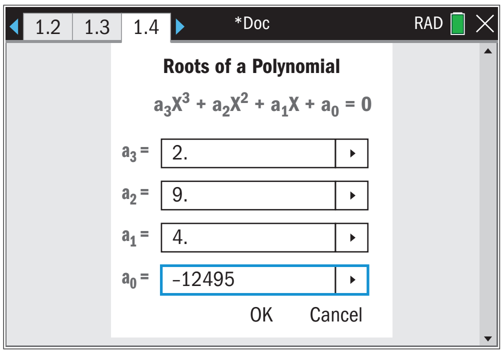
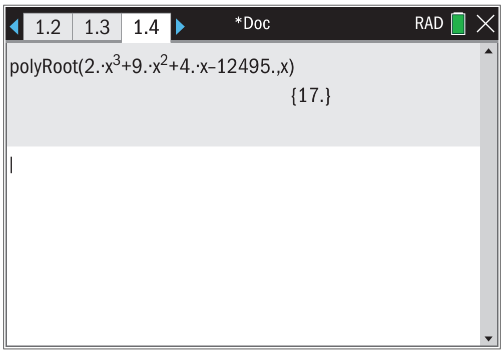
TOPIC 1 SL WORKED SOLUTIONS

Topic 1 SL Paper 1, Group 1

1. **a.** $V = n(n+4)(2n+1) = n\left(2n^2+9n+4\right) = 2n^3+9n^2+4n$

b. $2n^3+9n^2+4n = 12\,495$ has solution $n = 17$ cm as shown below:





2. Let r_M be the measured radius of the hemisphere, and r be the true radius

a. $r_M = 6.3 \text{ cm} \Rightarrow 6.25 \text{ cm} \le r < 6.35 \text{ cm}$

Let V_M be the volume resulting from the measured radius, and V be the true radius.

b. $V_M = \frac{2}{3} \times \pi \times 6.3^3$ cm and

$\frac{2}{3} \times \pi \times 6.25^3 \text{cm} \le V < \frac{2}{3} \times \pi \times 6.35^3$ cm

Hence the maximum percentage error is $\left| \dfrac{\frac{2}{3}\times\pi\times 6.3^3 - \frac{2}{3}\times\pi\times 6.25^3}{\frac{2}{3}\times\pi\times 6.25^3} \right| \times 100\% = 2.42\%$

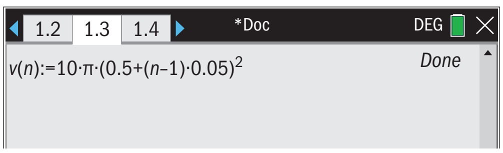
3. **a.** The fee in February 2026 will be $34 \times 1.02^2 = £35.37$ per month

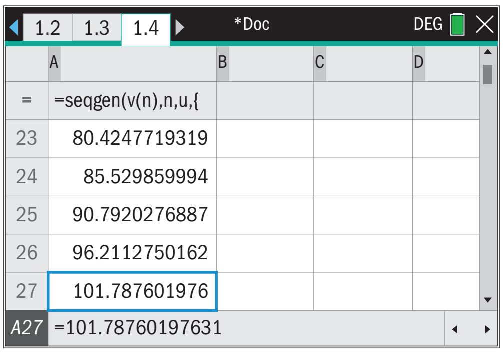
b. David will pay:

$34 \times 24 + 34 \times 1.02 \times 24 + 34 \times 1.02^2 \times 24 + 34 \times 1.02^3 \times 12 = £2930.26$

4. **a.** The columns have volumes in a sequence $V_n = 10\pi\left(0.5+(n-1)\times 0.05\right)^2$

Using a spreadsheet, the first column to have a volume greater than 100 m^3 is column number 27.





b. Using sigma notation, the GDC can find the answer efficiently

The total volume of concrete needed is 18 746 m^3 to the nearest m^3.

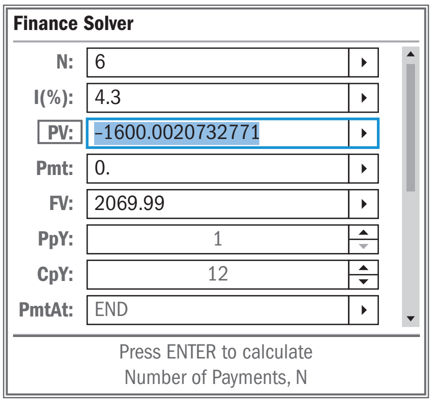
5. a. $254 \times 65 \times 154 = 2\,542\,540 \text{ mm}^3 \approx 2\,500\,000 \text{ mm}^3$ (2 sf)

b. $2.5 \times 10^6 \text{ mm}^3$

c. The least the container holds is $253.5 \times 64.5 \times 153.5 = 2\,509\,840.125 \text{ mm}^3$

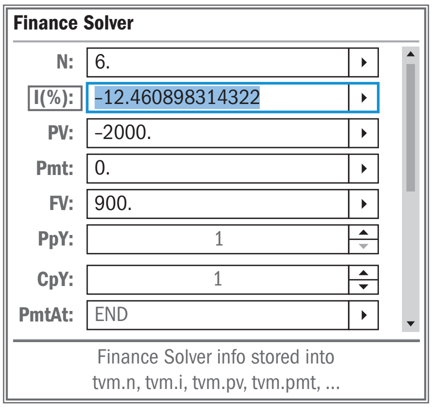
This is more than 2.5 litres since 2.5 litres = 2500 cm^3 = 2 500 000 mm^3 so the container can contain 2.5 litres of washing liquid.

6. a.



He invests € 1600.00

b.



The annual rate of depreciation is 12.5%

7. a. 252.65 million km = 252 650 000 km = 252 650 000 000 m = 2.5265×10^{11} m.

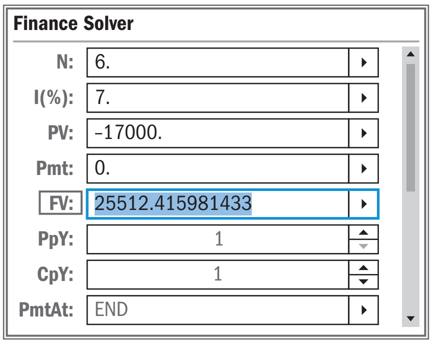
b. $60\,000 \text{ km h}^{-1} = \dfrac{60\,000 \times 1000}{60 \times 60} \text{ m s}^{-1} = 16\,666.67 \text{ m s}^{-1}$

In standard form, this is $1.67 \times 10^4 \text{ m s}^{-1}$

c. $\text{Time} = \dfrac{\text{distance}}{\text{speed}} = \dfrac{2.5265 \times 10^{11}}{1.666667 \times 10^4} = \dfrac{2.5265}{1.666667} \times 10^7 = 1.52 \times 10^7$ seconds.

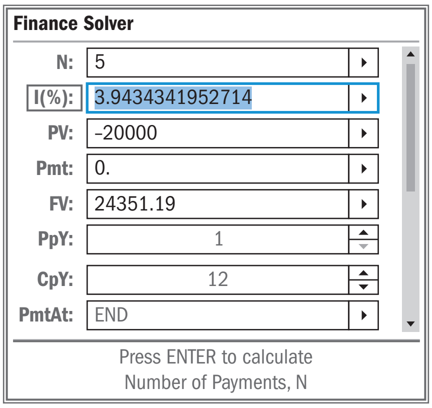
Topic 1 SL Paper 1, Group 2

8. a.



Pilar earns 25 512.42 – 17 000 = \$8512.42 USD interest.

b.



The interest rate of Ximena's account is 3.94%

9. a. $\log_{10}\left(\dfrac{52\,098}{0.001}\right) = 7.72$

b. $\log_{10}\left(\dfrac{x}{0.001}\right) = 8.9 \Rightarrow \dfrac{x}{0.001} = 10^{8.9} \Rightarrow x = 0.001 \times 10^{8.9} = 794\,328.234\,724\,28,$

which is 794 328 mm to the nearest mm.

10. a. $149\,600\,000 \times 1000 \times 100 = 14\,960\,000\,000\,000\,\text{cm}$

b. $14\,960\,000\,000\,000\,\text{cm} = 1.496 \times 10^{13}\ \text{cm}$

c. $2^x = 14\,960\,000\,000\,000 \Rightarrow x = \log_2 14\,960\,000\,000\,000 = 43.8$

Hence the value of p is 44.

11. a. This is an arithmetic sequence with first term $u_1 = 4$ and common difference $d = 0.5$.

Hence $u_{12} = 4 + 11 \times 0.5 = 9.5$ km in his 12th week of training.

b. $S_{20} = \sum_{r=1}^{20} (4 + (r-1) \times 0.5) = 175$ km

c. $u_n = 4 + (n-1) \times 0.5 \geq 42 \Rightarrow (n-1) \times 0.5 \geq 38 \Rightarrow n \geq 77$

Therefore at this rate Magzhan will first complete a full marathon during his 77th week of training.

12. **a.** $1624 = 1530 \times r^3 \Rightarrow r = \sqrt[3]{\dfrac{1624}{1530}} \approx 1.02$

The number of hires in 2025 is $1530 \times r^{10} = 1866$ hires.

b. $2000 = 1530 \times 1.02^{(n-1)} \Rightarrow n = 14.5$ so n must be 15 for the number of hires to exceed 2000.

Since $u_{15} = 1530 \times 1.02^{14}$, we require the 15th year in the sequence (where 2015 is the first year), which is 2029.

13. **a.** $17\,500 \times 0.95^7 = €12\,220.90$

b. $17\,500 \times 0.95^n = 0.4 \times 17\,500$ has solution 17.863752812425 hence 18 complete years must pass.

Topic SL Paper 1, Group 3

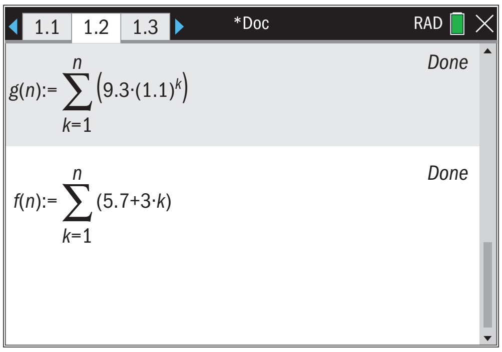
14. **a.** $g(n)$ represents an arithmetic series because $5.7 + 3k$ represents a series with a common difference.

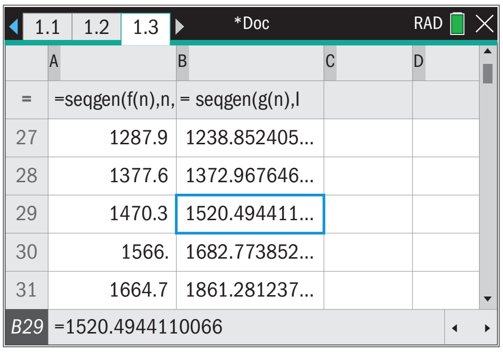
b. The first term is 8.7 and the common difference is 3.

c. $h(n)$ represents a geometric series because 9.3×1.1^k represents a series with a common ratio.

d. The first term is 10.23 and the common ratio is 1.1.

e.





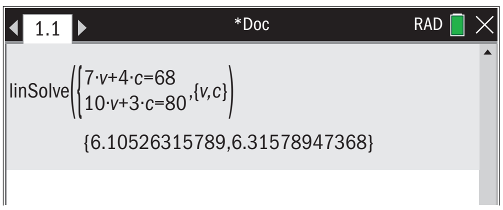
The solution $q = 29$ can be found from GDC as shown.

15. **a.** The context can be modelled with the system of equations

$7v + 4c = 68$

$10v + 3c = 80$

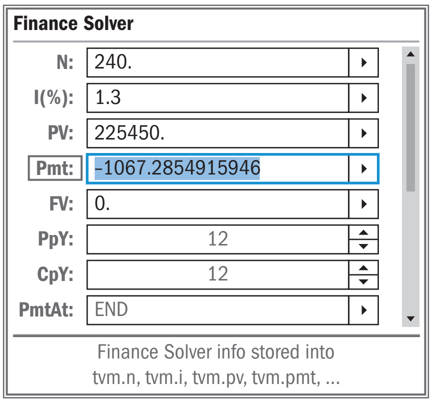
b. Use of GDC gives the solutions $v = 6.11$ and $c = 6.32$.

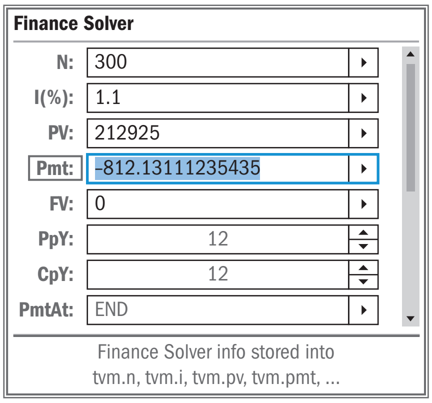


c. Hence $v = 6$ and $c = 6$ will not satisfy the requirements of the trip.
$v = 7$ and $c = 6$ would mean 73 people and 88 cases can be transported, whereas $v = 6$ and $c = 7$ would mean 70 people and 81 cases can be transported.
This is the most economical solution to the problem.

16. a. Bank A requires a loan of $0.9 \times £250\,500 = £225\,450$ over 240 payments.

Bank B requires a loan of $0.85 \times £250\,500 = £212\,925$ over 300 payments.





The monthly payment for the Bank A deal is £1067.29 and for Bank B £812.13.

b. The total repayment for Bank A is £256 149.60 and for Bank B £243 639.00

c. For each bank, total expense = deposit + loan repayment
For Bank A, expense = 25 050 + 256 149.60 = 281 200.60
For Bank B, expense = 37 575 + 243 639 = 281 214.00

Advantages of choosing A: smaller deposit; the loan is paid off 5 years sooner than in B; total expense is slightly less.
Disadvantages of choosing A: The monthly payments are higher than in B.

17. a. $c = \sqrt{100 + 7 - 2(10)\sqrt{7}\ \cos(60^\circ)} = \sqrt{107 - 10\sqrt{7}}$

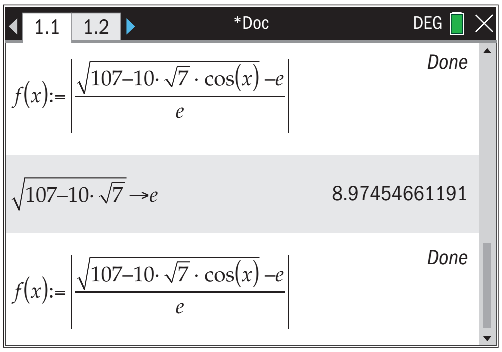
b. $\sqrt{7} \approx 2.65$ to three significant figures.

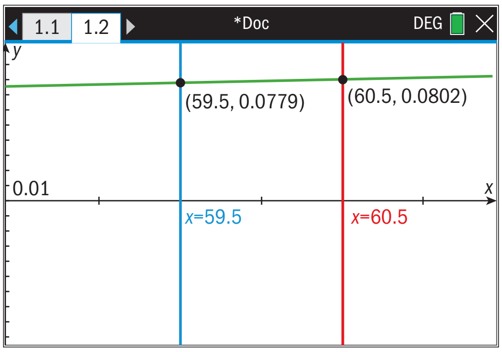
$$\left| \frac{\sqrt{100 + 2.65^2 - 2(10)(2.65)\cos(60^\circ)} - \sqrt{107 - 10\sqrt{7}}}{\sqrt{107 - 10\sqrt{7}}} \right| \times 100\% = 0.0124\%$$

c. $C = 60^\circ$ to the nearest degree so $59.5^\circ \le C_E < 60.5^\circ$

The percentage error on this domain is a function of C_E:

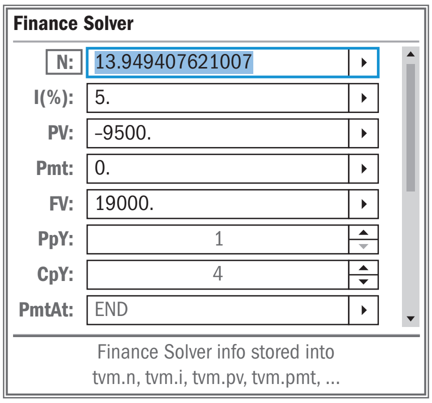
$$f(C_E) = \left| \frac{\sqrt{107 - 10\sqrt{7}\ \cos(C_E)} - \sqrt{107 - 10\sqrt{7}}}{\sqrt{107 - 10\sqrt{7}}} \right| \times 100\%$$





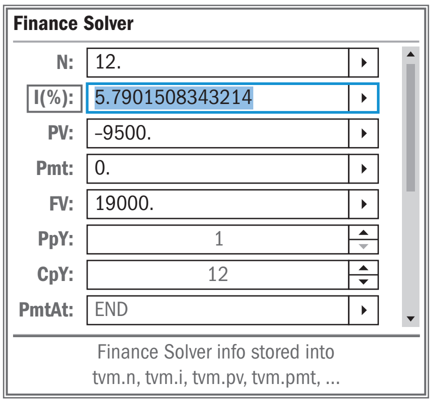
The maximum percentage error is therefore 0.0802 × 100% = 8.02%

18. a.



Chimdi must wait 14 years for her investment to double in value.

b.



Nicole's investment will double in value after 12 years with a nominal annual interest rate of 5.79%

Topic 1 SL Paper 2

1. a. $27 = y + 5x$.
$41 = y + 12x$

This is assuming that Denise increases the amount of subscriptions she sells each day by x.

b. $x = 2,\ y = 17$

c. If Denise sells 17 subscriptions on her first day and then increases the number of subscriptions she sells by just 1 each day for the 32 days, she will have sold $S_{32} = \frac{32}{2}(2\times 17 + (32-1)\times 1) = 16(34+31) = 1041$. This is the least she has sold, so she does sell more than 1000 subscriptions.

2. a. i. Option A: $21000(1.04) = €21\,840$

ii. Option B: $21000\left(1+\frac{3.8}{1200}\right)^{12} = €21\,812$

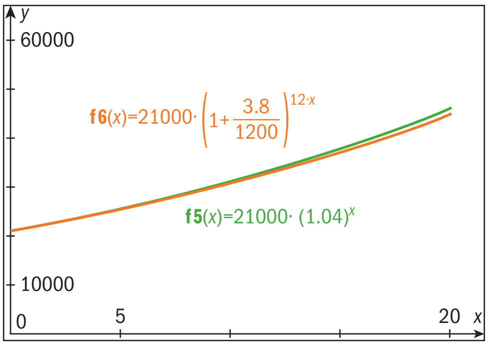
iii. Option C: $21000 + 1100 = €22\,100$

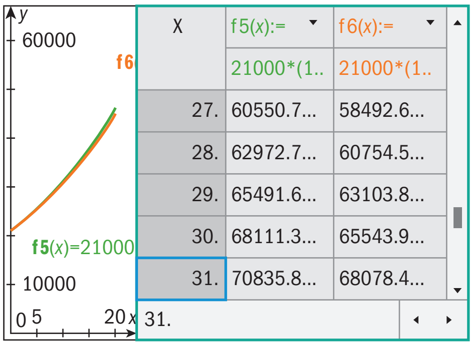
b. i. Option A: $V = 21000(1.04)^n$

ii. Option B: $V = 21000\left(1+\frac{3.8}{1200}\right)^{12n}$

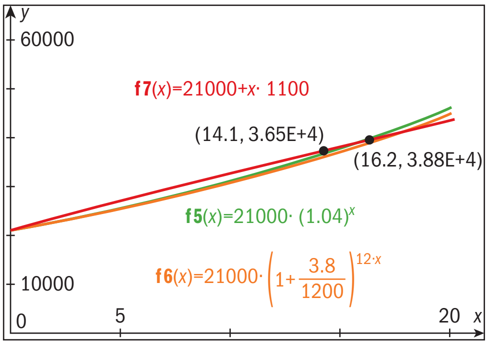
iii. Option C: $V = 21000 + 1100n$

c. Using the GDC to investigate, it is found that the value of Option A is always more than the value of Option B. So Beth's claim is false since each Option starts with €21 000.





d.



The GDC shows that after 17 years, Alex's investment would be worth more in both Option A and B than in C.

e. After 20 years in Option A, Alex's investment is worth

$$21000(1.04)^{20} = €46\,014$$

After 20 years in Option C, Alex's investment is worth

$$21000 + (20)(1100) = €43\,000$$

The difference in the interest earned is therefore €3014.

TOPIC 1 HL WORKED SOLUTIONS

Topic 1 HL Paper 1, Group 1

1. a. $\begin{pmatrix} 2 & 4 \\ 1.5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} 2-\lambda & 4 \\ 1.5 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \left|\begin{pmatrix} 2-\lambda & 4 \\ 1.5 & 1-\lambda \end{pmatrix}\right| = 0$

$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$ which when solved gives the eigenvalues 4 and –1.

If $\lambda = 4$ then $\begin{pmatrix} -2 & 4 \\ 1.5 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -2x+4y=0 \\ 1.5x-3y=0 \end{matrix} \Rightarrow x = 2y$

If $\lambda = -1$ then $\begin{pmatrix} 3 & 4 \\ 1.5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 3x+4y=0 \\ 1.5x+2y=0 \end{matrix} \Rightarrow -3x = 4y$

with associated eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ respectively.

b. $\mathbf{C} = \begin{pmatrix} 2 & 4 \\ 1.5 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}^{-1}$

c. $\mathbf{C}^n = \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 0.3 & 0.4 \\ -0.1 & 0.2 \end{pmatrix}$

2. a. $R = \log_{10}\left(\frac{I}{I_0}\right) \Rightarrow \frac{I}{I_0} = 10^R \Rightarrow I = I_0 10^R$.

Hence the ratio of the intensity of the San Francisco earthquake to that of the Dashur earthquake is $\frac{I_0 10^{8.25}}{I_0 10^{5.9}} = 10^{2.35} \approx 224$.

Hence the San Francisco earthquake was 224 times more intense than that of Dashur.

b. Let J represent the measure of the Japanese earthquake on the Richter scale and $J = \log_{10}\left(\frac{I}{I_0}\right) \Rightarrow I = I_0 10^J = 7.08(I_0 10^{8.25}) = 10^{\log_{10} 7.08} I_0 10^{8.25}$.

$I_0 10^J = 10^{\log_{10} 7.08} I_0 10^{8.25} \Rightarrow 10^J = 10^{8.25 + \log_{10} 7.08}$

Therefore $J = 8.25 + \log_{10} 7.08 \approx 9.10$.

3. a. $0.151\,515\,151\,515 =$

$0.15 + 0.0015 + 0.000\,015 + 0.000\,000\,15 + 0.000\,000\,000\,015 + \ldots$

which is a geometric series first term 0.15, common ratio 0.01 and 6 terms.

Hence

$$\sum_{r=1}^{6} 0.15 \times (0.01)^{n-1} = \sum_{r=1}^{6} 0.15 \times \left(10^{-2}\right)^{n-1} = \sum_{r=1}^{6} 0.15 \times 10^{2-2n}$$

b. $0.\overline{51} = 0.515151515151 = 0.51 + 0.0051 + 0.000051 + \ldots$

$= 0.51 + 0.51 \times 0.001 + 0.51 + 0.00001 + \ldots$

$0.\overline{51} = \sum_{r=1}^{\infty} 0.51 \times (0.01)^{r-1}$

c. $0.\overline{51} = \dfrac{0.51}{1-0.01} = \dfrac{51}{99} = \dfrac{17}{33}$

4. $\dfrac{z}{z+2\text{i}} = 4-7\text{i} \Rightarrow z = (z+2\text{i})(4-7\text{i}) \Rightarrow z = 4z - 7\text{i}z + 8\text{i} + 14$

$\Rightarrow z(-3+7\text{i}) = (14+8\text{i})$

Hence $z = \dfrac{14+8\text{i}}{-3+7\text{i}} = \dfrac{7}{29} - \dfrac{61}{29}\text{i}$

5. a. The roots of $3x^2 + x - 7 = 0$ are $x_1, x_2 = \dfrac{-1 \pm \sqrt{1^2 - 4(3)(-7)}}{2(3)} = \dfrac{-1 \pm \sqrt{85}}{6}$,

so that $x_1, x_2 = \dfrac{-1+\sqrt{85}}{6}, \dfrac{-1-\sqrt{85}}{6}$

b. The roots of $3x^2 + x + 7 = 0$ are $z_1, z_2 = \dfrac{-1 \pm \sqrt{1^2 - 4(3)(7)}}{2(3)} = \dfrac{-1 \pm \sqrt{-83}}{6}$, so that

$z_1, z_2 = \dfrac{-1+\sqrt{83}\text{i}}{6}, \dfrac{-1-\sqrt{83}\text{i}}{6}$

c. $\dfrac{-1+\sqrt{85}}{6} + \dfrac{-1-\sqrt{85}}{6} = \dfrac{-2}{6} = \dfrac{-1+\sqrt{83}\text{i}}{6} + \dfrac{-1-\sqrt{83}\text{i}}{6}$.

For both quadratic equations $3x^2 + x - 7 = 0$ and $3x^2 + x + 7 = 0$, it follows that $\dfrac{-b}{a} = \dfrac{-1}{3}$ so the conjecture is true.

6. a. $\sqrt[3]{9x^{14} \times 375x^{10} \times \left(2x^2\right)^3} = \sqrt[3]{9x^{14} \times 375x^{10} \times 8x^6} = \sqrt[3]{x^{30} \times 3^3 \times 2^3 \times 5^3} = 30x^{10}$

b. $\left(\dfrac{729x^{12}}{64y^3}\right)^{\frac{1}{3}} = \dfrac{729^{\frac{1}{3}}\left(x^{12}\right)^{\frac{1}{3}}}{\left(64y^3\right)^{\frac{1}{3}}} = \dfrac{9x^4}{4y}$

c. $\dfrac{\left(5y^{-1}x^2\right)^3}{\left(3y^2x^{-4}\right)^5} = \dfrac{125y^{-3}x^6}{243y^{10}x^{-20}} = \dfrac{125}{243}y^{-13}x^{26}$

Topic 1 HL Paper 1, Group 2

7. a. $\omega_1\omega_2 = 6\text{e}^{\frac{\pi}{12}\text{i}}$

b. $\left(\dfrac{\omega_1}{\omega_2}\right)^4 = \dfrac{16\text{cis}(\pi)}{81\text{cis}\left(\dfrac{-2\pi}{3}\right)} = \dfrac{16}{81}\text{e}^{\frac{5\pi}{3}\text{i}}$

c. $\left(\omega_1^{*}\right)^3 = \left(2\text{cis}\left(\dfrac{-\pi}{4}\right)\right)^3 = 8\text{e}^{-\frac{3\pi}{4}\text{i}}$

8. $V = 7\sin(20t+5) + 3\sin(20t+13)$

$$= \text{Im}\left(7\text{e}^{(20t+5)\text{i}}\right) + \text{Im}\left(3\text{e}^{(20t+13)\text{i}}\right)$$

$$= \text{Im}\left(7\text{e}^{(20t+5)\text{i}} + 3\text{e}^{(20t+13)\text{i}}\right)$$

$$= \text{Im}\left(\text{e}^{20t\text{i}}\left(7\text{e}^{5\text{i}} + 3\text{e}^{13\text{i}}\right)\right)$$

$$= \text{Im}\left(\text{e}^{20t\text{i}}\left(7.20\text{e}^{-0.858\text{i}}\right)\right)$$

$$= \text{Im}\left(7.20\text{e}^{(20t-0.858)\text{i}}\right)$$

$$= 7.20\sin(20t - 0.858)$$

Hence the maximum value of V is 7.20 and the phase shift is –0.858.

9. a. $\mathbf{XA} + \mathbf{XB} = \mathbf{C} - 3\mathbf{D}$

$$\Rightarrow \mathbf{X}(\mathbf{A}+\mathbf{B}) = \mathbf{C} - 3\mathbf{D}$$

$$\Rightarrow \mathbf{X} = (\mathbf{C} - 3\mathbf{D})(\mathbf{A}+\mathbf{B})^{-1}$$

$$= \begin{pmatrix} -12 & \pi \\ 6 & -3\sqrt{3} \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -12 & \pi \\ 6 & -3\sqrt{3} \end{pmatrix} \times \frac{-1}{13}\begin{pmatrix} 1 & -3 \\ -5 & 2 \end{pmatrix}$$

$$= \frac{-1}{13}\begin{pmatrix} -12-5\pi & 36+2\pi \\ 6+15\sqrt{3} & -18-6\sqrt{3} \end{pmatrix}$$

b. $\mathbf{BXA} = \mathbf{D} \Rightarrow \mathbf{BX} = \mathbf{DA}^{-1} \Rightarrow \mathbf{X} = \mathbf{B}^{-1}\mathbf{DA}^{-1} = \begin{pmatrix} 0 & 5 \\ 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 4 & 0 \\ -3 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 3 & 0 \end{pmatrix}^{-1}$

$$= \frac{1}{-10}\begin{pmatrix} 1 & -5 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ -3 & \sqrt{3} \end{pmatrix}\frac{1}{6}\begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix} = \frac{1}{-60}\begin{pmatrix} 19 & -5\sqrt{3} \\ -8 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix}$$

$$= \frac{1}{-60}\begin{pmatrix} 15\sqrt{3} & 38-10\sqrt{3} \\ 0 & -16 \end{pmatrix}$$

10. a. $-1 < 1.5(2^x) < 1 \Rightarrow \frac{-2}{3} < 2^x < \frac{2}{3}$. But the range of the exponential function is only positive numbers, so $0 < 2^x < \frac{2}{3}$, therefore $x < \log_2 \frac{2}{3}$.

b. $S_\infty = \frac{13}{1-1.5(2^x)} = 16 \Rightarrow 13 = 16 - 24(2^x) \Rightarrow 2^x = \frac{1}{8} \Rightarrow x = -3.$

11. a. The common difference d is $\log_{10}(a+9) - \log_{10} a = \log_{10}(a+20) - \log_{10}(a+9)$

Hence $\log_{10}\left(\frac{a+9}{a}\right) = \log_{10}\left(\frac{a+20}{a+9}\right)$, therefore $\frac{a+9}{a} = \frac{a+20}{a+9}$ so that

$$(a+9)^2 = a(a+20)$$

$$a^2 + 18a + 81 = a^2 + 20a \Rightarrow 2a = 81 \Rightarrow a = 40.5$$

b. $u_1 = \log_{10}(40.5),\ d = \log_{10}\left(\frac{49.5}{40.5}\right)$

$$S_{20} = 10\left(2\log_{10}(40.5) + 19(\log_{10}\left(\frac{49.5}{40.5}\right)\right) \approx 48.7$$

12. a. $\omega = \text{cis}\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) + \text{i}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}.$

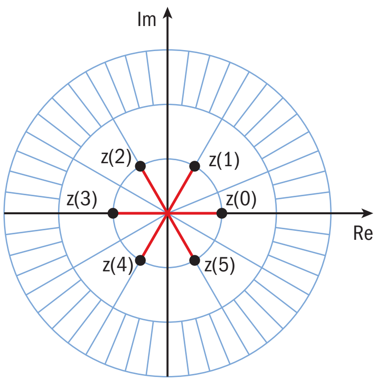
b. $\omega^2 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}\right)^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}$ in cartesian form

$\omega^2 = \text{cis}\left(\frac{2\pi}{3}\right)$ in modulus-argument form.

Since $\text{cis}\left(\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) + \text{i}\sin\left(\frac{2\pi}{3}\right)$, $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ and $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$ by equating real and imaginary parts.

c. $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5\} = \left\{1, \text{cis}\left(\frac{\pi}{3}\right), \text{cis}\left(\frac{2\pi}{3}\right), \text{cis}(\pi), \text{cis}\left(\frac{4\pi}{3}\right), \text{cis}\left(\frac{5\pi}{3}\right)\right\}$

d.



Applying symmetry in the Argand diagram, the vectors representing $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5\}$ have resultant 0 when added, since

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5$$

$$= 1 + \frac{1}{2} + \frac{\sqrt{3}}{2}\text{i} + \frac{-1}{2} + \frac{\sqrt{3}}{2}\text{i} + (-1)$$

$$+ \frac{-1}{2} + \frac{-\sqrt{3}}{2}\text{i} + \frac{1}{2} + \frac{-\sqrt{3}}{2}\text{i} = 0.$$

Topic 1 HL Paper 1, Group 3

13. a. $\log_{10}(ax+b) = 2 + 2\log_{10}(ax-b)$

$\Rightarrow \log_{10}(ax+b) = \log_{10}100 + \log_{10}(ax-b)^2$

$\Rightarrow \log_{10}(ax+b) = \log_{10}100(ax-b)^2 \Rightarrow 100(ax-b)^2 = (ax+b)$

Hence $100a^2x^2 - 200axb + 100b^2 - ax - b = 0$ so

$p = 100a^2$, $q = -200a$ and $r = 100b^2 - b$

b. $\ln(2-r) = 3 + 2(2)^3 = 19 \Rightarrow r = 2 - \text{e}^{19}$. $\text{e}^s = 3 + 2(1)^3 = 5 \Rightarrow s = \ln(5)$

c. $4^{2x+1} = 3^{1-x} \Rightarrow (2x+1)\ln(4) = (1-x)\ln(3) \Rightarrow 2\ln(4)x + \ln(4) = \ln(3) - \ln(3)x$

$$\Rightarrow x(2\ln(4) + \ln(3)) = \ln(3) - \ln(4) \Rightarrow x(\ln(16) + \ln(3)) = \ln\left(\frac{3}{4}\right) \Rightarrow x = \frac{\ln\left(\frac{3}{4}\right)}{\ln(48)}$$

14. a. $zw = (2 + 2\sqrt{3}\text{i})(\sqrt{2} - \sqrt{2}\text{i}) = (2\sqrt{2} + 2\sqrt{6}) + \text{i}(2\sqrt{6} - 2\sqrt{2})$

$$\frac{z}{w} = \frac{(2+2\sqrt{3}\text{i})}{(\sqrt{2}-\sqrt{2}\text{i})} = \frac{(2+2\sqrt{3}\text{i})(\sqrt{2}+\sqrt{2}\text{i})}{(\sqrt{2}-\sqrt{2}\text{i})(\sqrt{2}+\sqrt{2}\text{i})} = \frac{(2\sqrt{2}-2\sqrt{6}) + \text{i}(2\sqrt{6}+2\sqrt{2})}{4} = \frac{(\sqrt{2}-\sqrt{6})}{2} + \frac{\text{i}(\sqrt{6}+\sqrt{2})}{2}$$

b. $|z| = \sqrt{2^2 + (2\sqrt{3})^2} = 4$ $\quad |w| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2$

$\arg(z) = \arctan(\sqrt{3}) = \frac{\pi}{3}$ $\quad \arg(w) = \arctan(-1) = \frac{-\pi}{4}$

Hence $z = 4e^{\frac{\pi}{3}i}$ and $w = 2e^{\frac{-\pi}{4}i} \Rightarrow zw = 8e^{\frac{\pi}{12}i}$ and $\frac{z}{w} = 2e^{\frac{7\pi}{12}i}$

c. $(2\sqrt{2} + 2\sqrt{6}) + i(2\sqrt{6} - 2\sqrt{2}) = 8e^{\frac{\pi}{12}i} = 8\left(\cos\left(\frac{\pi}{12}\right) + i\left(\sin\left(\frac{\pi}{12}\right)\right)\right)$ hence by comparing real and imaginary parts, $\cos\left(\frac{\pi}{12}\right) = \frac{(2\sqrt{2} + 2\sqrt{6})}{8} = \frac{(\sqrt{2} + \sqrt{6})}{4}$.

15. a.

$$\mathbf{RP} = \begin{matrix} \\ T_1 \\ T_2 \\ T_3 \end{matrix} \begin{matrix} p_1 & p_2 \\ \end{matrix} \begin{pmatrix} 29 & 35 \\ 19 & 21 \\ 8 & 12 \end{pmatrix}$$

RP gives the number of points that the teams would score under each proposed points system

b. $\mathbf{XS} = \mathbf{E}$ has solution $\mathbf{X} = \mathbf{ES}^{-1}$. Hence $\mathbf{X} = \begin{pmatrix} 12000 & 15600 \\ 10000 & 20000 \\ 8000 & 14000 \end{pmatrix} \times \begin{pmatrix} 70 & 65 \\ 40 & 100 \end{pmatrix}^{-1}$.

Using a GDC, this gives the solution $\mathbf{X} = \begin{pmatrix} 131 & 71.0 \\ 45.5 & 170 \\ 54.5 & 105 \end{pmatrix}$ where each element is correct to three significant figures.

This matrix represents the number of points that each time would have to score in order to gain the amount of sponsorship money desired by each team.

16. $1 + 2\log_{10} y = \log_{10} 10x \Rightarrow \log_{10} 10 + \log_{10} y^2 = \log_{10} 10x$ hence $10y^2 = 10x$

$1 + \log_{10} x = \log_{10}(7 - 20y) \Rightarrow \log_{10} 10 + \log_{10} x = \log_{10}(7 - 20y)$ hence $10x \Rightarrow 7 - 20y$

So $7 - 20y = 10y^2 \Rightarrow 10y^2 + 20y - 7 = 0$ so $(3y - 1)(y + 7) = 0$.

This equation has solutions $y = -2.30$ and $y = 0.304$. However, $y = -2.30$ does not fit the original system so the only solution to the system is $y = 0.304$, $x = 0.0923$.

17. $\ln\left(\frac{y^2 z}{x}\right) = 3 \Rightarrow -\ln(x) + 2\ln(y) + \ln(z) = 3$

$\ln(x^2) + \ln(y^3) = 10 \Rightarrow 2\ln(x) + 3\ln(y) = 10$

$\ln(\sqrt{xyz}) = -1 \Rightarrow 0.5\ln(x) + 0.5\ln(y) + 0.5\ln(z) = -1$

Using a GDC to solve the system of linear equations:

$\ln(x) = -0.625$, $\ln(y) = 3.75$, $\ln(z) = -5.125$

Hence $x = e^{-0.625}$, $y = e^{3.75}$ and $z = e^{-5.125}$.

18. **a.** The completed matrix **P** is

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

b. Hence $\mathbf{P}^2$ is

$$\begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 3 & 3 & 2 \\ 1 & 3 & 6 & 1 \\ 3 & 2 & 1 & 5 \end{pmatrix}$$

c. There are 3 two-stage journeys from C to B:

$C \to D \to B$ via two different routes from C to D, and $C \to A \to B$.

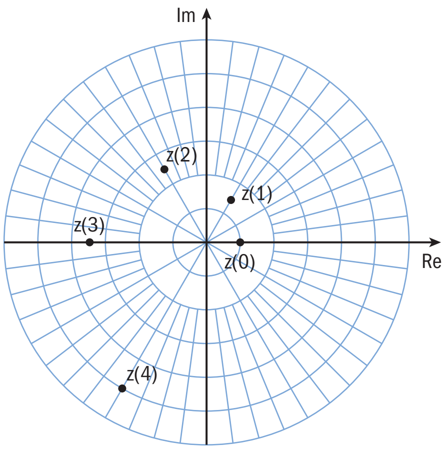
This is represented in the element in Row 3, column 2 of $\mathbf{P}^2$.

d. The elements of $\mathbf{P}^2$ give the number of two-stage journeys between each of the four towns.

Topic 1 HL Paper 2

1. **a.** $z(0)=1$, $z(1)=1.5e^{i\frac{\pi}{3}}$, $z(2)=2.25e^{i\frac{2\pi}{3}}$, $z(3)=-3.375$, $z(4)=5.0625e^{i\frac{4\pi}{3}}$

b.



c. In sector D, the argument of a complex number is between 0 and $-\frac{\pi}{2}$.

The first complex number in the sequence to have modulus greater than 50 and argument between 0 and $-\frac{\pi}{2}$ is $z(11)=86.5e^{-1.05i}$.

Hence, after 11 hours the UAV is first more than 50 km from its initial position **and** is in sector D.

d. $z(13)=1.5^{13}e^{\frac{13\pi}{3}i}=1.5^{13}e^{\frac{\pi}{3}i}=1.5^{13}\left(\cos\left(\frac{\pi}{3}\right)+i\left(\sin\left(\frac{\pi}{3}\right)\right)\right)=1.5^{13}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$z(0)=1+0i$. Hence the UAV travels on the vector $\begin{pmatrix} 1-\frac{1.5^{13}}{2} \\ -\frac{1.5^{13}\sqrt{3}}{2} \end{pmatrix}$ in order to return to its starting position.

e. At $t = 6$ hours, the arguments of $z(6)$ and $a(6)$ are 2π and 4π respectively. The modulus of $z(6)$ is 11.3906... km = 11 391 m to the nearest metre, and the modulus of $a(6)$ is 11.8539... km = 11 854 m to the nearest metre.

Although the two UAVs have the same argument at $t = 6$ hours, they are 463 metres apart at this time, and this does not qualify as a "near miss"

2. a.

$$\mathbf{E} = \begin{matrix} & \text{Cai} & \text{Tls} \\ \text{IT} & & \\ \text{sales} & & \\ \text{office} & & \end{matrix} \begin{pmatrix} 52 & 34 \\ 15 & 9 \\ 19 & 12 \end{pmatrix}$$

b. i. **S** would represent the salaries of IT, sales and office employees respectively.

ii. The dimensions of **S** are 1×3.

c. $x = 42\,900$

d. x represents the total salaries paid to the Cairo *BetaGamma* employees in one week.

e. $52 \times (66\,800 + 42\,900) = \text{USD}\$5\,704\,400$

f. $1.01^2 \times (900 \quad 700 \quad 500) \times \begin{pmatrix} 50 & 32 \\ 13 & 7 \\ 17 & 10 \end{pmatrix} = (63858.26 \quad 39477.87)$

The total paid to all *BetaGamma* employees in one year is

$52 \times 103\,336.13 = \text{USD}\$5\,373\,478.76$

Topic 1 HL Paper 3

1. a. i. $\frac{7}{6}$

ii. 0.5

iii. $\frac{1}{3}$

b. 64 pups, 10 young and 37 adults

c. Let the initial population vector be $q = \begin{pmatrix} 20 \\ 15 \\ 40 \end{pmatrix}$

After one year the population is $\mathbf{L}q$

After two years it is $\mathbf{L}(\ \mathbf{L}q\ ) = \mathbf{L}^2 q$

After three years is $\mathbf{L}(\ \mathbf{L}^2\ q\ ) = \mathbf{L}^3\ q$

d. i. $\begin{pmatrix} 0 & \frac{7}{6} & \frac{7}{6} \\ 0.5 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}^{20} \begin{pmatrix} 20 \\ 15 \\ 40 \end{pmatrix} = \begin{pmatrix} 971 \\ 416 \\ 555 \end{pmatrix}$

$T_{20} = 1942$

ii. $\begin{pmatrix} 4537 \\ 1944 \\ 2593 \end{pmatrix}$

$T_{30} = 9074$

e. **i.** $T_{30} = T_{20} r^{10}$

$r^{10} = \dfrac{9074}{1942}$

$= 4.67$ (3 sf)

$r = 4.67^{\frac{1}{10}} = 1.167$

ii. $5000 = 9074 \times 1.17^{n-30}$

41 years

f. $q_{30} = \dfrac{1}{9074}\begin{pmatrix} 4537 \\ 1944 \\ 2593 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.2142... \\ 0.2857... \end{pmatrix}$

g. **i.** $\begin{pmatrix} 0 & \frac{7}{6} & \frac{7}{6} \\ 0.5 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 0.5 \\ 0.2142... \\ 0.2857... \end{pmatrix} = \lambda\begin{pmatrix} 0.5 \\ 0.2142... \\ 0.2857... \end{pmatrix}$

Form any equation for example

$0.5 \times 0.5 = \lambda \times 0.2142...$

$\lambda = 1.166...$

ii. This is equal to the common ratio for the increase in the population size.

TOPIC 3 SL WORKED SOLUTIONS

Topic 3 SL Paper 1, Group 1

1. a. Area of triangle $=\frac{1}{2}\times 3\times 3\times \sin 140 = 2.892... \approx 2.89\text{ cm}^2$

b. Area of sector $=\frac{40}{360}\times \pi \times 3^2 = 3.14159... \approx 3.14\text{ cm}^2$

2. Distance $=\sqrt{(108.3-102.8)^2+(42.2-39.1)^2+(4.5-2.9)^2}$

$=\sqrt{42.42}\approx 6.51\text{ km}$

3. a. Volume $=\pi\times 8^2\times 10 = 2010.6... \approx 2011\text{ cm}^3$

b. Volume $=\frac{35}{360}\times 2010.6... = 195.47... \approx 195\text{ cm}^3$

4. a. Midpoint of $[AB]$ is $\left(\frac{2+(-1)}{2},\frac{8+10}{2}\right)=(0.5,9)$

b. Gradient of line joining A and $B=\frac{10-8}{-1-2}=-\frac{2}{3}$

Gradient of $l=\frac{3}{2}=1.5$

Equation of l is $y-9=1.5(x-0.5)$ or $y=1.5x+8.25$

5. a. $\tan 48^\circ=\frac{CD}{250}$

$CD=277.653...=278\text{ m}$ (3 sf)

b. Angle of depression from A is equal to the angle of elevation from B.

$\tan A\hat{B}C=\frac{\frac{4}{3}\times 277.653...}{250}$

or $90-\tan^{-1}\left(\frac{250}{\frac{4}{3}\times 277.653...}\right)$

gives angle of depression $=56.0^\circ$

6. a. $\frac{BC}{\sin 34^\circ}=\frac{5}{\sin 120^\circ}$

$BC=3.22850...=3.23\text{ cm}$ (3 sf)

b. Area $=\frac{1}{2}(5)(3.22850...)\sin 26^\circ = 3.54\text{ cm}^2$

Topic 3 SL Paper 1, Group 2

7. **a.** $y = 5,\ x = 4$ and $y = -\frac{3}{2}x + 11$

b. Using the formula for the area of a trapezoid and rectangle:

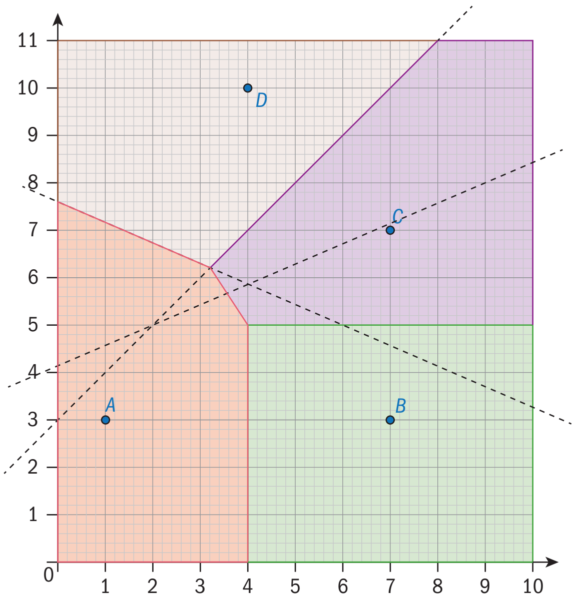
i. $A\colon\ \frac{1}{2} \times 4 \times (11 + 5) = 32\ \text{km}^2$

ii. $B\colon\ 6 \times 5 = 30\ \text{km}^2$

iii. $C\colon\ \frac{1}{2} \times 6 \times (10 + 6) = 48\ \text{km}^2$

The third area can also be found by subtracting the first two found from 110.

c.



8. Midpoint of the tunnel is at $\left(\frac{10.1 + 8.5}{2}, \frac{3.2 + 1.7}{2}, \frac{-0.8 - 0.9}{2}\right) = (9.3, 2.45, -0.85)$

Distance from C to this point is $\sqrt{(9.0 - 9.3)^2 + (2.0 - 2.45)^2 + (0 + 0.85)^2}$

$\approx 1.01\ \text{km}$

9. **a.** $BC = \sqrt{(250 + 120)^2 + (120 - 230)^2 + (75 - 140)^2} = 391.4\ldots \approx 391\ \text{m}$

b. $AC = \sqrt{120^2 + 230^2 + 140^2} = 294.7\ldots \approx 295\ \text{m}$

Area $= \frac{1}{2} \times 391 \times 295 \times \sin 47° = 42\,196 \approx 42\,200\ \text{m}^2$

10. **a.** $20 = 2r + \frac{\theta}{360} 2\pi r \Rightarrow \theta = \frac{180(20 - 2r)}{\pi r}$

b. $A = \frac{\theta}{360} \times \pi r^2$

$= \frac{1}{360} \times \frac{180(20 - 2r)}{\pi r} \times \pi r^2 = \frac{(20 - 2r)r}{2}$

$= (10 - r)r = 10r - r^2$

c. Plot $A = 10r - r^2$ on the GDC or use the symmetry of a quadratic to obtain the value $r = 5$ cm.

11. a. $100 = \frac{1}{3}\pi r^2(8)$

$r = 3.45$ (cm) (3.45494… (cm))

b. $l^2 = 8^2 + (3.45494\ldots)^2$

$l = 8.71$ (cm) (8.71416… (cm))

c. $\pi \times 3.45494\ldots \times 8.71416\ldots$

$= 94.6\text{cm}^2$ ($94.5836\ldots \text{cm}^2$)

12. a. $AC^2 = 8^2 + 6^2$

$AC = 10$

$VM^2 = 13^2 - 5^2$

$VM = 12$ (cm)

b. $\frac{1}{3} \times 8 \times 6 \times 12$

$= 192\,\text{cm}^3$

Topic 3 SL Paper 1, Group 3

13. a. $AB^2 = 6^2 + 8^2 - 2 \times 6 \times 8 \cos 60 = 52$

$AB = \sqrt{52} = 7.2111\ldots \approx 7.21$ m

b. $\cos\left(B\hat{O}A\right) = \frac{4^2 + 4^2 - 52}{2 \times 4 \times 4} = -0.625$

$\Rightarrow B\hat{O}A = 128.68\ldots \approx 129°$

Arc length $AB = \frac{128.68\ldots}{360} \times 2\pi \times 4 = 8.9837\ldots \approx 8.98$ m

Amount of edging needed $= 8.98 + 7.21 \approx 16.2$ m

14. a. i. $OC = r\cos 60 = 0.5r$

ii. Area of triangle $= \frac{1}{2} \times r \times 0.5r \times \sin 60 = 0.2165\ldots r^2$

b. Area of sector $= \frac{60}{360} \times \pi \times r^2 = \frac{\pi}{6} r^2$ $\left(\text{or } 0.5235\ldots r^2\right)$

Hence $0.5235\ldots r^2 - 0.2165\ldots r^2 = 50$

$0.307\ldots r^2 = 50 \Rightarrow r^2 = 162.8\ldots$

$r = 12.759\ldots \approx 12.8$ cm

15. a. i. 15 m

ii. $\frac{1}{3} \times 8^2 \times 15 = 320\,\text{m}^3$

b. $a = 4$

c. The light is half way between the midpoint of [AB] and the vertex.

Midpoint of [AB] has coordinates $(4, 0, 0)$.

Coordinates of light $= \left(\frac{4+4}{2}, \frac{4+0}{2}, \frac{15+0}{2}\right) = (4, 2, 7.5)$

d. Height of face is the distance between V and $(4,0,0)$

$= \sqrt{(4-4)^2+(4-0)^2+(15-0)^2} = \sqrt{241} = 15.52... \approx 15.5$ m

Surface area of pyramid $= \left(4\times\frac{1}{2}\times 8\times 15.5\right) = 248.38... \approx 248$ m^3

16. Attempt to use tan, or sine rule, in triangle BXN or BXS

$NX = 80\tan 55°\left(= \frac{80}{\tan 35°}\right) = 114.25$

$SX = 80\tan 65°\left(= \frac{80}{\tan 25°}\right) = 171.56$

Attempt to use cosine rule

$SN^2 = 171.56^2 + 114.25^2 - 2 \times 171.56 \times 114.25 \cos 70°$

$SN = 171$ m

17. a. 3.2

b. $a = \frac{1}{2}\times 4\times 6 = 12$ km^2

c. $a_C = \frac{1}{2}\times 4\times 2 = 4$ km^2

By symmetry $a_A = a_B = \frac{12-4}{2} = 4$ km^2

$P_D = \frac{6.5\times 4+7.2\times 4+3.2\times 4}{12} = 5.633... \approx 5.63$

18. a. Volume of water $= \pi\times 8^2\times 12$

$= 768\pi$

$= 2412.74... = 2410$ cm^3 (3 sf).

b. $\frac{4}{3}\pi\times 2.9^3 + 768\pi = \pi\times 8^2 h$ or $\frac{4}{3}\pi\times 2.9^3 = \pi\times 8^2(h-12)$

gives $h = 12.5081... = 12.5$ cm (3 sf)

Topic 3 SL Paper 2

1. a. i. Midpoint is (6, 3).

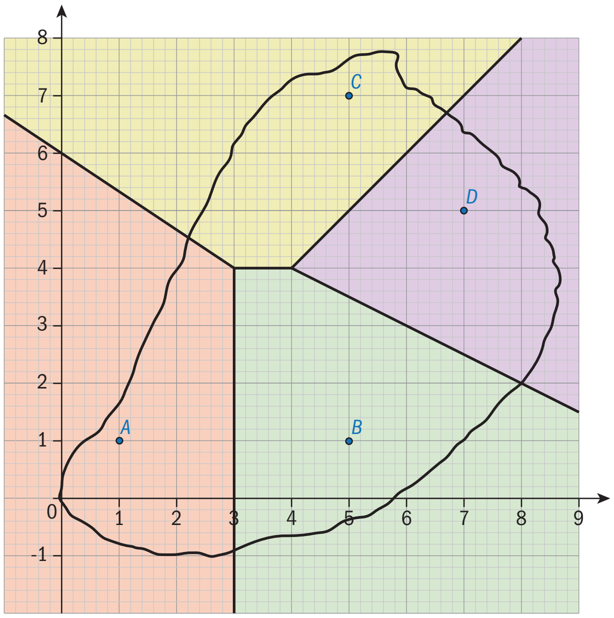
ii. Gradient is 2.

b. Either $y-3 = -\frac{1}{2}(x-6) \Rightarrow y = -\frac{1}{2}x+6$

or $y = -\frac{1}{2}x + c \Rightarrow 3 = -\frac{1}{2}\times 6 + c \Rightarrow c = 6$

c. $y = x$

d.



e. (3,4)

f. $\sqrt{13} \approx 3.61$ km from A, B and C and $\sqrt{17} \approx 4.12$ km from D

2. a. $BD^2 = 40^2 + 84^2$

$BD = 93.0376\ldots$

$= 93$

b. $\cos B\hat{C}D = \dfrac{115^2 + 60^2 - 93^2}{2 \times 115 \times 60}$

$= 53.7°\ (53.6679\ldots°)$

c. $\dfrac{1}{2}(40)(84) + \dfrac{1}{2}(115)(60)\sin(53.6679\ldots)$

$= 4460\ \text{m}^2\ (4459.30\ldots\ \text{m}^2)$

d. i. $\dfrac{(40+60)(84+115)}{4}$

$= 4980\ \text{m}^2\ (4975\ \text{m}^2)$

ii. $\left|\dfrac{4975 - 4459.30\ldots}{4459.30\ldots}\right| \times 100$

$= 11.6\ (\%)\ (11.5645\ldots)$

3. a. i. $222 = \dfrac{1}{2}x(x+3) + (x+3)(x+5)$

OR

$222 = (x+3)(2x+5) - 2\left(\dfrac{1}{4}\right)x(x+3)$

ii. $222 = \dfrac{1}{2}x^2 + \dfrac{3}{2}x + x^2 + 3x + 5x + 15$

$3x^2 + 19x - 414 = 0$

b. Solving the equation in (a. ii) by factorising or use of *GDC*

$x = 9\left(\text{and } x = -\frac{46}{3}\right)$

$CD = 12$ (cm)

c. Divide triangle *ABE* into 2 right-angled triangles.

The base of each triangle is $\frac{1}{2}(9+3) = 6$

$\tan\left(\frac{B\hat{A}E}{2}\right) = \frac{6}{9}$

$B\hat{A}E = 67.3801\ldots°$

$= 67.4°$

d. $2\sqrt{9^2 + 6^2} + 12 + 2(14)$

$= 61.6$ (cm) (61.6333… (cm))

e. $F\hat{B}C = 90 + \left(\frac{180 - 67.4}{2}\right)(= 146.3°)$

OR

$180 - \frac{67.4}{2}$

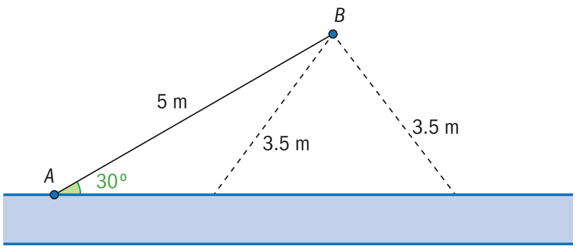
$CF^2 = 8^2 + 14^2 - 2(8)(14)\cos(146.3°)$

$CF = 21.1$ (cm) (21.1271…)

TOPIC 3 HL WORKED SOLUTIONS

Topic HL Paper 1, Group 1

1. a.



b. $\frac{\sin\theta}{5} = \frac{\sin 30°}{3.5} \Rightarrow \sin\theta = 0.714...$

$\theta = 45.58...$ or $180 - 45.58... = 134.4...$

For the smaller triangle the third angle is $180 - 30 - 134.4... = 15.58...$

Area $= \frac{1}{2} \times 5 \times 3.5 \times \sin(15.58...) \approx 2.35 \text{ m}^2$

2. a. $\begin{pmatrix} 2 & 1 \\ a & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} b \\ 3 \end{pmatrix}$

$8 + 3 = b \Rightarrow b = 11$

$4a - 3 = 3 \Rightarrow a = \frac{6}{4} = 1.5$

b. $\begin{pmatrix} 2 & 1 \\ 1.5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -7 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1.5 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ -7 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

Point is $(-2, 4)$

3. a. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 105 \\ 226 \\ 12 \end{pmatrix} + 0.5\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow a = -210,\ b = -452,\ c = -24$

b. Speed $= \sqrt{210^2 + 452^2 + 24^2} \approx 499$ or 500 km h^{-1}

4. a. i. $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

ii. $\overrightarrow{AC} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$

b. Area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \left|\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}\right| = \frac{1}{2} \times \left|\begin{pmatrix} -8 \\ 1 \\ -2 \end{pmatrix}\right|$

$= \frac{1}{2}\sqrt{8^2 + 1^2 + 2^2} = \frac{1}{2} \times \sqrt{69} \approx 4.15$

5. a. $\dot{\boldsymbol{x}} = \begin{pmatrix} 3t^2 + c_1 \\ 2t + c_2 \end{pmatrix}$

At $t = 0$, $\dot{\boldsymbol{x}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow c_1 = 2, \;\; c_2 = 1$

Hence $\dot{\boldsymbol{x}} = \begin{pmatrix} 3t^2 + 2 \\ 2t + 1 \end{pmatrix}$

b. $\boldsymbol{x} = \begin{pmatrix} t^3 + 2t + c_3 \\ t^2 + t + c_4 \end{pmatrix}$

At $t = 0$, $\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow c_3 = 0, \;\; c_4 = 0$

Hence $\boldsymbol{x} = \begin{pmatrix} t^3 + 2t \\ t^2 + t \end{pmatrix}$

c. At $t = 2$ $\boldsymbol{x} = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$

Distance $= \sqrt{12^2 + 6^2} = \sqrt{180} \approx 13.4$ m

6. a. Prim's or Kruskal's algorithm

b. **EITHER**

using Prim's algorithm, starting at A

Edge	Cost
AC	4
CD	3
CF	4
FE	4
AB	5

lowest cost road system contains roads *AC*, *CD*, *CF*, *FE* and *AB* cost is 20

OR

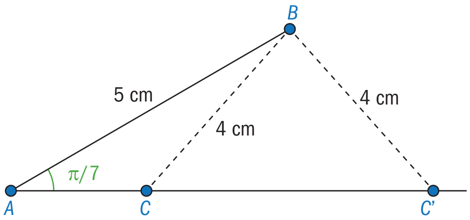
using Kruskal's algorithm

Edge	Cost
CD	3
CF	4
FE	4
AC	4
CB	5

lowest cost road system contains roads *CD*, *CF*, *FE*, *AC* and *AB* cost is 20

Topic 3 HL Paper 1, Group 2

7. a.



Let $AC = x$

$4^2 = x^2 + 5^2 - 2 \times x \times 5\cos\left(\frac{\pi}{7}\right)$

$\Rightarrow x^2 - 10\cos\left(\frac{\pi}{7}\right)x + 9 = 0$

$\Rightarrow x = 1.144... \approx 1.14 \text{ m} \text{ or } 7.865\ldots \approx 7.87 \text{ cm}$

b. $\text{Area} = \frac{1}{2} \times 5 \times 7.865... \times \sin\left(\frac{\pi}{7}\right) \approx 8.53 \text{ cm}^2$

8. a. $\text{Area} = \frac{1}{2}(5-3)(6-2) = 4$

b. $\text{Area of image} = \det\begin{pmatrix} 2\sin\theta & -2\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \times 4$

$= \left(2\sin^2\theta + 2\cos^2\theta\right) \times 4 = 8\left(\sin^2\theta + \cos^2\theta\right) = 8$

9. a. $A = \frac{1}{2} \times 10^2 \times \theta - \frac{1}{2} \times 10^2 \times \sin\theta = 50\theta - 50\sin\theta$

b. $\text{Unshaded area} = \frac{\pi \times 10^2}{2} - 50(\theta - \sin\theta) = 50(\pi - \theta + \sin\theta)$

Solve $50(\theta - \sin\theta) = \frac{1}{2} \times 50(\pi - \theta + \sin\theta)$

$\theta \approx 1.969$

10. $\sqrt{m^2 + n^2} = 1 \Rightarrow m^2 + n^2 = 1$

$\boldsymbol{u} \cdot \boldsymbol{v} = 0$

$\Rightarrow -3 \times 0 + 1 \times m + 1 \times n = 0 \Rightarrow m + n = 0$

$\Rightarrow m = -n$

$(-n)^2 + n^2 = 1 \Rightarrow 2n^2 = 1 \Rightarrow n = \pm\sqrt{0.5} = \pm 0.70710\ldots$

Either $n = \sqrt{0.5}$, $m = -\sqrt{0.5}$ or $n = -\sqrt{0.5}$, $m = \sqrt{0.5}$

11. **a.** The graph has no Euler circuit because not all the vertices are even.

b. The four vertices of odd degree are A, D, F, J. These can be paired up as:

AD JF 15 + 15 = 30

AJ DF 30 + 13 = 43

AF JD 22 + 15 = 37

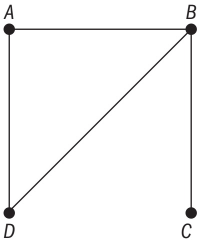
Least possible cost is 152 + 30 = 182 USD

Routes to be repeated are AD, JI and IF

c. Start at A and finish at J or vice versa.

Extra cost is DF = 13 USD so would save 30 – 13 = 17 USD

12. **a.**



b.

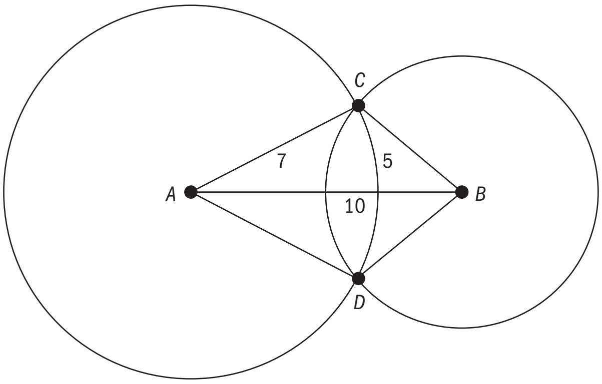
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 & 0 \end{pmatrix} \end{array}$$

c. $\begin{pmatrix} 0.25 \\ 0.375 \\ 0.125 \\ 0.25 \end{pmatrix}$

d. He spends most time in room B and least time in room C. We are assuming that when he goes into a room he spends an equal amount of time in each one.

Topic 3 HL Paper 1, Group 3

13.



Use of cosine rule:

$$C\hat{A}B = \arccos\left(\frac{49+100-25}{2\times7\times10}\right) = 0.48276\ldots (= 27.660\ldots°)$$

$$C\hat{B}A = \arccos\left(\frac{25+100-49}{2\times5\times10}\right) = 0.70748\ldots (= 40.535\ldots°)$$

Attempt to subtract triangle area from sector area:

$$\text{area} = \frac{1}{2}\times 49\left(2C\hat{A}B - \sin 2C\hat{A}B\right) + \frac{1}{2}\times 25\left(2C\hat{B}A - \sin 2C\hat{B}A\right)$$

$= 3.0579\ldots + 5.3385\ldots$

$= 8.85\,(\text{km}^2)$

14. a. i. $\begin{pmatrix} 1.1 & 0 \\ 0 & 1.1 \end{pmatrix}$

ii. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

b. $\mathbf{T} = \begin{pmatrix} 1.1 & 0 \\ 0 & 1.1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1.1 \\ 1.1 & 0 \end{pmatrix}$

i. $\mathbf{T}^n = \begin{pmatrix} 0 & 1.1^n \\ 1.1^n & 0 \end{pmatrix}$ when n is odd

ii. $\mathbf{T}^n = \begin{pmatrix} 1.1^n & 0 \\ 0 & 1.1^n \end{pmatrix}$ when n is even

c. $1.1^n > 20 \Rightarrow n \geq 32$

As 32 is even the coordinates are $(1.1^{32}, 0) \approx (21.1, 0)$

15. **a.** $\sqrt{5.2^2+3.9^2}=6.5$

b. Bearing $=\arctan\left(\frac{5.2}{3.9}\right)=053.1°$

c. Time $=\frac{12}{6.5}=1.846\ldots$ hours $\approx$ 1 hour, 51 minutes

d. Displacement from the port when $t=0$ is $-\frac{12}{6.5}\begin{pmatrix}5.2\\3.9\end{pmatrix}$

Displacement at time t is

$$\boldsymbol{r}=-\frac{12}{6.5}\begin{pmatrix}5.2\\3.9\end{pmatrix}+t\begin{pmatrix}5.2\\3.9\end{pmatrix}=\begin{pmatrix}-9.6\\7.2\end{pmatrix}+t\begin{pmatrix}5.2\\3.9\end{pmatrix}$$

16. **a.** $\begin{pmatrix}5\cos 30\\5\sin 30\end{pmatrix}\approx\begin{pmatrix}4.33\\2.5\end{pmatrix}$

b. Integrate acceleration to obtain velocity and velocity to obtain displacement. Use the initial conditions to find the values of the constant.

$$\boldsymbol{a}=\begin{pmatrix}0\\-9.8\end{pmatrix}\Rightarrow\boldsymbol{v}=\begin{pmatrix}4.33\\2.5-9.8t\end{pmatrix}$$

$$\boldsymbol{r}=\begin{pmatrix}4.33t\\1.5+2.5t-4.9t^2\end{pmatrix}$$

c. For student B's beanbag initial displacement is $\begin{pmatrix}20\\1.0\end{pmatrix}$ and the velocity is

$\begin{pmatrix}-6\cos\theta\\6\sin\theta\end{pmatrix}$ where θ is the acute angle made with the horizontal.

$$\boldsymbol{r}_B=\begin{pmatrix}20-(6\cos\theta)t\\1.0+(6\sin\theta)t-4.9t^2\end{pmatrix}$$

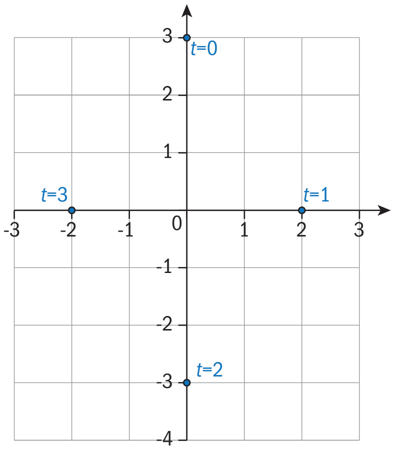
If they collide $20-(6\cos\theta)t=4.33\ldots t\Rightarrow t=\dfrac{20}{6\cos\theta-4.33\ldots}$

$1.0+(6\sin\theta)t-4.9t^2=1.5+2.5t-4.9t^2\Rightarrow(6\sin\theta-2.5)t=0.5$

Substitute in $(6\sin\theta-2.5)\dfrac{20}{6\cos\theta-4.33\ldots}=0.5$

Solve to find $\theta\approx 24.9°$

17. **a.**



b. $\boldsymbol{v} = \begin{pmatrix} \pi\cos\left(\frac{\pi t}{2}\right) \\ -\frac{3\pi}{2}\sin\left(\frac{\pi t}{2}\right) \end{pmatrix}$

c. $|\boldsymbol{v}| = \sqrt{\left(\pi\cos\left(\frac{\pi t}{2}\right)\right)^2 + \left(\frac{3\pi}{2}\sin\left(\frac{\pi t}{2}\right)\right)^2} = \frac{\pi}{2}\sqrt{4\cos^2\left(\frac{\pi t}{2}\right) + 9\sin^2\left(\frac{\pi t}{2}\right)}$

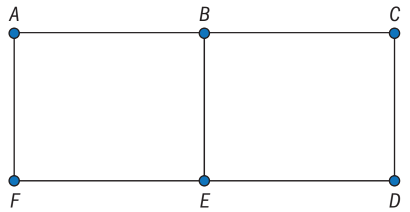
$= \frac{\pi}{2}\sqrt{4 + 5\sin^2\left(\frac{\pi t}{2}\right)}$ [using $\cos^2\left(\frac{\pi t}{2}\right) = 1 - \sin^2\left(\frac{\pi t}{2}\right)$]

d. Maximum when $\sin^2\left(\frac{\pi t}{2}\right) = 1 \Rightarrow \sin\left(\frac{\pi t}{2}\right) = \pm 1$

$\Rightarrow t = 1$ and $t = 3$

e. These are the two points when the planet is closest to the star.

18. a.



b. i. From the matrix raised to the power 6, the number of walks from *A* to *A* is 50

ii. 2 walks; it is only possible if no vertices are repeated

c. *ABCD*, *ABED*, *AFED*

d. $P(ABCD) = \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{12}$

$P(ABED) = \frac{1}{2} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{18}$

$P(AFED) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{12}$

Total probability $= \frac{1}{12} + \frac{1}{18} + \frac{1}{12} = \frac{2}{9}$

Topic 3 HL Paper 2

1. a. $(-3, 1, 2)$

b. i. $\overrightarrow{AB} = 5\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ -5 \end{pmatrix}$

ii. $\left|\overrightarrow{AB}\right| = \sqrt{10^2 + 10^2 + 5^2} = 15$

c. $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

d. $\boldsymbol{r} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + (t-3)\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

e. $\overline{AB} \bullet \overline{AC} = \begin{pmatrix} 10 \\ 10 \\ -5 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} = 90$

$\cos B\hat{A}C = \frac{90}{15\sqrt{56}}\left(= \frac{6}{\sqrt{56}} = \frac{3}{\sqrt{14}}\right) \approx 0.802$

Note: could have used $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

f. Let the distance between P_1 and P_2 be x.

Using the cosine rule $x^2 = 15^2 + \left(\sqrt{56}\right)^2 - 2 \times 15 \times \sqrt{56} \times \frac{6}{\sqrt{56}} = 101$

$\Rightarrow x = \sqrt{101} \approx 10.0$ m

2. a. Order of edge selection is *AD*, *DB*, *BC*

Weight of minimum spanning tree is 9 + 11 + 8 = 28 km

b.

	A	*B*	*C*	*D*	*E*
A		12	15	9	5
B	12		8	11	8
C	15	8		18	16
D	9	11	18		14
E	5	8	16	14	

Route *EAC* = 5 + 15 = 20, *EBC* = 8 + 8 = 16 so 16 km is the least distance.

Route *EAD* = 5 + 9 = 14, *EBD* = 8 + 11 = 19 so 14 km is the least distance.

c. *AEBCDA*; length = 5 + 8 + 8 + 18 + 9 = 48 km

d. Lower bound is 28 + 5 + 8 = 41 km

e. Weight of minimum spanning tree in the remaining graph = 29 km
Hence lower bound = 29 + 8 + 8 = 45 km

f. Let the length of the journey be x.

Then $45 \le x \le 48$

Topic 3 HL Paper 3

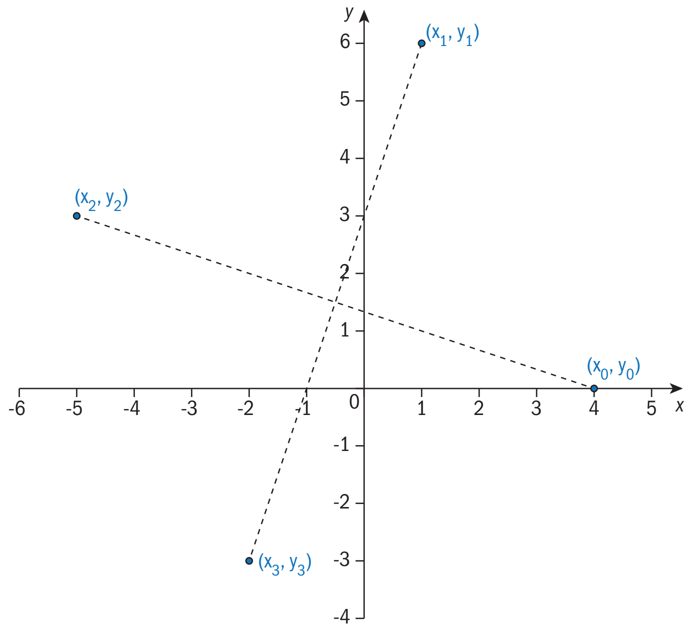
1. a. A rotation of $90°$ anticlockwise about $(0, 0)$

b. i. $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

ii. $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$

iii. $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -5 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

c.



d. Midpoint of $(4, 0)$ and $(-5, 3)$ is $\left(\frac{4-5}{2}, \frac{0+3}{2}\right) = (-0.5, 1.5)$

e. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -0.5 \\ 1.5 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1.5 \\ -0.5 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 1.5 \end{pmatrix}$

f. $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

g. $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 4 \\ -2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \end{pmatrix}$

A is $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} + \begin{pmatrix} -8 \\ 4 \end{pmatrix}$

h. $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} -8 \\ 4 \end{pmatrix} = \begin{pmatrix} -8-x_0 \\ 4-y_0 \end{pmatrix}$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -8-x_0 \\ 4-y_0 \end{pmatrix} + \begin{pmatrix} -8 \\ 4 \end{pmatrix} = \begin{pmatrix} 8+x_0-8 \\ -4+y_0+4 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

i. Because two rotations of 180° will return any point to its original position

j. When n is odd $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -8-x_0 \\ 4-y_0 \end{pmatrix}$

k. (a, b) is invariant

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$

$$a = (\cos\theta)a - (\sin\theta)b + x \Rightarrow x = a(1-\cos\theta) + b(\sin\theta)$$

$$b = (\sin\theta)a + (\cos\theta)b + y \Rightarrow y = b(1-\cos\theta) - a(\sin\theta)$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} a(1-\cos\theta) + b\sin\theta \\ b(1-\cos\theta) - a(\sin\theta) \end{pmatrix}$$

TOPIC 4 SL WORKED SOLUTIONS

Topic 4 SL Paper 1, Group 1

1. a. €1.89 in Germany, €1.84 in Italy

   b. One-tailed

   c. i. H_0: The mean cost is the same in Germany and Italy.

      H_1: The mean cost in Italy is less than the mean cost in Germany.

      ii. p-value = 0.388

      iii. p-value = 0.388 > 0.05, so insufficient evidence to reject H_0: The mean cost is the same.

2. There are 53 – 20 = 33 lockers in total.

   a. The event "49 or higher" = {49, 50, 51, 52, 53} so the required probability is $\frac{5}{33}$

   b. The event "26 or less" = {21, 22, 23, 24, 25, 26} so the required probability is $\frac{6}{33}$

   c. The event "a multiple of 9" = {27, 36, 45} so the required probability is $\frac{3}{33}$

   d. Apply the complementary event: the required probability is $1-\frac{3}{33}=\frac{30}{33}$

   e. The event "a factor of 120" = {24, 30, 40} so the required probability is $\frac{3}{33}$

3. a. $4a+0.47+0.17+0.09+0.02+0.01=1$ hence $a=0.06$

   b. $P(T=4|\,T\geq 3)=\frac{P((T=4)\cap(T\geq 3))}{P(T\geq 3)}=\frac{P(T=4)}{P(T\geq 3)}=\frac{0.09}{0.17+0.09+0.06+0.02+0.01}=\frac{9}{35}$

4. a. $a=12$, $b=7$

   b. i. H_0: The choice of main course is independent of gender.

      H_1: The choice of main course is not independent of gender.

      ii. $\chi^2 = 4.0335\ldots \approx 4.03$

      iii. 4.03 < 7.81 so the result is not significant at the 5% level so insufficient evidence to reject the null hypothesis that the choice of main course is independent of gender.

      (degrees of freedom = 3)

5. **a.** 31

b.

Mid-value	Frequency
62.5	3
67.5	3
72.5	8
77.5	10
82.5	5
87.5	2

$\bar{x} = 75.2$ minutes, $\sigma = 6.58$ minutes

c. 105.2 minutes, 6.58 minutes

6. The sample space diagram shows the possible outcomes of the total T.

T	**2**	**3**	**5**	**8**
1	3	4	6	9
1	3	4	6	9
2	4	**5**	7	**10**
3	**5**	6	8	11
5	7	8	**10**	13
8	**10**	11	13	16

a. $P(T \text{ is prime}) = \frac{10}{24} = \frac{5}{12}$

b. $P(T \text{ is prime or a factor of } 10) = \frac{13}{24}$

c. $P(T \text{ is prime or a multiple of } 4) = \frac{16}{24}$

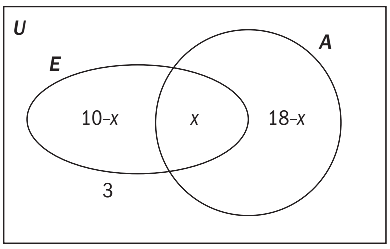
Topic 4 SL Paper 1, Group 2

7. **a.** $a = 4$

b. An outlier is any value greater than $1.5 \times 3.3 + 7.3 = 12.25$

So 13.2 should be shown as an outlier.

8. Represent the context with a Venn diagram



Hence write down the equation $10 - x + x + 18 - x + 3 = 30$, which has solution $x = 1$.

a. Using Venn diagram,

$P(\text{choose a student who studies both Economics and Art}) = \frac{1}{30}$

b. $P(\text{Art} \mid \text{Economics}) = \frac{P(\text{Art and Economics})}{P(\text{Economics})} = \frac{1}{10}$

c. $P(\text{Art}) \times P(\text{Economics}) = \frac{18}{30} \times \frac{10}{30} = \frac{1}{5}$

$\neq P(\text{choose a student who studies both Economics and Art}) = \frac{1}{30}$

Hence the events are not independent.

(Equivalently, $P(\text{Art}) = \frac{18}{30} \neq P(\text{Art} \mid \text{Economics})$ hence the events are not independent.)

9. a. A, B, C

b. B, D, C

10. $V \sim N(499.3, 3.7^2)$. Let v be the quantity required. Then $P(V < v) = 0.04$.
Use the inverse normal feature of your GDC to find $v = 493\,\text{ml}$.

11. a. i. $\frac{85}{200} \times 50 = 21.25 \approx 21$

ii. $50 - 21 = 29$

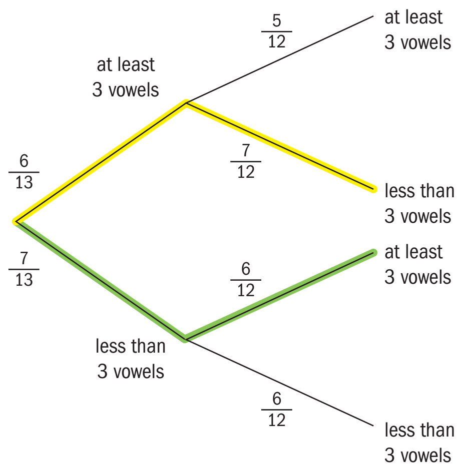
b. Randomly choose a student from grade 9 and then choose every $\frac{85}{21} \approx 4\text{th}$ student from that student using the list provided by the Principal.

c. The data is not symmetrical which indicates the population may not be normally distributed.

12. a. $P(\text{at least 3 vowels}) = \frac{4+1+1}{13} = \frac{6}{13}$

b. $25 \times \frac{6}{13} \approx 11.5$

c.



The event "exactly one of the two names chosen has at least 3 vowels" means either the first name chosen has at least three vowels and the second does not, or the first does not have at least three vowels and the second does. These combined events are highlighted in the diagram above.

Hence the required probability is $\frac{6}{13}\times\frac{7}{12}+\frac{7}{13}\times\frac{6}{12}=\frac{84}{156}=\frac{7}{13}$

Topic 4 SL Paper 1, Group 3

13. a. Let the score be u_n where $u_n = 8+(n-1)2 = 6+2n$

b. $\bar{x}=\frac{S_n}{n}=\frac{\frac{n}{2}(16+2(n-1))}{n}=\frac{1}{2}(14+2n)=7+n>20$

$\Rightarrow n \geq 14$

$0.8 + 5p + 6q + 0.7 + 2.4 = 6.2$ so $5p + 6q = 2.3$

Solve the system to find $q = 0.3$ and $p = 0.1$.

14. a. $0.2 + p + q + 0.1 + 0.3 = 1$ so $p + q = 0.4$.

$0.8 + 5p + 6q + 0.7 + 2.4 = 6.2$ so $5p + 6q = 2.3$

Solve the system to find $q = 0.3$ and $p = 0.1$.

b. The ticket should cost USD 6.20 in order to make the expected gain zero.

c. The probability of winning at least USD 7 is 0.4. The number of times at least USD 7 is won in 10 games (X) is distributed binomially: $X \sim \text{B}(10,0.4)$. The probability required is $P(X=4)=0.251$.

15. a. Ranking from lowest to highest

	A	B	C	D	E	F	G	H
Mass (m)	1	2	3	4	5	6	7	8
Height (h)	1	2	3	4	7	5.5	5.5	8

$r_s = 0.922$

This is a strong correlation and so the height of the plants does generally increase with the amount of nutrients provided.

b. $h = 0.826m + 5.44$

c. The amount of growth in cm per gram of nutrient (0.826 cm g^{-1})

d. $h = 0.826\times 20 + 5.44 = 21.96$, so $c = 21.96 \approx 22.0$

16. a. $X \sim \text{B}(20,0.4)$ and $Y \sim \text{B}(100,0.08)$. In both cases, the assumption is that the trials are independent and the probability of scoring a point is fixed.

b. $\text{E}(X)=20\times 0.4=8$ $\quad\quad$ $\text{E}(Y)=100\times 0.08=8$

c. $\text{Var}(X)=20\times 0.4\times 0.6=4.8$ $\quad\quad$ $\text{Var}(Y)=100\times 0.08\times 0.92=7.36$

d. Since $\text{E}(X)=\text{E}(Y)$, the mean of the distributions are equal, meaning that on average the expected number of points scored for each distribution is equal.

Since $\text{Var}(Y)$ is larger than $\text{Var}(X)$, the number of points scored in Y will be spread more widely than when playing X.

17. a. $T \sim N(10, 2.5^2)$ $P(0 < T < 15) = 0.977.$

b. If Y represents the number of games at which Brian does not find his seat by the start of the game, then $Y \sim B(20, 0.022781748\ldots)$, using the unrounded answer from part (a) to avoid unnecessary rounding errors.

This is assuming that Brian attends each of the 20 games and that the probability of him finding his seat by the start of the game remains constant.

i. $P(Y \geq 2) = 0.0752$

ii. $P(Y = 2) = 0.0651$

18. a. H_0: X is distributed as $B(2, 0.5)$.

H_1: X is not distributed as $B(2, 0.5)$.

b. Assuming $B(2, 0.5)$

Number of times stopped	0	1	2
Frequency	$(0.5)^2 \times 60 = 15$	$2 \times 0.5 \times 0.5 \times 60 = 30$	$(0.5)^2 \times 60 = 15$

c. p-value = 0.0273 < 0.05. so sufficient evidence to reject H_0

d. $p > 0.5$ (lights are on red more than green), the incidents of red are not independent, the trials are not identical e.g. more traffic in the evening.

Topic 4 SL Paper 2

1. a. $a = 6.96103$, $b = -454.805$

$a = 6.96$, $b = -455$ (accept $6.96x - 455$)

b. $P = 6.96(270) - 455$

$= 1424.67$

$P = 1420$ (g)

c. 40 (hives)

d. i. 128 + 40

168 hives have a production less than k

$k = 1640$

ii. 200 − 168

32 (hives)

e. $X \sim B(n, p), \binom{n}{r} p^r (1-p)^{n-r}$

$n = 40$ and $p = 0.75$ and $r = 30$

$P(X = 30) = 0.144364 \approx 0.144$

2. **a.** Let M denote the height of a randomly selected male.

Then $M \sim \mathrm{N}(170, 11^2)$

$P(M > 175) = 0.325$

b. Let F denote the height of a randomly selected female.

Then $F \sim \mathrm{N}(163, 10^2)$

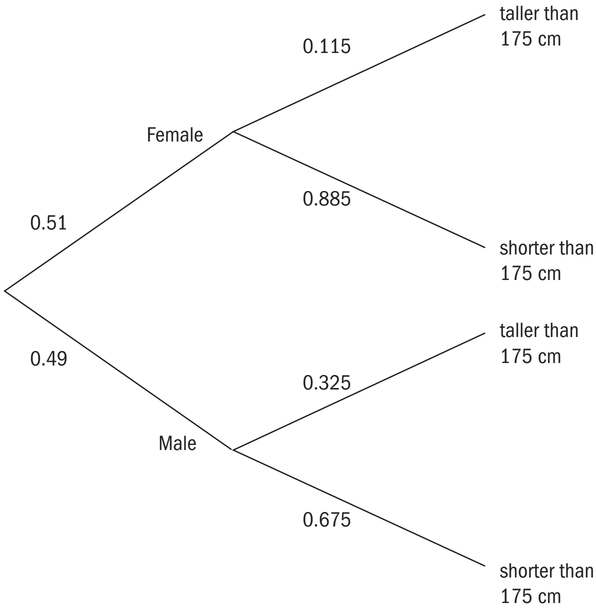
$P(F < t) = 0.15$. Use the inverse norm function of your GDC to find $t = 153\,\text{cm}$

c. The interquartile range is $\text{IQR} = Q_3 - Q_1$ where $P(M < Q_1) = 0.25$.

Hence $Q_1 = 162.58061275605$

Since the normal curve is symmetric, $\text{IQR} = 2 \times (170 - 162.58..) = 14.8\,\text{cm}$

d.



The tree diagram shows that the probability required is

$0.51 \times 0.115 + 0.49 \times 0.325 = 0.218$

e. Let B represent the event "the student is taller than 175 cm" and let A represent the event "the student is male".

The probability required is $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.49 \times 0.325}{0.218} = 0.731$.

TOPIC 4 HL WORKED SOLUTIONS

Topic 4 HL Paper 1, Group 1

1. a. A T distribution with 9 degrees of freedom or $T(9)$

   b. $10.3 < \mu < 14.1$

2. a. $\text{E}(A) = \text{Var}(A) = 3.1$ and $\text{E}(B) = \text{Var}(B) = 2.7$

   $\text{E}(A - 2B) = \text{E}(A) - 2\text{E}(B) = -2.3$

   $\text{Var}(A - 2B) = \text{Var}(A) + 4\text{Var}(B) = 13.9$

   b. Since $\text{E}(A - 2B)$ and $\text{Var}(A - 2B)$ are different, $A - 2B$ cannot follow a Poisson distribution since the mean and the variance are always equal for every Poisson distribution.

3. a. The sample is large enough for the central limit theorem to apply

   b. $\text{H}_0: \mu = 0$ $\qquad$ $\text{H}_1: \mu \neq 0$

   $s_{n-1} = \sqrt{\frac{40}{39}} \times 1.2 = 1.215...$

   p-value $= 0.00770... < 0.05$ so the null hypothesis that the mean is equal to 0 is rejected.

   c. It is likely there is a systematic error that is causing Antoine to underestimate the results.

4. $\text{E.}(X) = 4.9$, $\text{E}(Y) = 7 \times 0.35 = 2.45$ and $\text{E}(Z) = 61.$

   $\text{Var}(X) = 4.9$, $\text{Var}(Y) = 7 \times 0.35 \times 0.65 = 1.5925$ and $\text{Var}(Z) = 25$

   a. $\text{E}(2Y + 7) = 2 \times 2.45 + 7 = 11.9$

   b. $\text{Var}(8 - 2X) = 4 \times \text{Var}(X) = 19.6$

   c. $\text{Var}(X + 2Y - Z) = \text{Var}(X) + 4\ \text{Var}(Y) + \text{Var}(Z) = 36.27$

5. a. $\text{H}_0: p = 0.5$ $\qquad$ $\text{H}_1: p > 0.5$

   b. Let X be the number Heads.

   $P(X \geq a) < 0.05$, from the GDC, $a = 32$

   Hence the critical region is $X \geq 32$

   c. Accept H_0 if $X \leq 31$

   $P(X \leq 31 \mid p = 0.6) = 0.664$

6. The unbiased estimate of the population mean is the mean of the sample which is 248.3.

   The unbiased estimate of the population variance is found from the GDC as $6.5667512684905^2 = 43.1$, or equivalently $6.229767250869^2 \times \frac{10}{9} = 43.1$.

7. Using the χ^2 goodness of fit test

sample mean = 1.26315…

H_0: Distribution is Poisson. H_1: Distribution is not Poisson.

Expected values for Po(1.26)

Number of goals	0	1	2	3	4	≥5
Frequency	10.7	13.6	8.57	3.61	1.14	0.38

Need to combine the final 3 columns for the observed and expected tables.

Number of goals	0	1	2	≥3
Observed	9	14	12	3
Expected	10.7	13.6	8.57	5.13

Degrees of freedom $= 4 - 1 - 1 = 2$

p-value $\approx 0.28 > 0.1$, not significant even at the 10% level so insufficient evidence to reject H_0.

8. a. Let V denote the number of admissions to the emergency room during Vicky's shift.

Assuming the average rate is the same, $V \sim \text{Po}(4 \times 4.7)$.

Hence $P(V \geq 20) = 0.421$

b. Let S denote the number of Saturday evening shifts on which there are at least 20 admissions to the emergency room. Then assuming that 0.421 is constant for all 5 shifts, and that the number of admissions on each evening is independent, then $S \sim \text{B}(5, 0.421)$. $P(S = 2) = 0.344$ is the maximum of the probabilities, so the most likely number of evenings is 2.

9. a. Po(8.1), this is assuming that the number of problems in office A is independent of the number of problems in office B.

b. H_0: $\mu = 2 \times 8.1 = 16.2$ H_1: $\mu < 16.2$

$P(X \leq 12 \mid \mu = 16.2) \approx 0.180 > 0.05$

Hence insufficient evidence at the 5% level to reject H_0 that the mean rate has stayed the same.

10. a. Let F denote the number of flaws in a silk sheet of area 10 square metres.

Then $F \sim \text{Po}(1.8)$. $P(F \geq 1) = 0.8347$

b. Let R denote the profit from one silk sheet of area 10 square metres.

Then $\text{E}(R) = 350 \times P(F = 0) + 150 \times P(F = 1) - 100 \times P(F \geq 2) = €48.77$

Hence the expected profit from 70 sheets is €3400 to the nearest 100 Euros.

11. a. Quota

b. $\frac{20}{50} = 0.40$

c. $H_0: p = 0.2$ $\qquad$ $H_1: p > 0.2$

Let X be the number of people who respond to the survey and under H_0 assume $X \sim B(50, 0.2)$

$P(X \geq 20) = 0.000\,932 < 0.01$

Hence, sufficient evidence to reject H_0 and to accept the alternative that the proportion of people who would visit the coffee shop is greater than 0.2

d. The question is too vague. It does not ask how often they might visit the coffee shop.

12. a. Let $\bar{A}$ denote the mean height of the 70 corn plants fertilized with type A.

Then $\bar{A} \sim N\left(253, \frac{10^2}{70}\right)$. Hence $P\left(\bar{A} > 255\right) = 0.0471$.

b. Let $\bar{B}$ denote the mean height of the 80 corn plants fertilized with type B.

Then $\bar{B} \sim N\left(250, \frac{12^2}{80}\right)$. The required probability is $P\left(\bar{A} > \bar{B}\right) = P\left(\bar{A} - \bar{B} > 0\right)$.

Let $D = \bar{A} - \bar{B}$. Then $D \sim N\left(253 - 250, \frac{10^2}{70} + \frac{12^2}{80}\right)$. Hence $P(D > 0) = 0.953$.

Topic 4 HL Paper 1, Group 3

13. a. $P\left(\bar{X} < a\right) = 0.05$

$\bar{X} \sim N\left(1000, \frac{50}{\sqrt{15}}\right)$

$a \approx 979$

b. H_0 is accepted if $\bar{X} > 979$ g

$P\left(\bar{X} > 979 \mid \mu = 980\right) = 0.531$

c. The number of times the machine fails is B(3, 0.469…)

$\left(1 - (1 - 0.469...)^3\right) \approx 0.850$

14. a.

$$\mathbf{T} = \begin{array}{c} \\ \text{Future phone} \end{array} \begin{array}{c} \\ \text{Pi} \\ \text{Mu} \\ \text{Fi} \end{array} \overset{\substack{\text{Current phone} \\ \text{Pi} \quad \text{Mu} \quad \text{Fi}}}{\begin{pmatrix} 0.76 & 0.11 & 0.2 \\ 0.10 & 0.7 & 0.05 \\ 0.14 & 0.19 & 0.75 \end{pmatrix}}$$

b. $\begin{pmatrix} 0.76 & 0.11 & 0.2 \\ 0.10 & 0.7 & 0.05 \\ 0.14 & 0.19 & 0.75 \end{pmatrix}^3 \begin{pmatrix} 0.32 \\ 0.4 \\ 0.28 \end{pmatrix} = \begin{pmatrix} 0.378 \\ 0.25 \\ 0.372 \end{pmatrix}$.

Hence Fi's claim is not correct because Pi are predicted to have a slightly larger share of the market, assuming that the current transition probabilities remain applicable for the three years.

15. a. i. The curve has a single maximum.

ii. The curve has a maximum and is not symmetric about the maximum.

b. For $y = -0.4x^2 + 2.9x$

$$SS_{res} = (2.5-2)^2 + (5.1-4.5)^2 + (4.5-4.3)^2 + (3.0-1.7)^2$$

$$= 0.5^2 + 0.6^2 + 0.2^2 + 1.3^2 \approx 2.34$$

For $y = -0.155x^3 + 0.944x^2$

$$SS_{res} = (0.789-2)^2 + (4.311-4.5)^2 + (4.225-4.3)^2 + (0.504-1.7)^2$$

$$= 1.21^2 + 0.189^2 + 0.075^2 + 1.196^2 \approx 2.94$$

Hence $y = -0.4x^2 + 2.9x$ is the better fit.

c. $y = -0.0992x^3 + 0.555x^2 + 0.319x + 1.23$

d. $R^2 = 1$, the curve is a perfect fit to the points.

e. When $t = 0$ the height is 0 for both the given curves but is equal to 1.23 for the regression curve.

16. a. T denote the total mass of the 5 randomly chosen strawberries.

Then $T = T_1 + T_2 + T_3 + T_4 + T_5$.

Consequently $T \sim \mathrm{N}(60, 5 \times 2.7^2)$. Hence $P(T \geq 70) = 0.0488$

b. Let S denote the mass of the randomly chosen strawberry and R denote the mass of the randomly chosen raspberry.

Then the probability required is $P(S > 4R) = P(S - 4R > 0)$.

Consider the random variable $Y = S - 4R$. Assuming S and R are independent,

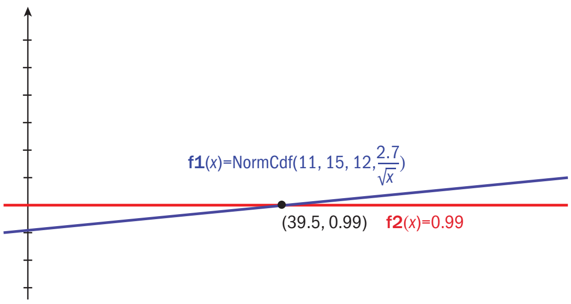
$$Y \sim \mathrm{N}(12 - 4 \times 4, 2.7^2 + 16 \times 0.5^2), \text{ giving } Y \sim \mathrm{N}(-4, 11.29).$$

Therefore the probability required is 0.3616.

c. Let $\bar{X}$ denote the mean weight of the n strawberries. Then $\bar{X} \sim \mathrm{N}\left(12, \frac{2.7^2}{n}\right)$.

If the probability the container is rejected is 0.01 then $P(11 < \bar{X} < 15) = 0.99$.

Applying technology it can be seen that $n = 39$ or that $n = 40$ both give the required probability to two decimal places.



17. a. Total number of posters accepted = 124, proportion = $\frac{124}{250} = 0.496$

(or $\bar{x} = 5p = 2.48 \Rightarrow p = 0.496$)

b.

Number of posters accepted	0	1	2	3	4	5
Expected frequency	1.626	8.001	15.748	15.498	7.626	1.501

Combining columns:

Number of posters accepted	0&1	2	3	4&5
Observed frequency	15	10	7	18
Expected frequency	9.627	15.748	15.498	9.127

H_0: The results follow a binomial distribution.

H_1: The results do not follow a binomial distribution.

Degrees of freedom = 4 –1 –1 = 2

p-value = 0.000 102 < 0.05 so the null hypothesis is rejected.

The results do not follow a binomial distribution and hence this is likely to mean that the selection of posters is not independent of the sample they are in.

18. a. Assuming that in the following season the sunflowers grow according to the same distribution, then the heights of the sunflowers H follow a normal distribution $H \sim \mathrm{N}(189.5, 15.3^2)$.

Hence the expected number of sunflowers growing higher than 195 cm is

$310 \times P(H > 195) = 111.$

b. The heights Izzy measured were 3 cm too large.

Consider the random variable $C = H - 3$.

This would transform the data to the correct values for the heights of the sunflowers.

$\mathrm{E}(H-3) = 189.5 - 3 = 186.5$

$\mathrm{Var}(H-3) = \mathrm{Var}(H) = 15.3^2 = 234$

c. The true answer to (a) is therefore 90.

Topic 4 HL Paper 2

1. a. 0.916, a strong positive correlation

b. H_0: $\rho = 0$, H_1: $\rho \neq 0$

p-value = 0.0290 < 0.05 hence sufficient evidence at the 5% significance level to reject H_0 and to accept that there is sufficient evidence of correlation between the results of the two surveys.

c. Test-retest

d. This is a paired-sample t-test.

Employee	A	B	C	D	E
First survey	7.2	4.1	6.1	5.4	3.9
Second survey	7.3	5.2	6.3	6.7	4.2
Difference	0.1	1.1	0.2	1.3	0.3

Let μ_D be the mean difference.

H_0: $\mu_D = 0$, H_1: $\mu_D > 0$

p-value = 0.0368 < 0.05, hence sufficient evidence at the 5% level to accept H_1 that the cafeteria food has improved.

2. a. Let H denote the weight of a randomly chosen item of hand luggage.

Then $H \sim N(9.4, 2.8^2)$. Hence $P(9 < H < 12) = 0.380$

b. Assuming that the weights of the 8 items of hand luggage are independent of each other, let T denote the total weight of 8 randomly chosen items of hand luggage.

Then $T \sim N(8 \times 9.4, 8 \times 2.8^2)$.

Hence $P(T > 100) = 0.000\ 870$.

c. Let L denote the weight of a randomly chosen large case.

Then $L \sim N(25, 7^2)$.

The event is $P(L > 3H) = P(L - 3H > 0)$.

The random variable $(L - 3H)$ has distribution

$(L - 3H) \sim N(25 - 3 \times 9.4,\ \ 2.8^2 + 9 \times 7^2)$.

Hence $P(L - 3H > 0) = 0.440$.

d. Let S denote the total weight of the randomly chosen sample of three large cases and two items of hand luggage.

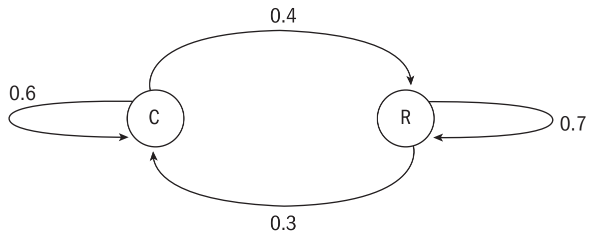
Then $S \sim N(2 \times 9.4 + 3 \times 25, 2 \times 2.8^2 + 3 \times 7^2)$.

The probability required is $P(S > 105) = 0.190$.

Topic 4 HL Paper 3

1. a. This context can be modelled with a Markov chain because the probabilities of transitioning from each current state to each future state depend what the current state is.

b.



c. Current state

$$\mathbf{G} = \begin{array}{c} \text{Future} \\ \text{state} \end{array} \begin{array}{c} C \\ R \end{array} \overset{\begin{array}{cc} C & R \end{array}}{\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}}$$

d. $\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}\begin{pmatrix} 17\,530 \\ 8956 \end{pmatrix} = \begin{pmatrix} 13\,204.8 \\ 13\,281.2 \end{pmatrix}$, so after one year, ClearGym will have 13 200 customers and ResultsNow will have 13 300 customers, to the nearest 100 customers.

e. $\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0.6-\lambda & 0.3 \\ 0.4 & 0.7-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow (0.6-\lambda)(0.7-\lambda) - 0.12 = 0$$

$$\Rightarrow 10\lambda^2 - 13\lambda + 3 = 0 \Rightarrow \lambda = 0.3 \text{ or } \lambda = 1$$

$$\lambda = 1 \Rightarrow \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 0.4x - 0.3y = 0 \Rightarrow 3y = 4x$$

Hence the eigenvector associated with $\lambda = 1$ is $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

$$\lambda = 0.3 \Rightarrow \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.3\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow x + y = 0 \Rightarrow y = -x$$

Hence the eigenvector associated with $\lambda = 0.3$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

f. Hence $\mathbf{G} = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix}\begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{-3}{7} \end{pmatrix}$

g. $\mathbf{G}^n = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix}^n\begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{-3}{7} \end{pmatrix}$

$$= \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1^n & 0 \\ 0 & 0.3^n \end{pmatrix}\begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{-3}{7} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0.3^n \\ 4 & -0.3^n \end{pmatrix}\begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{-3}{7} \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 3 & 0.3^n \\ 4 & -0.3^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -3 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 3+4(0.3^n) & 3-3(0.3^n) \\ 4-4(0.3^n) & 4+3(0.3^n) \end{pmatrix}$$