition state diagram the sum of the out degrees of the directed edges from a vertex (state) must add up to +1. [1]

Markscheme

You must leave the state along one of the edges directed out of the vertex. ***R1***

[1 mark]

(b) Write down the transition matrix **M**, for this Markov chain problem. [3]

Markscheme

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$ ***M1A2***

[3 marks]

(c.i) Find the steady state probability vector for this Markov chain problem. [4]

Markscheme

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \Rightarrow 0 = w,\ \frac{w}{2} + \frac{x}{2} = x,\ \frac{w}{2} + \frac{y}{2} = y,\ \frac{x}{2} + \frac{y}{2} + z = z$$ ***M1***

$\Rightarrow w = 0,\ x = 0, ,\ y = 0,\ z = 1$ since $w + x + y + z = 1$ so steady state vector is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. ***A1R1A1***

[4 marks]

(c.ii) Explain which part of the transition state diagram confirms this. [1]

Markscheme

There is a loop with probability of 1 from state D to itself. ***A1***

[1 mark]

(d) Explain why having a steady state probability vector means that the matrix **M** must have an eigenvalue of $\lambda = 1$. [2]

Markscheme

Let the steady state probability vector be **s** then **Ms** = 1**s** showing that (\lambda = 1\) is an eigenvalue with associated eigenvector of **s**. ***A1R1***

[2 marks]

After n throws the probability vector, for the 4 states, is given by $\mathbf{v}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix}$ where the numbers represent the probability of being in that particular state, e.g. b_n is the probability of being in state B after n throws.

Initially $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

(e) Find $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4$. [4]

Markscheme

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 0 \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{2}{4} \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} 0 \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{6}{8} \end{pmatrix},\ \mathbf{v}_4 = \begin{pmatrix} 0 \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{14}{16} \end{pmatrix}$$ ***A1A1A1A1***

[4 marks]

(f) Hence, deduce the form of $\mathbf{v}_n$. [2]

Markscheme

$$\mathbf{v}_n = \begin{pmatrix} 0 \\ \frac{1}{2^n} \\ \frac{1}{2^n} \\ \frac{2^n-2}{2^n} \end{pmatrix}$$ ***A2***

[2 marks]

(g) Explain how your answer to part (f) fits with your answer to part (c). [2]

Markscheme

$\lim\limits_{n\to\infty} \mathbf{v}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ the steady state probability vector ***M1R1***

[2 marks]

(h) Find the minimum number of tosses of the coin that Abi will have to make to be at least 95% certain of having finished the game by reaching state C. [4]

Markscheme

Require $\frac{2^n-2}{2^n} \geqslant 0.95 \Rightarrow \frac{2}{2^n} \leqslant 0.05 \Rightarrow n = 6$ (e.g. by use of table) ***R1M1A2***

[4 marks]

55. [Maximum mark: 30] EXM.3.AHL.TZ0.5

This question will diagonalize a matrix and apply this to the transformation of a curve.

Let the matrix $M = \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix}$.

(a) Find the eigenvalues for M. For each eigenvalue find the set of associated eigenvectors. [8]

Markscheme

$\begin{vmatrix} \frac{5}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} - \lambda \end{vmatrix} = 0 \Rightarrow \left(\frac{5}{2} - \lambda\right)^2 - \left(\frac{1}{2}\right)^2 = 0 \Rightarrow \frac{5}{2} - \lambda = \pm\frac{1}{2} \Rightarrow \lambda = 2 \text{ or } 3$ ***M1M1A1A1***

$\lambda = 2 \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow q = -p$ eigenvalues are of the form $t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ***M1A1***

$\lambda = 3 \quad \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow q = p$ eigenvalues are of the form $t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ***M1A1***

[8 marks]

(b) Show that the matrix equation $(x \;\; y)\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = (6)$ is equivalent to the Cartesian equation $\frac{5}{2}x^2 + xy + \frac{5}{2}y^2 = 6.$ [2]

Markscheme

$(x \;\; y)\begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (6) \Rightarrow \left(\frac{5}{2}x + \frac{1}{2}y \quad \frac{1}{2}x + \frac{5}{2}y\right)\begin{pmatrix} x \\ y \end{pmatrix} = (6)$ ***M1A1***

$\Rightarrow \left(\frac{5}{2}x^2 + \frac{1}{2}xy + \frac{1}{2}xy + \frac{5}{2}y^2\right) = (6) \Rightarrow \frac{5}{2}x^2 + xy + \frac{5}{2}y^2 = 6.$ ***AG***

[2 marks]

(c.i) Show that $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ are unit eigenvectors and that they correspond to different eigenvalues. [2]

Markscheme

$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ corresponding to $\lambda = 2$, $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ corresponding to $\lambda = 3$ ***R1R1***

[2 marks]

(c.ii) Hence, show that $M\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$. [1]

Markscheme

$$M\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix} = 2\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix} \text{ and } M\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = 3\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} \Rightarrow M\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$$

A1AG

[1 mark]

Let $\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = \mathbf{R}^{-1}$.

(d.i) Find matrix **R**. [2]

Markscheme

Determinant is 1. $\mathbf{R} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$ ***M1A1***

[2 marks]

(d.ii) Show that $\mathbf{M} = \mathbf{R}^{-1}\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\mathbf{R}$. [1]

Markscheme

$\mathbf{MR}^{-1} = \mathbf{R}^{-1}\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}$ so post multiplying by $\mathbf{R}$ gives $\mathbf{M} = \mathbf{R}^{-1}\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\mathbf{R}$ ***M1AG***

[1 mark]

Let $\mathbf{R}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}X\\ Y\end{pmatrix}$.

(e.i) Verify that $(X \quad Y) = (x \quad y)\mathbf{R}^{-1}$. [3]

Markscheme

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \Rightarrow (X \quad Y) = \left(\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \quad \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)$$

M1A1

and $(x \quad y)\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \left(\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \quad \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)$ completing the proof ***A1AG***

[3 marks]

(e.ii) Hence, find the Cartesian equation satisfied by X and Y. [2]

Markscheme

$(x \quad y)\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = (6) \Rightarrow (x \quad y)\mathbf{R}^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{R}\begin{pmatrix} x \\ y \end{pmatrix} = (6) \Rightarrow (X \quad Y)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = (6)$

$\Rightarrow (2X^2 + 3Y^2) = (6) \Rightarrow \frac{X^2}{3} + \frac{Y^2}{2} = 1$ ***M1A1***

[2 marks]

Let $\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$.

(f) Find the Cartesian equation satisfied by u and v and state the geometric shape that this curve represents. [2]

Markscheme

$\frac{X}{\sqrt{3}} = u,\ \frac{Y}{\sqrt{2}} = v \Rightarrow u^2 + v^2 = 1$, a circle (centre at the origin radius of 1) ***A1A1***

[2 marks]

(g) State geometrically what transformation the matrix $\mathbf{R}$ represents. [2]

Markscheme

A rotation about the origin through an angle of 45° anticlockwise. ***A1A1***

[2 marks]

Hence state the geometrical shape represented by (h.i) the curve in X and Y in part (e) (ii), giving a reason. [2]

Markscheme

an ellipse, since the matrix represents a vertical and a horizontal stretch ***R1A1***

[2 marks]

(h.ii) the curve in x and y in part (b). [1]

Markscheme

an ellipse ***A1***

[1 mark]

(i) Write down the equations of two lines of symmetry for the curve in x and y in part (b). [2]

Markscheme

$y = x, y = -x$ ***A1A1***

[2 marks]