Quadratics - review (TL) [116 marks]

1. [Maximum mark: 5] EXN.2.SL.TZ0.5

The quadratic equation $(k-1)x^2 + 2x + (2k-3) = 0$, where $k \in \mathbb{R}$, has real distinct roots.

Find the range of possible values for k. [5]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

attempts to find an expression for the discriminant, Δ, in terms of k **(M1)**

$\Delta = 4 - 4(k-1)(2k-3) \ \left(= -8k^2 + 20k - 8\right)$ **(A1)**

Note: Award **M1A1** for finding $x = \frac{-2 \pm \sqrt{4-4(k-1)(2k-3)}}{2(k-1)}$.

attempts to solve $\Delta > 0$ for k **(M1)**

Note: Award **M1** for attempting to solve $\Delta = 0$ for k.

$\frac{1}{2} < k < 2$ **A1A1**

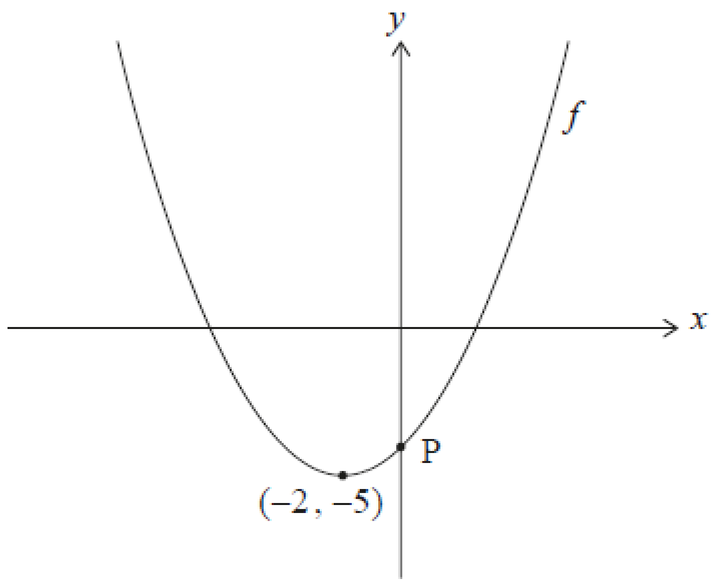
Note: Award **A1** for obtaining critical values $k = \frac{1}{2}, 2$ and **A1** for correct inequality signs.

[5 marks]

2. [Maximum mark: 16] 23M.1.SL.TZ2.7

The following diagram shows part of the graph of a quadratic function f.

The vertex of the parabola is $(-2,\ -5)$ and the y-intercept is at point P.



(a) Write down the equation of the axis of symmetry. [1]

Markscheme

$x = -2$ (must be an equation) ***A1***

[1 mark]

The function can be written in the form $f\left(x\right) = \frac{1}{4}(x - h)^2 + k$, where h, $k \in \mathbb{Z}$.

(b) Write down the values of h and k. [2]

Markscheme

$h = -2, k = -5$ ***A1A1***

[2 marks]

(c) Find the y-coordinate of P. [2]

Markscheme

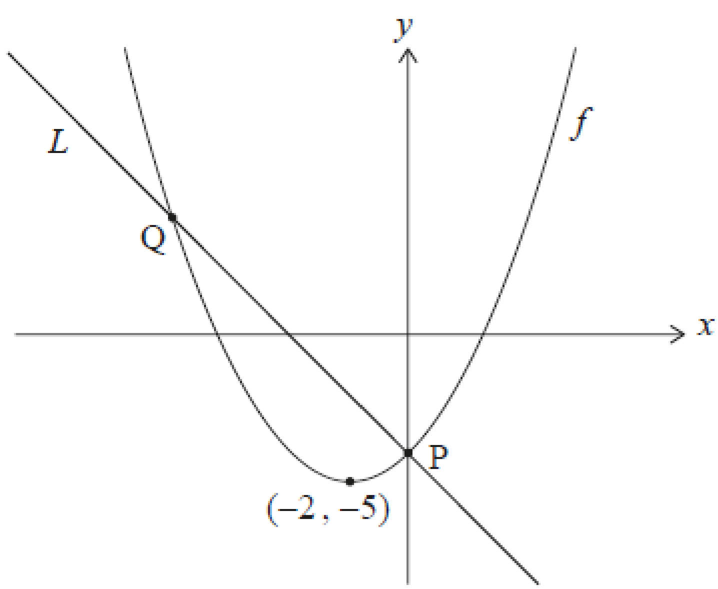
substituting $x = 0$ into $f(x)$ ***(M1)***

$y = \frac{1}{4}(0 + 2)^2 - 5$

$y = -4$ (accept $\mathrm{P}(0, \ -4)$) ***A1***

[2 marks]

In the following diagram, the line L is normal to the graph of f at point P.



(d) Find the equation of the line L, in the form $y = ax + b$. [3]

Markscheme

$f'(x) = \frac{1}{2}(x+2)$ $\left(= \frac{1}{2}x + 1\right)$ ***(A1)***

substituting $x = 0$ into their derivative ***(M1)***

$f'(0) = 1$

gradient of normal is -1 (may be seen in their equation) ***A1***

$y = -x - 4$ (accept $a = -1, b = -4$) ***A1***

Note: Award ***A0*** for $L = -x - 4$ (without the $y =$).

[4 marks]

The line L intersects the graph of f at a second point, Q, as shown above.

(e) Calculate the distance between P and Q. [8]

Markscheme

equating their $f(x)$ to their L ***(M1)***

$\frac{1}{4}(x+2)^2 - 5 = -x - 4$

$\frac{1}{4}x^2 + 2x = 0$ (or equivalent) ***(A1)***

valid attempt to solve their quadratic ***(M1)***

$\frac{1}{4}x(x+8) = 0$ OR $x(x+8) = 0$

$x = -8$ ***A1***

Note: Accept both solutions $x = -8$ and $x = 0$ here, $x = -8$ may be seen in working to find coordinates of Q or distance.

substituting their value of x (not $x = 0$) into their $f(x)$ or their L
(M1)

$y = -(-8) - 4$ OR $y = \frac{1}{4}(-8+2)^2 - 5$

$\mathrm{Q}(-8,\ 4)$ ***A1***

correct substitution into distance formula ***(A1)***

$$\sqrt{(-8-0)^2 + \left(4-(-4)\right)^2}$$

$\text{distance} = \sqrt{128}\ \left(= 8\sqrt{2}\right)$ ***A1***

[8 marks]

3. [Maximum mark: 6] 23M.1.AHL.TZ1.4

Find the range of possible values of k such that $\mathrm{e}^{2x} + \ln k = 3\mathrm{e}^{x}$ has at least one real solution. [6]

Markscheme

recognition of quadratic in e^{x} ***(M1)***

$(\mathrm{e}^{x})^2 - 3\mathrm{e}^{x} + \ln k\left(= 0\right)$ OR $A^2 - 3A + \ln k(= 0)$

recognizing discriminant ≥ 0 (seen anywhere) ***(M1)***

$(-3)^2 - 4\left(1\right)\left(\ln k\right)$ OR $9 - 4 \ln k$ ***(A1)***

$\ln k \leq \frac{9}{4}$ ***(A1)***

$\mathrm{e}^{9/4}$ (seen anywhere) ***A1***

$0 < k \leq \mathrm{e}^{9/4}$ ***A1***

[6 marks]

4. [Maximum mark: 7] 23M.1.AHL.TZ2.5

The functions f and g are defined for $x \in \mathbb{R}$ by

$f(x) = ax + b$, where $a, b \in \mathbb{Z}$

$g(x) = x^2 + x + 3$.

Find the two possible functions f such that $(g \circ f)(x) = 4x^2 - 14x + 15$.

[7]

Markscheme

attempts to form $(g \circ f)(x)$ ***(M1)***

$[f(x)]^2 + f(x) + 3$ OR $(ax + b)^2 + ax + b + 3$

$a^2x^2 + 2abx + b^2 + ax + b + 3 (= 4x^2 - 14x + 15)$ ***(A1)***

equates their corresponding terms to form at least one equation ***(M1)***

$a^2x^2 = 4x^2$ OR $a^2 = 4$ OR $2abx + ax = -14x$ OR $2ab + a = -14$ OR $b^2 + b + 3 = 15$

$a = \pm 2$ (seen anywhere) ***A1***

attempt to use $2ab + a = -14$ to pair the correct values (seen anywhere) ***(M1)***

$f(x) = 2x - 4$ (accept $a = 2$ with $b = -4$), $f(x) = -2x + 3$ (accept $a = -2$ with $b = 3$) ***A1A1***

[7 marks]

5. [Maximum mark: 16] 22N.1.SL.TZ0.7

(a) The graph of a quadratic function f has its vertex at the point $(3,\ 2)$ and it intersects the x-axis at $x = 5$. Find f in the form $f\left(x\right) = a(x - h)^2 + k$. [3]

Markscheme

correct substitution of $h = 3$ and $k = 2$ into $f(x)$ ***(A1)***

$f(x) = a(x - 3)^2 + 2$

correct substitution of $(5,\ 0)$ ***(A1)***

$0 = a(5 - 3)^2 + 2\ \ \left(a = -\frac{1}{2}\right)$

Note: The first two A marks are independent.

$f(x) = -\frac{1}{2}(x - 3)^2 + 2$ ***A1***

[3 marks]

The quadratic function g is defined by $g(x) = px^2 + (t - 1)x - p$ where $x \in \mathbb{R}$ and $p,\ t \in \mathbb{R},\ p \neq 0$.

In the case where $g(-3) = g(1) = 4$,

(b.i) find the value of p and the value of t. [4]

Markscheme

METHOD 1

correct substitution of $(1,\ 4)$ *(A1)*

$p+(t-1)-p=4$

$t=5$ ***A1***

substituting their value of t into $9p-3(t-1)-p=4$ *(M1)*

$8p-12=4$

$p=2$ ***A1***

METHOD 2

correct substitution of ONE of the coordinates $(-3,\ 4)$ or $(1,\ 4)$ *(A1)*

$9p-3(t-1)-p=4$ OR $p+(t-1)-p=4$

valid attempt to solve their two equations *(M1)*

$p=2,\ t=5$ ***A1A1***

$\left(g(x)=2x^2+4x-2\right)$

[4 marks]

(b.ii) find the range of g. [3]

Markscheme

attempt to find the x-coordinate of the vertex *(M1)*

$x=\frac{-3+1}{2}(=-1)$ OR $\frac{-4}{2\times2}$ OR $4x+4=0$ OR $2(x+1)^2-4$

y-coordinate of the vertex $=-4$ *(A1)* correct range *A1*

$[-4,\ +\infty[$ OR $y \geq -4$ OR $g \geq -4$ OR $[-4,\ \infty)$

[3 marks]

(c) The linear function j is defined by $j(x) = -x + 3p$ where $x \in \mathbb{R}$ and $p \in \mathbb{R},\ p \neq 0.$

Show that the graphs of $j(x) = -x + 3p$ and $g(x) = px^2 + (t-1)x - p$ have two distinct points of intersection for every possible value of p and t. [6]

Markscheme

equating the two functions or equations *(M1)*

$g(x) = j(x)$ OR $px^2 + (t-1)x - p = -x + 3p$

$px^2 + tx - 4p = 0$ *(A1)*

attempt to find discriminant (do not accept only in quadratic formula) *(M1)*

$\Delta = t^2 + 16p^2$ *A1*

$\Delta = t^2 + 16p^2 > 0$, because $t^2 \geq 0$ and $p^2 > 0$, therefore the sum will be positive *R1R1*

Note: Award ***R1*** for recognising that Δ is positive and ***R1*** for the reason.

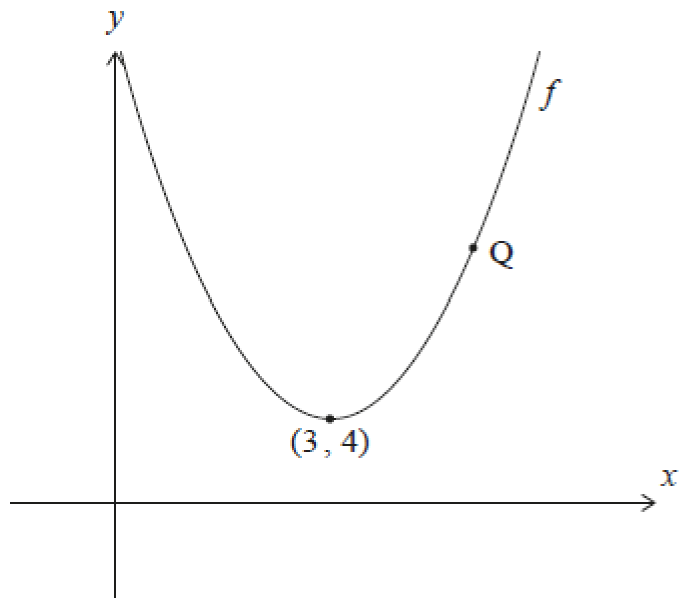
There are two distinct points of intersection between the graphs of g and j. *AG*

[6 marks]

6. [Maximum mark: 15] 22M.1.SL.TZ2.7

The following diagram shows part of the graph of a quadratic function f.

The graph of f has its vertex at $(3,\ 4)$, and it passes through point Q as shown.



(a) Write down the equation of the axis of symmetry. [1]

Markscheme

$x = 3$ ***A1***

Note: Must be an equation in the form "$x =$". Do not accept 3 or $\frac{-b}{2a} = 3$.

[1 mark]

The function can be written in the form $f\left(x\right) = a(x - h)^2 + k$.

(b.i) Write down the values of h and k. [2]

Markscheme

$h=3,\ k=4$ (accept $a(x-3)^2+4$) ***A1A1***

[2 marks]

(b.ii) Point Q has coordinates $(5,\ 12)$. Find the value of a. [2]

Markscheme

attempt to substitute coordinates of Q ***(M1)***

$12=a(5-3)^2+4,\ \ 4a+4=12$

$a=2$ ***A1***

[2 marks]

The line L is tangent to the graph of f at Q.

(c) Find the equation of L. [4]

Markscheme

recognize need to find derivative of f ***(M1)***

$f\prime(x)=4(x-3)$ or $f\prime(x)=4x-12$ ***A1***

$f\prime(5)=8$ (may be seen as gradient in their equation) ***(A1)***

$y-12=8(x-5)$ or $y=8x-28$ ***A1***

Note: Award ***A0*** for $L = 8x - 28$.

[4 marks]

Now consider another function $y = g(x)$. The derivative of g is given by $g\prime(x) = f(x) - d$, where $d \in \mathbb{R}$.

(d) Find the values of d for which g is an increasing function. [3]

Markscheme

METHOD 1

Recognizing that for g to be increasing, $f(x) - d > 0$, or $g\prime > 0$ ***(M1)***

The vertex must be above the x-axis, $4 - d > 0,\ d - 4 < 0$ ***(R1)***

$d < 4$ ***A1***

METHOD 2

attempting to find discriminant of $g\prime$ ***(M1)***

$(-12)^2 - 4(2)(22 - d)$

recognizing discriminant must be negative ***(R1)***

$-32 + 8d < 0$ OR $\Delta < 0$

$d < 4$ ***A1***

[3 marks]

(e) Find the values of x for which the graph of g is concave-up. [3]

Markscheme

recognizing that for g to be concave up, $g'' > 0$ ***(M1)***

$g'' > 0$ when $f' > 0,\ 4x - 12 > 0,\ x - 3 > 0$ ***(R1)***

$x > 3$ ***A1***

[3 marks]

7. [Maximum mark: 4] 22M.2.AHL.TZ1.8

Consider the equation $kx^2 - (k+3)x + 2k + 9 = 0$, where $k \in \mathbb{R}$.

(a) Write down an expression for the product of the roots, in terms of k. [1]

Markscheme

product of roots $= \frac{2k+9}{k}$ ***A1***

[1 mark]

(b) Hence or otherwise, determine the values of k such that the equation has one positive and one negative real root. [3]

Markscheme

recognition that the product of the roots will be negative ***(M1)***

$\frac{2k+9}{k} < 0$

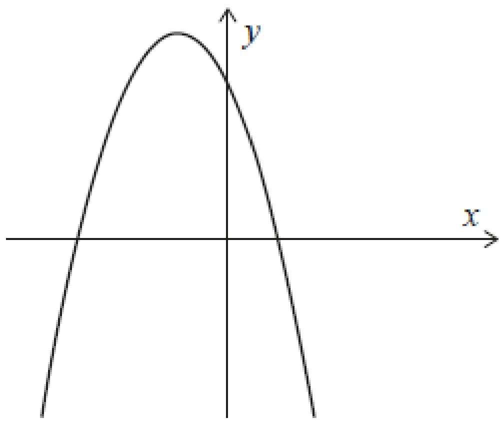
critical values $k = 0, -\frac{9}{2}$ seen ***(A1)***

$-\frac{9}{2} < k < 0$ ***A1***

[3 marks]

8. [Maximum mark: 7] 21N.1.SL.TZ0.1

Consider the function $f(x) = -2(x-1)(x+3)$, for $x \in \mathbb{R}$. The following diagram shows part of the graph of f.



For the graph of f

(a.i) find the x-coordinates of the x-intercepts. [2]

Markscheme

setting $f(x) = 0$ ***(M1)***

$x = 1,\ x = -3$ (accept $(1, 0), (-3, 0)$) ***A1***

[2 marks]

(a.ii) find the coordinates of the vertex. [3]

Markscheme

METHOD 1

$x = -1$ ***A1***

substituting their x-coordinate into f ***(M1)***

$y = 8$ ***A1***

$(-1, 8)$

METHOD 2

attempt to complete the square ***(M1)***

$-2\left((x+1)^2 - 4\right)$ ***(M1)***

$x = -1,\ y = 8$ ***A1A1***

$(-1, 8)$

[3 marks]

(b) The function f can be written in the form $f(x) = -2(x-h)^2 + k$.

Write down the value of h and the value of k. [2]

Markscheme

$h = -1$ ***A1***

$k = 8$ ***A1***

[2 marks]

9. [Maximum mark: 7] 21N.1.AHL.TZ0.7

The equation $3px^2 + 2px + 1 = p$ has two real, distinct roots.

(a) Find the possible values for p. [5]

Markscheme

attempt to use discriminant $b^2 - 4ac(> 0)$ ***M1***

$(2p)^2 - 4(3p)(1 - p)(> 0)$

$16p^2 - 12p(> 0)$ ***(A1)***

$p(4p - 3)(> 0)$

attempt to find critical values $\left(p = 0,\ p = \frac{3}{4}\right)$ ***M1***

recognition that discriminant > 0 ***(M1)***

$p < 0$ or $p > \frac{3}{4}$ ***A1***

Note: Condone 'or' replaced with 'and', a comma, or no separator

[5 marks]

(b) Consider the case when $p = 4$. The roots of the equation can be expressed in the form $x = \frac{a \pm \sqrt{13}}{6}$, where $a \in \mathbb{Z}$. Find the value of a. [2]

Markscheme

$p = 4 \Rightarrow 12x^2 + 8x - 3 = 0$

valid attempt to use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (or equivalent) ***M1***

$x = \frac{-8 \pm \sqrt{208}}{24}$

$x = \frac{-2 \pm \sqrt{13}}{6}$

$a = -2$ ***A1***

[2 marks]

10. [Maximum mark: 14] 21M.1.SL.TZ1.7

Let $f(x) = mx^2 - 2mx$, where $x \in \mathbb{R}$ and $m \in \mathbb{R}$. The line $y = mx - 9$ meets the graph of f at exactly one point.

(a) Show that $m = 4$. [6]

Markscheme

METHOD 1 (discriminant)

$mx^2 - 2mx = mx - 9$ ***(M1)***

$mx^2 - 3mx + 9 = 0$

recognizing $\Delta = 0$ (seen anywhere) ***M1***

$\Delta = (-3m)^2 - 4(m)(9)$ (do not accept only in quadratic formula for x) ***A1***

valid approach to solve quadratic for m ***(M1)***

$9m(m-4) = 0$ OR $m = \frac{36 \pm \sqrt{36^2 - 4 \times 9 \times 0}}{2 \times 9}$

both solutions $m = 0, 4$ ***A1***

$m \neq 0$ with a valid reason ***R1***

the two graphs would not intersect OR $0 \neq -9$

$m = 4$ ***AG***

METHOD 2 (equating slopes)

$mx^2 - 2mx = mx - 9$ (seen anywhere) ***(M1)***

$f\prime(x) = 2mx - 2m$ ***A1***

equating slopes, $f\prime(x) = m$ (seen anywhere) ***M1***

$2mx - 2m = m$

$x = \frac{3}{2}$ ***A1***

substituting their x value ***(M1)***

$\left(\frac{3}{2}\right)^2 m - 2m \times \frac{3}{2} = m \times \frac{3}{2} - 9$

$\frac{9}{4}m - \frac{12}{4}m = \frac{6}{4}m - 9$ ***A1***

$\frac{-9m}{4} = -9$

$m = 4$ ***AG***

METHOD 3 (using $\frac{-b}{2a}$)

$mx^2 - 2mx = mx - 9$ ***(M1)***

$mx^2 - 3mx + 9 = 0$

attempt to find x-coord of vertex using $\frac{-b}{2a}$ ***(M1)***

$\frac{-(-3m)}{2m}$ ***A1***

$x = \frac{3}{2}$ ***A1***

substituting their x value ***(M1)***

$\left(\frac{3}{2}\right)^2 m - 3m \times \frac{3}{2} + 9 = 0$

$\frac{9}{4}m - \frac{9}{2}m + 9 = 0$ ***A1***

$-9m = -36$

$m = 4$ ***AG***

[6 marks]

The function f can be expressed in the form $f(x) = 4(x - p)(x - q)$, where $p,\ q \in \mathbb{R}$.

(b) Find the value of p and the value of q. [2]

Markscheme

$4x(x - 2)$ ***(A1)***

$p = 0$ and $q = 2$ OR $p = 2$ and $q = 0$ ***A1***

[2 marks]

The function f can also be expressed in the form $f(x) = 4(x - h)^2 + k$, where $h,\ k \in \mathbb{R}$.

(c) Find the value of h and the value of k. [3]

Markscheme

attempt to use valid approach ***(M1)***

$\frac{0+2}{2},\ \frac{-(-8)}{2\times 4},\ f(1),\ 8x - 8 = 0$ OR
$4\left(x^2 - 2x + 1 - 1\right)\left(= 4(x - 1)^2 - 4\right)$

$h = 1,\ k = -4$ ***A1A1***

[3 marks]

(d) Hence find the values of x where the graph of f is both negative and increasing. [3]

Markscheme

EITHER

recognition $x = h$ to 2 (may be seen on sketch) ***(M1)***

OR

recognition that $f(x) < 0$ and $f\prime(x) > 0$ ***(M1)***

THEN

$1 < x < 2$ ***A1A1***

Note: Award ***A1*** for two correct values, ***A1*** for correct inequality signs.

[3 marks]

11. [Maximum mark: 6] 21M.2.SL.TZ2.5

The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = 6x^2 - 12x + 1$ and $g(x) = -x + c$, where $c \in \mathbb{R}$.

(a) Find the range of f. [2]

Markscheme

attempting to find the vertex ***(M1)***

$x = 1$ OR $y = -5$ OR $f(x) = 6(x-1)^2 - 5$

range is $y \geq -5$ ***A1***

[2 marks]

(b) Given that $(g \circ f)(x) \leq 0$ for all $x \in \mathbb{R}$, determine the set of possible values for c. [4]

Markscheme

METHOD 1

$(g \circ f)(x) = -(6x^2 - 12x + 1) + c \ \left(= -\left(6(x-1)^2 - 5\right) + c\right)$

(A1)

EITHER

relating to the range of f OR attempting to find $g(-5)$ ***(M1)***

$5 + c \leq 0$ ***(A1)***

OR

attempting to find the discriminant of $(g \circ f)(x)$ *(M1)*

$144 + 24(c - 1) \leq 0 \;\; (120 + 24c \leq 0)$ *(A1)*

THEN

$c \leq -5$ *A1*

METHOD 2

vertical reflection followed by vertical shift *(M1)*

new vertex is $(1,\, 5 + c)$ *(A1)*

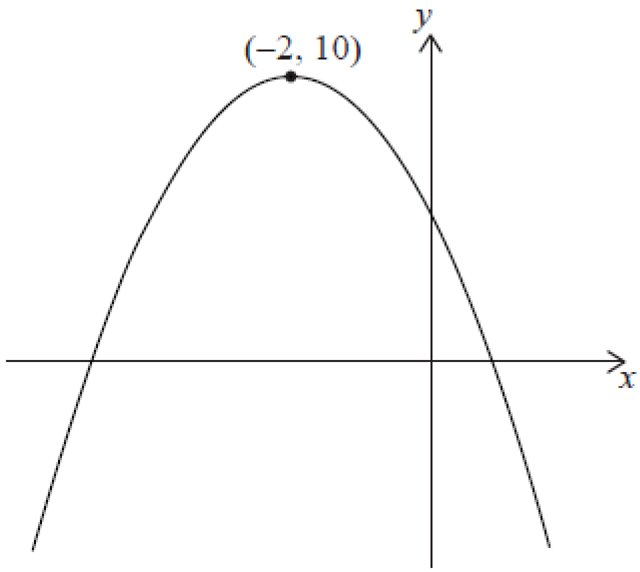
$5 + c \leq 0$ *(A1)*

$c \leq -5$ *A1*

[4 marks]

12. [Maximum mark: 6] 20N.1.SL.TZ0.T_11

The diagram shows the graph of the quadratic function $f(x) = ax^2 + bx + c$, with vertex $(-2,\ 10)$.



The equation $f(x) = k$ has two solutions. One of these solutions is $x = 2$.

(a) Write down the other solution of $f(x) = k$. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure. It appeared in a paper that permitted the use of a calculator, and so might not be suitable for all forms of practice.

$(x =)\ (-2) - 4$ **OR** $(x =)\ (-2) - (2 - (-2))$ ***(M1)***

Note: Award ***(M1)*** for correct calculation of the left symmetrical point.

$(x =)\ -6$ ***(A1) (C2)***

[2 marks]

(b) Complete the table below placing a tick (✔) to show whether the unknown parameters a and b are positive, zero or negative.
The row for c has been completed as an example.

	positive	zero	negative
a			
b			
c	✓		

[2]

Markscheme

	positive	zero	negative
a			✓
b			✓

(A1)(A1)

(C2)

Note: Award *(A1)* for each correct row.

[2 marks]

(c) State the values of x for which $f(x)$ is decreasing. [2]

Markscheme

$x > -2$ **OR** $x \geq -2$ *(A1)(A1) (C2)*

Note: Award *(A1)* for -2 seen as part of an inequality, *(A1)* for completely correct notation. Award ***(A1)(A1)*** for correct equivalent statement in words, for example “decreasing when x is greater than negative 2”.

[2 marks]

13. [Maximum mark: 7] 19N.1.SL.TZ0.S_3

Let $g(x) = x^2 + bx + 11$. The point $(-1, 8)$ lies on the graph of g.

(a) Find the value of b. [3]

Markscheme

valid attempt to substitute coordinates ***(M1)***

eg $g(-1) = 8$

correct substitution ***(A1)***

eg $(-1)^2 + b(-1) + 11 = 8$, $1 - b + 11 = 8$

$b = 4$ ***A1 N2***

[3 marks]

(b) The graph of $f(x) = x^2$ is transformed to obtain the graph of g.

Describe this transformation. [4]

Markscheme

valid attempt to solve ***(M1)***

eg $(x^2 + 4x + 4) + 7$, $h = \frac{-4}{2}$, $k = g(-2)$

correct working ***A1***

eg $(x + 2)^2 + 7$, $h = -2$, $k = 7$

translation or shift (do not accept move) of vector $\begin{pmatrix} -2 \\ 7 \end{pmatrix}$ (accept left by 2 and up by 7) ***A1A1 N2***

[4 marks]