SL PRACTICE QUESTIONS

PAPER 1, GROUP 1

1. A cuboid has width n, length $n+4$ and height $2n+1$, where $n \in \mathbb{Z}^+$

a. Show that the volume V of the cuboid can be written as $V = 2n^3 + 9n^2 + 4n$

b. Hence or otherwise, find the value of n for which $V = 12\,495$

2. A bowl is in the shape of a hemisphere. Its radius is measured as 6.3 cm, correct to 1 decimal place.

a. Determine the upper and lower bounds of the radius of the hemisphere.

b. Hence, find the maximum percentage error of the volume of the hemisphere.

3. When David joins a gym in January 2021, the monthly fee is £34. The fee increases by 2% every **two** years.

a. Calculate how much the monthly fee will be in February 2026.

b. Find the **total amount** David will pay for his gym membership if he stays for exactly 7 years.

4. A sequence of 80 cylindrical concrete columns is planned in the garden of a new university campus. The height of each column is 10 m and the radii are in an arithmetic sequence with first term 0.5 m and common difference 0.05 m.

a. Find the first column to have volume greater than 100 m^3.

b. Find the total amount of concrete needed for the 80 columns, to the nearest m^3.

5. Erin measures the height, width and length of a cuboid-shaped food container as 254 mm, 65 mm and 154 mm respectively.

a. Using these measurements, find the volume of the container in mm^3, giving your answer correct to two significant figures.

b. Write down your answer to part **a** in the form $a \times 10^n$, $1 \le a < 10$, $n \in \mathbb{Z}$

c. Given that the measurements of height, width and length are correct to the nearest mm, determine if the container will definitely be able to hold 2.5 litres of washing liquid.

6. Callum is saving money to buy a digital camera. He invests p Euros (€) in an account that pays an annual interest rate of 4.3%, compounded monthly.
After 6 years he has €2069.99 in his account.

a. Calculate the value of p. Give your answer correct to 2 decimal places.

Callum bought a digital camera for €2000 and sold it six years later for €900.

b. Find the rate at which the value of camera depreciated each year.

7. In this question, assume that the distance from Earth to Mars is 252.65 million km and that the space probe travels in a straight line.

a. Find the distance to Mars in metres and express your answer in standard form.

A space probe leaves Earth at a speed of 60 000 kilometres per hour.

b. Find the speed of the space probe in metres per second, and express your answer in the form $a \times 10^n$, $1 \le a < 10$, $n \in \mathbb{Z}$.

c. Use your answers to parts **a** and **b** to find the time taken in seconds for the probe to travel the distance to Mars.

GROUP 2

8. Pilar deposits $17,000 in a bank account which pays a nominal interest rate of 7%, compounded yearly.

a. Find how much interest Pilar has earned after 6 years.

Ximena deposited $20,000 in a bank account. Her account pays a nominal annual interest rate of r %, compounded monthly. After five years, the total amount in Ximena's account is $24,351.19

b. Find r, correct to 2 decimal places.

9. Eito is exploring the mathematics of earthquakes. The magnitude R of an earthquake can be modelled by the formula $R = \log_{10}\left(\frac{x}{0.001}\right)$, where x is a reading from a seismograph in mm which represents the size of the movement in the Earth caused by the earthquake.

a. Find the magnitude of an earthquake for which $x = 52\,098$ mm.

b. An earthquake has a magnitude of 8.9. Find the size of the movement in the Earth, x, correct to the nearest mm.

10. The distance from the Earth to the Sun is approximately 149 600 000 km.

a. Find the distance from the Earth to the Sun in cm.

b. Write your answer to part **a** in the form $a \times 10^n,\ 1 \le a < 10,\ n \in \mathbb{Z}$

Madita draws a graph of the function $f(x) = 2^x$, $-2 \le x \le p$ with a scale 1 cm = 1 unit on each axis.

c. The distance from the Earth to the Sun is equal to $f(p)$. Find p correct to the nearest centimetre.

11. Magzhan is going to train for a marathon. He plans to run 4 km in his first week of training, then increase the amount he runs by 0.5 km every week.

a. Find how many km Magzhan will run in his 12th week of training.

b. Using sigma notation, write down an expression for the total distance Magzhan has run after 20 weeks in training, and calculate this distance.

c. A marathon is approximately 42 km long. How many weeks of training must Magzhan complete before he runs at least 42 km in one week of training?

12. Bicycle sharing app Ebike is analysing its business performance in a city.

In 2015, the number of bicycles hired by Ebike customers was 1530. In 2018, the figure was 1624.

a. Assuming that the number of hires follows a geometric sequence, predict the number of hires in 2025 to the nearest whole number.

b. Predict the year in which the number of hires first exceeds 2000.

13. A new café opens in Galoisville. The owners invest €17,500 in the kitchen equipment. The value of the equipment decreases by 5% each year.

a. Find the value of the kitchen equipment after seven years.

b. The managers decide to re-fit the kitchen when the value of the equipment falls below 40% of its original value. Find how many complete years will pass before the kitchen is re-fitted.

GROUP 3

14. Three functions g, f and h are defined for $n \in \mathbb{Z}^+$ as:

$g(n) = \sum_{k=1}^{n}(5.7 + 3k)$, $f(n) = \sum_{k=1}^{n}\left(-0.5 + 0.7^k\right)$ and

$h(n) = \sum_{k=1}^{n}\left(9.3 \times 1.1^k\right)$

a. Identify which function represents an arithmetic series and justify your answer.

b. Hence write down the first term and common difference of the arithmetic sequence.

c. Identify which function represents a geometric series and justify your answer.

d. Hence write down the first term and common ratio of the geometric sequence.

e. Find the least value of q for which $h(n) > g(n)$ for all $n > q$.

15. A trip to a remote jungle region is planned. 68 people and 80 cases must be transported on narrow roads. The organizers of the trip can hire cars or vans. Each car carries 4 people and 3 cases, and each van can carry 7 people and 10 cases.

a. If v is the number of vans hired, and c the number of cars hired, write down a system of two equations that represent this context.

b. Use technology to solve the system.

c. Hence find the number of vans and the number of cars that should be hired, given that a van costs more to hire than a car.

16. Emilia is comparing loan deals. She wishes to buy an apartment for £250,500.

Bank A requires a 10% deposit, then a loan for 20 years at a nominal interest rate of 1.3%, compounded monthly. Repayments are made each month.

Bank B requires a 15% deposit, then a loan for 25 years at a nominal interest rate of 1.1%, compounded monthly. Repayments are made each month.

a. Calculate the monthly repayments for each loan.

b. Hence find the total repayment for each loan.

c. Hence describe the advantages and disadvantages of Emilia choosing the deal from Bank A in preference to the deal from Bank B.

17. Teodora is planning a rock garden in the shape of a triangle ABC. To find side c she uses the formula $c=\sqrt{a^2+b^2-2ab\cos C}$.

a. If $a=10$ m, $b=\sqrt{7}$ m and $C=60°$, find the exact value of c.

b. If Teodora approximates $\sqrt{7}$ m to three significant figures and uses this value instead of $\sqrt{7}$, find the percentage error in the value of c.

c. If $a=10$ m and $b=\sqrt{7}$ m are exact, but $C=60°$ is accurate only to the nearest degree, find the maximum percentage error in the value of c.

18. Chimdi invests \$9500 in a savings account that pays a nominal annual interest rate of 5%, compounded quarterly.

a. Find how many years Chimdi must wait until the value of her investment doubles.

Nicole invests \$9500 in a saving account which pays a nominal interest rate of r%, compounded monthly.

b. Find the minimum nominal annual interest rate needed for Nicole to double her money in 12 years.

PAPER 2

1. Denise works for a car sharing subscription scheme. On her first day at work, she sells y subscriptions. Her manager asks her to increase the number of subscriptions she sells by the same amount, x, each working day.

Denise sells 27 subscriptions on the sixth working day, and 41 on the thirteenth working day.

a. Write down a system of equations, in terms of y and x, for the number of subscriptions Denise sells on the sixth day and for the number of subscriptions Denise sells on the thirteenth day, stating any assumptions you make.

b. Hence, predict the value of y and the value of x.

Assume that on each of her first 32 working days, Denise increases the number of subscriptions she sells by at least 1 but no more than 3 each day and that she sells y subscriptions on her first day.

c. Determine if Denise will sell at least 1000 subscriptions in total in her first 32 working days.

2. Alex has €21,000 to invest. He considers three options:

Option A: Invest in a savings account paying an annual interest rate of 4%, compounded yearly

Option B: Invest in a savings account paying a nominal annual interest rate of 3.8%, compounded monthly

Option C: Invest in a fund that guarantees a payment of €1100 per year.

a. To the nearest €, calculate the value of Alex's investment after one year if he chooses:

i. Option A

ii. Option B

iii. Option C

b. Write down an expression for the value V of Alex's investment after n years if he chooses:

i. Option A

ii. Option B

iii. Option C

Alex is told by his friend Beth that his investment will grow in value faster in Option B than in Option A, since the compounding periods are more frequent.

c. Determine if Beth's claim is correct and justify your answer.

d. Find the number of complete years for which Alex's investment would be worth more in **both** Options A and B than in Option C.

e. Find, to the nearest €, the difference in the total interest Alex will earn after 20 years by choosing option A instead of Option C.

HL PRACTICE QUESTIONS

PAPER 1, GROUP 1

1. a. Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{C}=\begin{pmatrix}2 & 4\\ 1.5 & 1\end{pmatrix}$

   b. Hence, write down matrices **P** and **D** such that $\mathbf{C}=\mathbf{PDP}^{-1}$

   c. Find a general expression for $\mathbf{C}^n$ in terms of n.

2. The magnitude R of an earthquake on the Richter scale is measured by the formula $R=\log_{10}\left(\frac{I}{I_0}\right)$ where I is the intensity of the earthquake and I_0 is the intensity of ground movement on a normal day.

   a. The San Francisco, California earthquake of 1906 measured 8.25 on the Richter scale, and the earthquake near Dashur, Egypt in 1992 measured 5.9 on the Richter scale. Calculate how many times more intense the San Francisco earthquake was in comparison to the Dashur earthquake.

   b. The earthquake near Japan's north-east coast in 2011 was 7.08 times more intense than the San Francisco earthquake of 1906. Calculate the measure of the Japanese earthquake on the Richter scale.

3. a. Show that the number 0.151 515 151 515 can be written as

   $$\sum_{r=1}^{6}0.15\times10^{2-2n}$$

   b. Show that the number $0.\overline{51}$ can be written as

   $$\sum_{r=1}^{\infty}0.51\times(0.01)^{r-1}$$

   c. Hence find the exact value of $0.\overline{51}$

4. Solve the equation $\frac{z}{z+2\text{i}}=4-7\text{i}$

5. a. Find the exact values of the roots of $3x^2+x-7=0$

   b. Find the exact values of the roots of $3x^2+x+7=0$

   c. Verify that for each equation, the conjecture below is true:

   The roots of $ax^2+bx+c=0$ add to $\frac{-b}{a}$

6. Simplify these algebraic expressions:

   a. $\sqrt[3]{9x^{14}\times375x^{10}\times\left(2x^2\right)^3}$

   b. $\left(\frac{729x^{12}}{64y^3}\right)^{\frac{1}{3}}$

   c. $\frac{\left(5y^{-1}x^2\right)^3}{\left(3y^2x^{-4}\right)^5}$

GROUP 2

7. Given $\omega_1=2\text{cis}\left(\frac{\pi}{4}\right)$ and $\omega_2=3\text{cis}\left(\frac{-\pi}{6}\right)$, find expressions in Euler form for the following:

   a. $\omega_1\omega_2$

   b. $\left(\frac{\omega_1}{\omega_2}\right)^4$

   c. $\left(\omega_1^*\right)^3$

8. Two voltage sources V_1 and V_2 are connected in a circuit so that the total voltage is $V=V_1+V_2$. If $V_1=7\sin(20t+5)$ and $V_2=3\sin(20t+13)$, find an expression for the total voltage in the form $V=A\sin(20t+B)$.

9. $\mathbf{A}=\begin{pmatrix}2 & -2\\ 3 & 0\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}0 & 5\\ 2 & 1\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}0 & \pi\\ -3 & 0\end{pmatrix}$, and

   $\mathbf{D}=\begin{pmatrix}4 & 0\\ -3 & \sqrt{3}\end{pmatrix}$

   a. Find the 2×2 matrix **X** that satisfies the equation $\mathbf{XA}+\mathbf{XB}=\mathbf{C}-3\mathbf{D}$.

   b. Solve the equation $\mathbf{BXA}=\mathbf{D}$, expressing all answers as matrices in the form $q\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, where $q\in\mathbb{Q}$ and $a,b,c,d\in\mathbb{R}$ are all exact values.

10. A geometric series has common ratio $1.5\left(2^x\right)$

    a. Find the values of x for which the sum to infinity of the series exists.

    b. If the first term of the sequence is 13, and the sum to infinity of the series is 16, find the value of x.

11. $\log_4 a$, $\log_4(a+9)$ and $\log_4(a+20)$ are the first three terms of an arithmetic sequence.

    a. Find the value of a.

    b. Hence, find the sum of the first 20 terms of the sequence.

12. Consider $\omega = \text{cis}\left(\frac{\pi}{3}\right)$

a. Given that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, express w in Cartesian form.

b. Find ω^2 in Cartesian form and in modulus-argument form. Hence, find exact values for $\cos\left(\frac{2\pi}{3}\right)$ and $\sin\left(\frac{2\pi}{3}\right)$.

c. Express each element of the set $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5\}$ in modulus-argument form.

d. Hence, with reference to an Argand diagram or otherwise, show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = 0$

GROUP 3

13. a. Given that $\log_3(ax+b) = 2 + 2\log_3(ax-b)$, use the laws of logarithms to find p, q and r in terms of a and b in the quadratic equation $px^2 + qx + c = 0$.

b. Find the exact values of r and s if the piecewise function $f(x)$ is continuous:

$$f(x) = \begin{cases} e^{sx} & x < 1 \\ 3 + 2x^3 & 1 \le x < 2 \\ \ln(x - r) & x \ge 2 \end{cases}$$

c. Find the exact solution of the equation $4^{2x+1} = 3^{1-x}$. Express the answer in the form $\frac{\ln a}{\ln b}$, where $a, b \in \mathbb{Q}$.

14. a. Given $z = 2 + 2\sqrt{3}\text{i}$ and $\omega = \sqrt{2} - \sqrt{2}\text{i}$, find zw and $\frac{z}{w}$ in Cartesian form.

b. Express z and ω in Euler form. Hence, find zw and $\frac{z}{w}$ in Euler form.

c. By comparing the two forms of zw, find the exact value of $\cos\left(\frac{\pi}{12}\right)$

15. The organizers of a sports league are investigating a proposed change to how points are awarded.

Three teams T_1, T_2 and T_3 have their results (win, draw or lose) represented in the matrix **R**:

$$\mathbf{R} = \begin{matrix} T_1 \\ T_2 \\ T_3 \end{matrix} \overset{\begin{matrix} \text{win} & \text{draw} & \text{lose} \end{matrix}}{\begin{pmatrix} 8 & 3 & 1 \\ 4 & 5 & 3 \\ 2 & 4 & 6 \end{pmatrix}}$$

The proposed changes, p_1 and p_2, to the points awarded are represented in the matrix **P**:

$$\mathbf{P} = \begin{matrix} \text{win} \\ \text{draw} \\ \text{lose} \end{matrix} \overset{\begin{matrix} p_1 & p_2 \end{matrix}}{\begin{pmatrix} 3 & 4 \\ 2 & 1 \\ -1 & 0 \end{pmatrix}}$$

a. Find the matrix **RP** and interpret the elements of the matrix.

The proposed changes, p_1 and p_2, to the points awarded have sponsorship payment plans s_1 and s_2 respectively, based on the number of points scored.

Matrix **E** shows the amount in £ that T_1, T_2 and T_3 would like to win under sponsorship plans s_1 and s_2.

$$\mathbf{E} = \begin{matrix} T_1 \\ T_2 \\ T_3 \end{matrix} \overset{\begin{matrix} s_1 & s_2 \end{matrix}}{\begin{pmatrix} 12{,}000 & 15{,}600 \\ 10{,}000 & 20{,}000 \\ 8000 & 14{,}000 \end{pmatrix}}$$

The sponsorship payments for each point awarded under changes p_1 and p_2 are represented by the matrix **S**.

$$\mathbf{S} = \begin{matrix} p_1 \\ p_2 \end{matrix} \overset{\begin{matrix} s_1 & s_2 \end{matrix}}{\begin{pmatrix} 70 & 65 \\ 40 & 100 \end{pmatrix}}$$

b. Solve the equation $\mathbf{XS} = \mathbf{E}$ for a 3×2 matrix $\mathbf{X}$ and interpret its elements in context.

16. Solve the simultaneous equations

$$1 + 2\log_{10} y = \log_{10} 10x$$

$$1 + \log_{10} x = \log_{10}(7 - 20y)$$

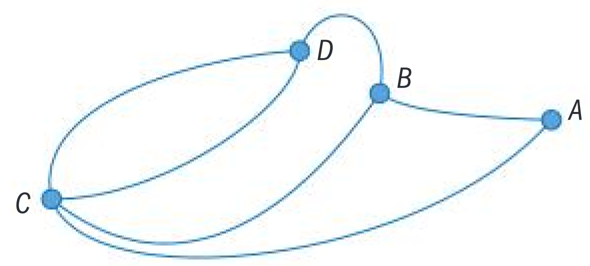
17. Solve the simultaneous equations

$$\ln\left(\frac{y^2 z}{x}\right) = 3$$

$$\ln(x^2) + \ln(y^3) = 10$$

$$\ln\left(\sqrt{xyz}\right) = -1$$

18. The sketch below shows the roads connecting four villages A, B, C and D.



The matrix **P** shows the number of one-stage journeys between the four villages:

$$\mathbf{P} = \text{from}\begin{matrix} \\ A \\ B \\ C \\ D \end{matrix}\overset{\text{to}}{\begin{matrix} A & B & C & D \\ \end{matrix}}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

a. Complete the matrix **P**.

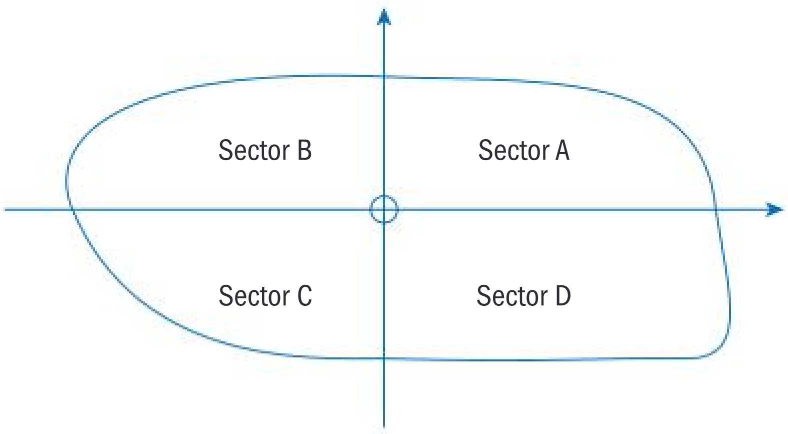
b. Find $\mathbf{P}^2$

c. Find the number of two-stage journeys from C to B and identify this number in the elements of $\mathbf{P}^2$.

d. Interpret the elements of $\mathbf{P}^2$.

PAPER 2

1. An Unmanned Aerial Vehicle (UAV) is surveying a large forested area, divided into sectors as shown.

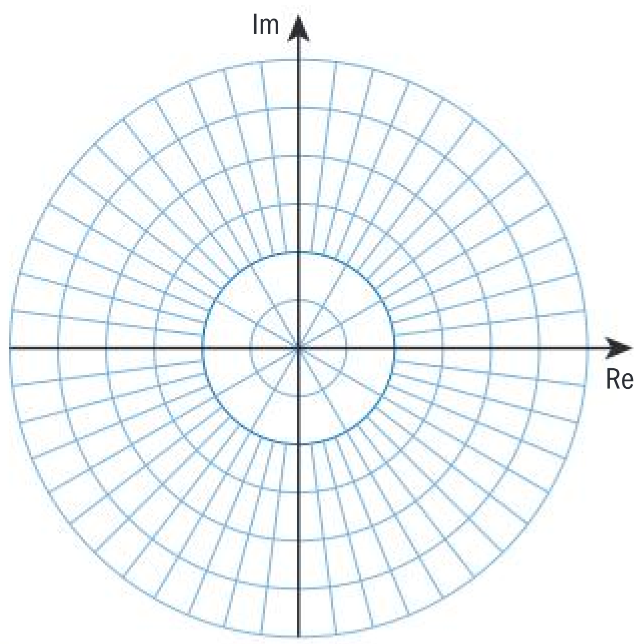


The position z after t hours of the UAV named *Zappy* is given by $z(t) = \left(1.5e^{i\frac{\pi}{3}}\right)^t$

Distances are measured in km.

a. Find the values of $z(0)$, $z(1)$, $z(2)$, $z(3)$, $z(4)$.

b. Hence, sketch the positions of *Zappy* at $t = 0, 1, 2, 3$ and 4 hours on this Argand diagram:



c. Determine after how many hours *Zappy* is first more than 50 km from its initial position **and** is in sector D.

d. If, at $t = 13$ hours, *Zappy* is programmed to return directly to its starting position, find the components of the vector that represents this journey.

e. A second UAV named *Alpha* takes off at the same time as *Zappy*. The position of *Alpha* after t hours is given by $a(t) = \left(\sqrt{1.51}e^{i\frac{\pi}{3}}\right)^{2t}$. Flight engineer Nico states that *Alpha* and *Zappy* are in danger of a near miss after 6 hours – meaning that they will be within 100 m of each other. Flight engineer Zank disagrees. Determine which flight engineer is correct.

2. The *BetaGamma* Information Technology store has two megastores, one in Cairo, Egypt and the other in Toulouse, France. Each megastore has three types of employee who have three different salaries, one for each type of employee. For the financial year of 2020, employees who work on the IT desk earn \$900 per week, those working in the sales room earn \$700 per week and those who work in the office earn \$500 per week.

The number of workers at each site is given in the following table:

	IT desk	Sales room	Office
Cairo	52	15	19
Toulouse	34	9	12

a. Construct and label a 3×2 matrix **E** to represent the number of employees of each type in each megastore.

b. i. What would the matrix $\mathbf{S} = (900 \quad 700 \quad 500)$ represent?

ii. What are the dimensions of **S**?

c. If $\mathbf{SE} = (66\,800 \quad x)$, calculate the value of x.

d. Describe what the value of x represents.

e. Find the total amount of money that *BetaGamma* must pay in one year to its employees in both megastores in the financial year of 2020.

The managers of *BetaGamma* predict that each year after 2020, all salaries will increase by 1% and one of each type of employee will retire in each megastore.

f. Predict the total amount of money that *BetaGamma* must pay in one year to its employees in both megastores in the financial year of 2022.

PAPER 3

1. *The aim of this question is to explore the long-term population of a colony of seals using matrices.*

The female population of a colony of seals can be divided into three groups: pups, young seals and adults. *Pups* are seals aged 0-1 years, *young seals* are those aged 1-2 years and an *adult seal* is any seal over 2 years old.

If, in year n, the population of female pups is p_n, female young seals is y_n and female adults is a_n, then the population of females in year $n+1$ can be found from:

$$\begin{pmatrix} p_{n+1} \\ y_{n+1} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{7}{6} & \frac{7}{6} \\ 0.5 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} p_n \\ y_n \\ a_n \end{pmatrix}$$

The matrix $\mathbf{L} = \begin{pmatrix} 0 & \frac{7}{6} & \frac{7}{6} \\ 0.5 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$ is called the Leslie matrix for the population. It is used to model the average change in the number of females in a population.

a. By considering the entries in **L**, write down:

i. the average number of young born to a young female

ii. the probability a pup survives and moves into the young seals group

iii. the fraction of adults that die each year.

Scientists begin measuring the population and their initial data is $p_0 = 20,\ y_0 = 15$ and $a_0 = 40$

b. Find the number of seals in each group one year later. Give your answers to the nearest integer.

c. Explain why the population after 3 years can be found from:

$$\begin{pmatrix} 0 & \frac{7}{6} & \frac{7}{6} \\ 0.5 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}^3 \begin{pmatrix} 20 \\ 15 \\ 40 \end{pmatrix}$$

Let T_n be the **total** female population after n years.

d. Use an extension of the above result to find:

i. T_{20}

ii. T_{30}

It is given that for $n \geq 20$ the sequence T_n is approximately geometric.

e. i. Use your answer to part **d** to find the common ratio for the sequence T_n for $n \geq 20$

ii. Hence, find when the total female population will first pass 50 000.

The vector $\boldsymbol{q}_n$ gives the proportion of the total population in each of the groups in year n.

f. Find the vector $\boldsymbol{q}_{30}$

It is given that each element of $\boldsymbol{q}_{30}$ is approximately equal to the corresponding element in $\boldsymbol{q}$, an eigenvector for **L**.

g. i. Use your answer to part **f** and the relation $\mathbf{L}\boldsymbol{q} = \lambda\boldsymbol{q}$ to find λ the corresponding eigenvalue.

ii. Comment on your answer.

SL PRACTICE QUESTIONS

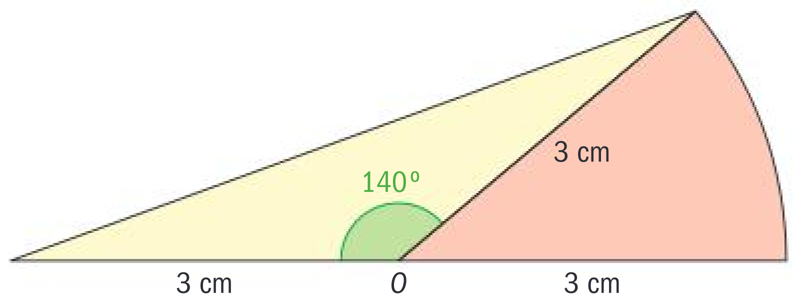
PAPER 1, GROUP 1

1. A logo for a sports team is made up of a sector of a circle and a triangle in two colours, as shown. The sector of the circle has a centre at O and the circle has a radius of 3 cm.

Find the area of:

a. the triangular section

b. the sector.



2. The coordinates of the summits of two adjacent mountains are (102.8, 39.1, 2.9) and (108.3, 42.2, 4.5), where all distances are in kilometres.

Find the straight-line distance between the two summits.

3. A cake is in the shape of a cylinder with a radius of 8 cm and a height of 10 cm.

a. Find the volume of the cake.

A piece of the cake is removed. The cross-section of the piece is a sector of a circle and forms an angle of 35° at the centre of the cake.

b. Find the volume of the piece.

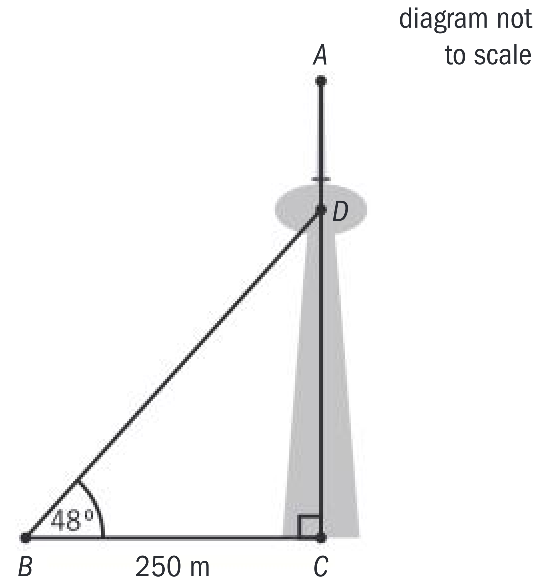
4. Let A be the point with coordinates (2, 8) and B be the point with coordinates (–1, 10).

a. Find the midpoint of the line segment joining A and B.

Let l be the perpendicular bisector of the line segment joining A and B.

b. Find the equation of l.

5. A is a point at the top of a vertical communications tower with its base at C. The tower has an observation deck D, three-quarters of the way to the top of the tower, A.

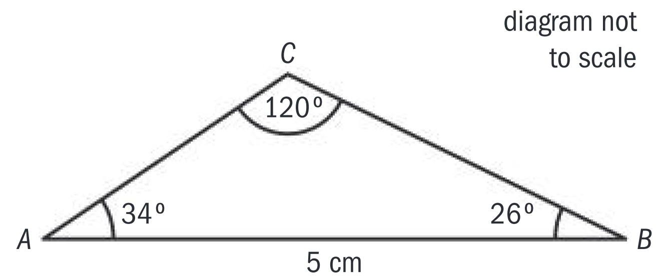


From point B, on horizontal ground 250 m from C, the angle of elevation to D is 48°

a. Calculate CD, the height of the observation deck above the ground.

b. Calculate the angle of depression from A to B.

6. A triangular postage stamp, ABC, is shown in the diagram below, such that $AB = 5$ cm, $B\hat{A}C = 34°$, $A\hat{B}C = 26°$ and $A\hat{C}B = 120°$

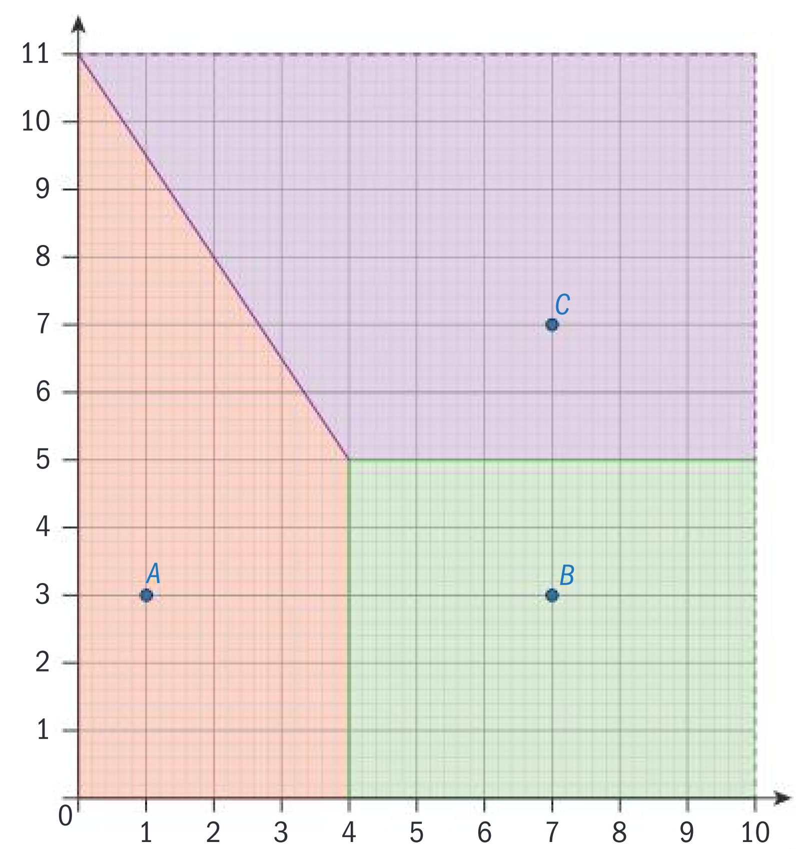


a. Find the length of BC.

b. Find the area of the postage stamp.

GROUP 2

7. The Voronoi diagram below shows three wells, A, B and C, set in a rectangular area of land. Each unit on the axes represents 1 km.



a. Write the equations of each of the lines forming the three boundaries within the Voronoi diagram.

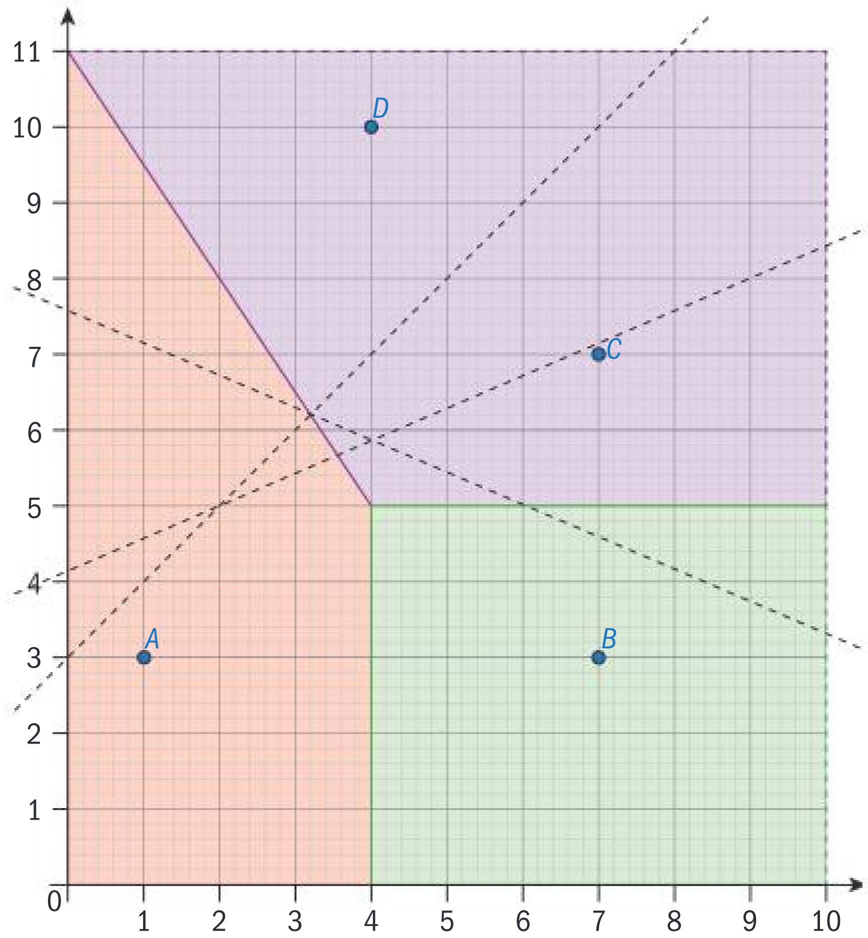
b. Find the area of the region containing:

i. A

ii. B

iii. C

A fourth well is created at the point D. The perpendicular bisectors between D and each of A, B and C are shown as dashed lines on the diagram.



c. On a copy of the diagram, complete the Voronoi diagram for the four wells.

8. An underground tunnel goes from point A with coordinates (10.1, 3.2, −0.8) to a point B with coordinates (8.5, 1.7, −0.9)

A relief tunnel is to go from a point on the surface at C(9.0, 2.0, 0) to the midpoint of the tunnel joining A and B. All positions are given in kilometres.

Find the length of the relief tunnel.

9. A geologist is trying to work out the approximate surface area of a triangular glacier. The coordinates of the three corners of the glacier are A(0, 0, 0), B(250, 120, 75) and C(−120, 230, 140), where distances are in metres.

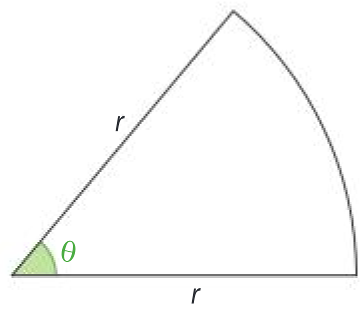
From C, he can see both A and B and measures the angle $A\hat{C}B$ as 47°

Find:

a. the distance from B to C

b. the area covered by the glacier.

10. Jorge wishes to bend a 20 cm length of wire into the shape of a circular sector. Let the radius of the sector be r cm and the angle made at the centre by the sector be θ.



a. Find an expression for θ in terms of r.

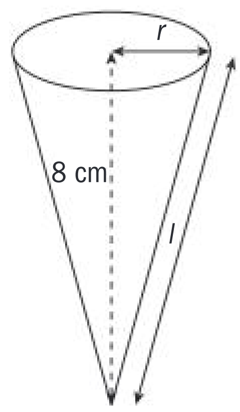
Jorge wants the sector to contain the largest possible area.

b. Show that the area of the sector A can be given by the expression:

$A = 10r - r^2$

c. Hence find the value of r which gives the maximum area.

11. A type of candy is packaged in a right circular cone with a volume of 100 cm^3 and vertical height of 8 cm.

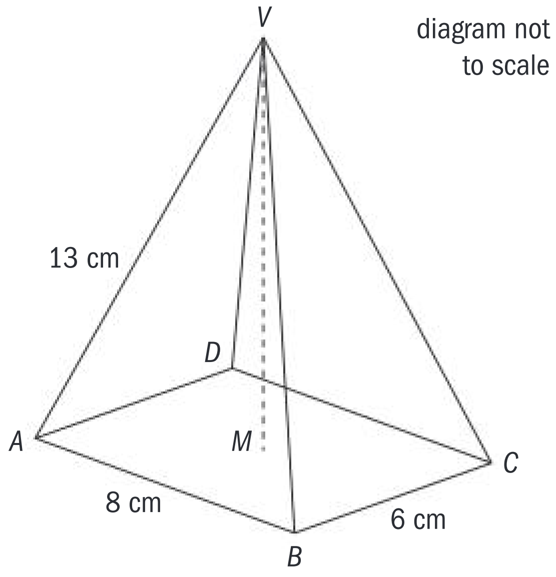


a. Find the radius, r, of the circular base of the cone.

b. Find the slant height, l, of the cone.

c. Find the curved surface area of the cone.

12. A right pyramid has apex V and rectangular base $ABCD$, with $AB = 8$ cm, $BC = 6$ cm and $VA = 13$ cm. The vertical height of the pyramid is VM.

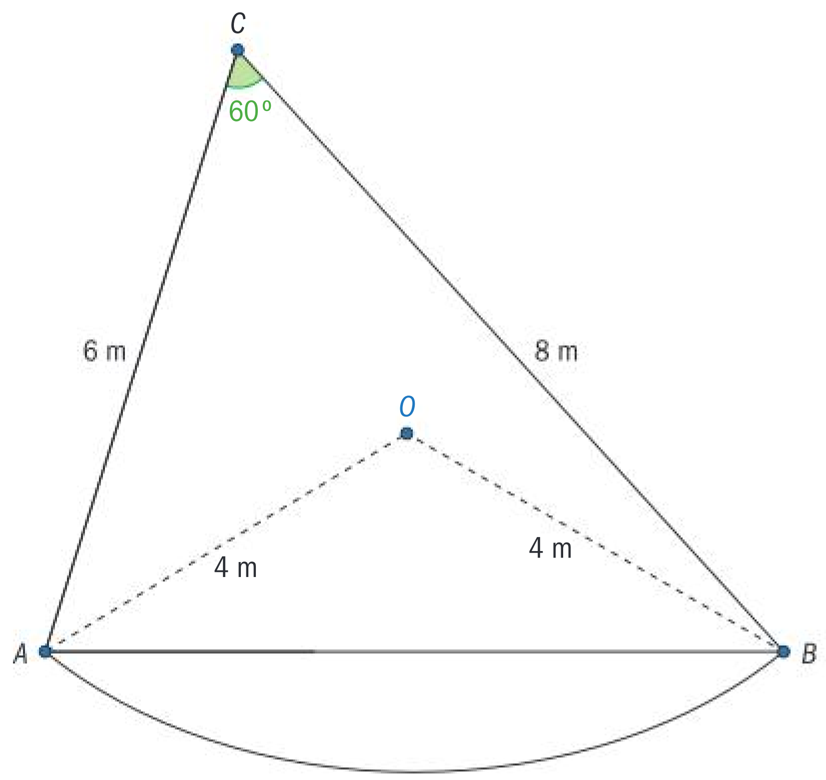


a. Calculate VM.

b. Calculate the volume of the pyramid.

GROUP 3

13. A garden consists of a lawn in the shape of a triangle, shown as ABC in the diagram, and a flower bed formed by the arc of a circle, centre O with a radius of 4 m. The length of AC is 6 m, the length of BC is 8 m and angle $A\hat{C}B = 60°$



The owner of the garden wants to buy some decorative edging to go around the flower bed.

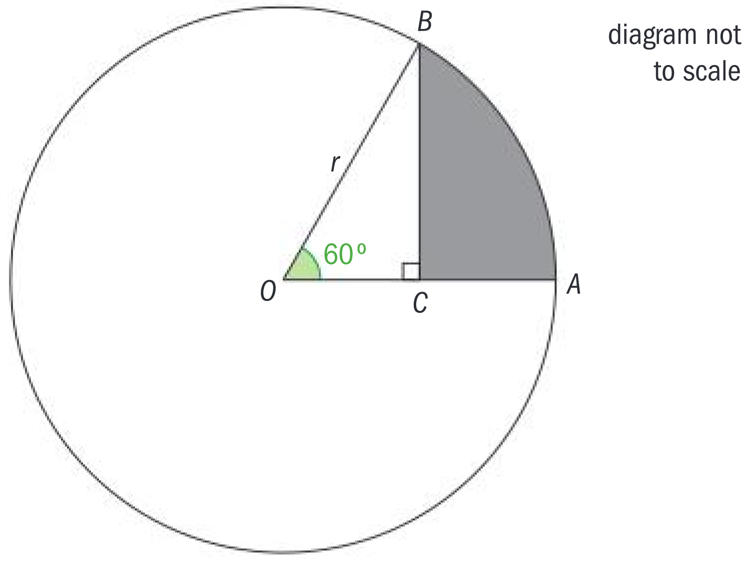
a. Find the length of the side AB.

b. Hence, find the length of edging the owner needs to buy.

14. The diagram shows a circle with centre O and radius r cm.

Points A and B lie on the circumference of the circle and $A\hat{O}B = 60°$.

The point C is on $[OA]$ such that $B\hat{C}O = 90°$.



a. Find an expression, in terms of r, for:

i. OC

ii. the area of the triangle OBC.

The area of the shaded region is 50 cm^3

b. Find the value of r.

15. A large sculpture is in the shape of a right square-based pyramid. Two adjacent corners of the base, A and B, have coordinates (0, 0, 0) and (8, 0, 0) respectively. The vertex of the pyramid, V, has coordinates (a, a, 15). All distances are in metres.

a. i. Write down the height of the pyramid.

ii. Find the volume of the pyramid.

b. Write down the value of a.

The pyramid has a light positioned half way up the triangular face which has $[AB]$ as its base, and is centred horizontally on that face.

c. Find the coordinates of the light.

The sides of the pyramid are covered in steel panels.

d. Find the area of the panels used.

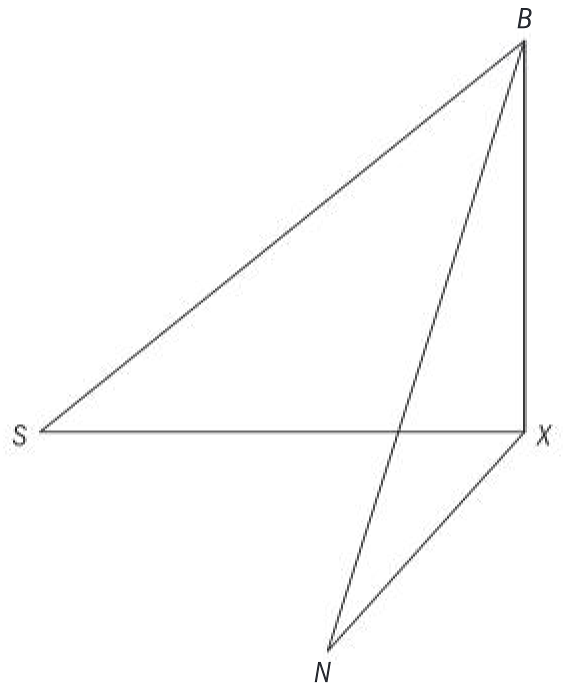
16. Barry is at the top of a cliff, standing 80 m above sea level. He observes two yachts in the sea.

Seaview (S) is at an angle of depression of 25°

Nauti Buoy (N) is at an angle of depression of 35°

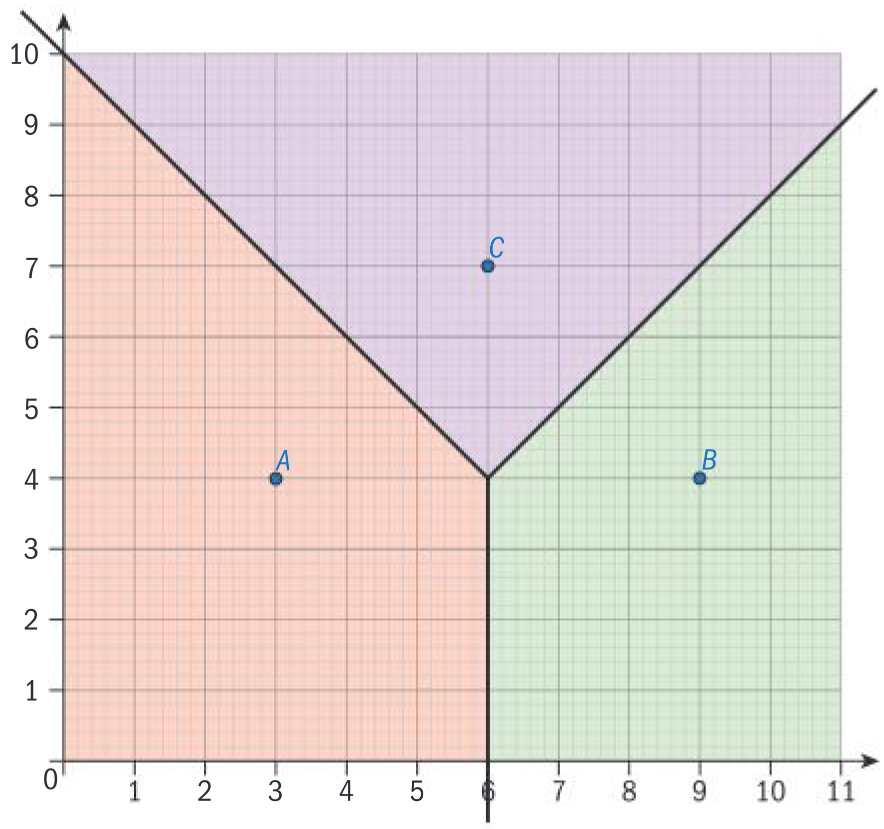
The three-dimensional diagram shows Barry (B) and the two yachts at S and N.

X lies at the foot of the cliff, and $S\hat{X}N = 70°$



Find the distance between the two yachts, accurate to three significant figures.

17. Pollution levels in a city are recorded at three environmental stations A, B and C. The environmental stations are shown on the Voronoi diagram with coordinates $A(3, 4)$, $B(9, 4)$ and $C(6, 7)$, where distances are given in kilometres.



On a particular day, the levels of pollution according to the Air Quality Health Index are recorded as 6.5 at A, 7.2 at B and 3.2 at C.

The level of pollution at any point in the city is taken to be the same as the value recorded at the nearest station to that point.

A house is situated at the point D with coordinates (6, 5)

a. State the level of pollution that would be assigned to D.

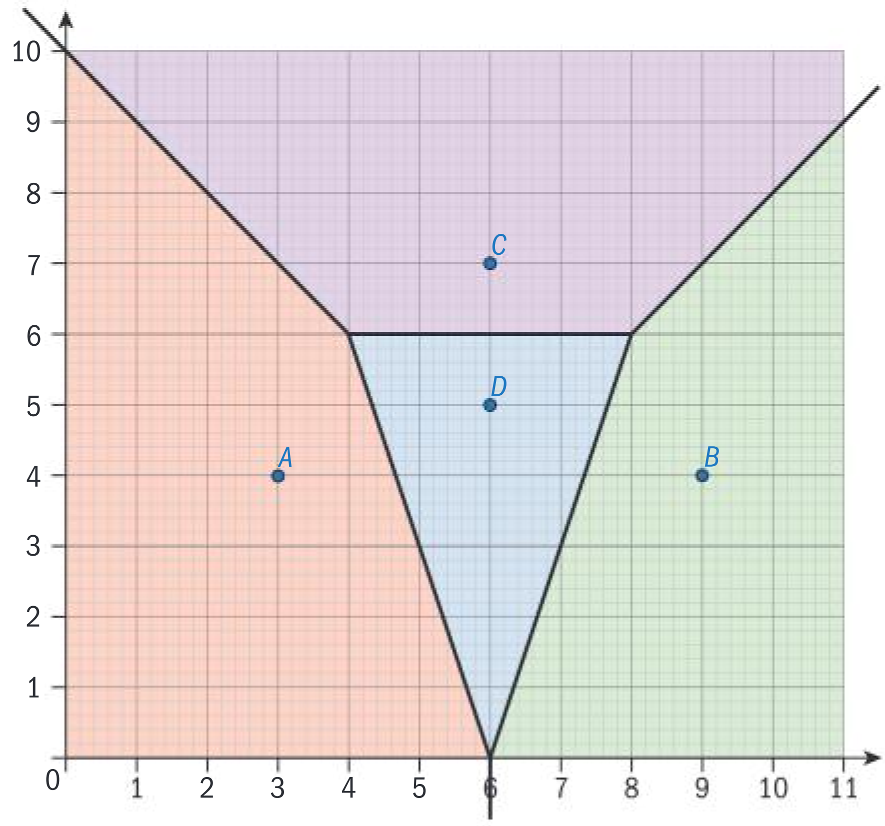
The house owner at D claims that the pollution levels are much worse than the assigned value.

An alternative method is suggested to give a more accurate measure. A new cell for the Voronoi diagram will be created around D. The level of pollution, P_D, for the point D will then be found using the formula:

$$P_D = \frac{6.5a_A + 7.2a_B + 3.2a_C}{a}$$

where a is the area of the new cell and a_A, a_B and a_C are the areas of the regions within the new cell that used to belong to stations A, B and C respectively.

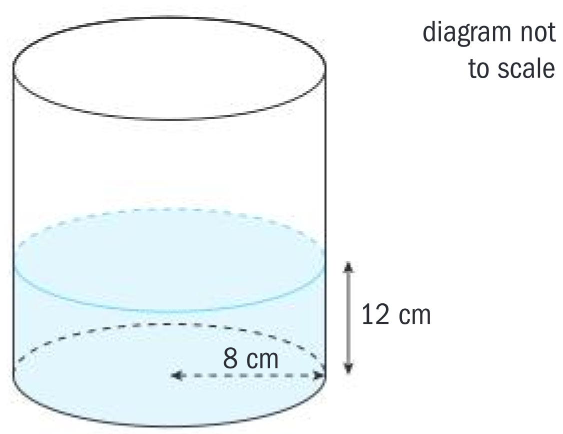
The new Voronoi diagram is shown below with vertices at (4, 6), (8, 6) and (6, 0).



b. Find the value of a

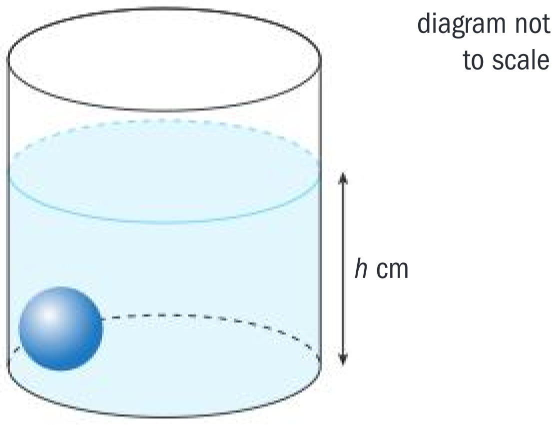
c. Find the value of P_D

18. A cylindrical container with a radius of 8 cm is placed on a flat surface. The container is filled with water to a height of 12 cm, as shown.



a. Find the volume of water in the container.

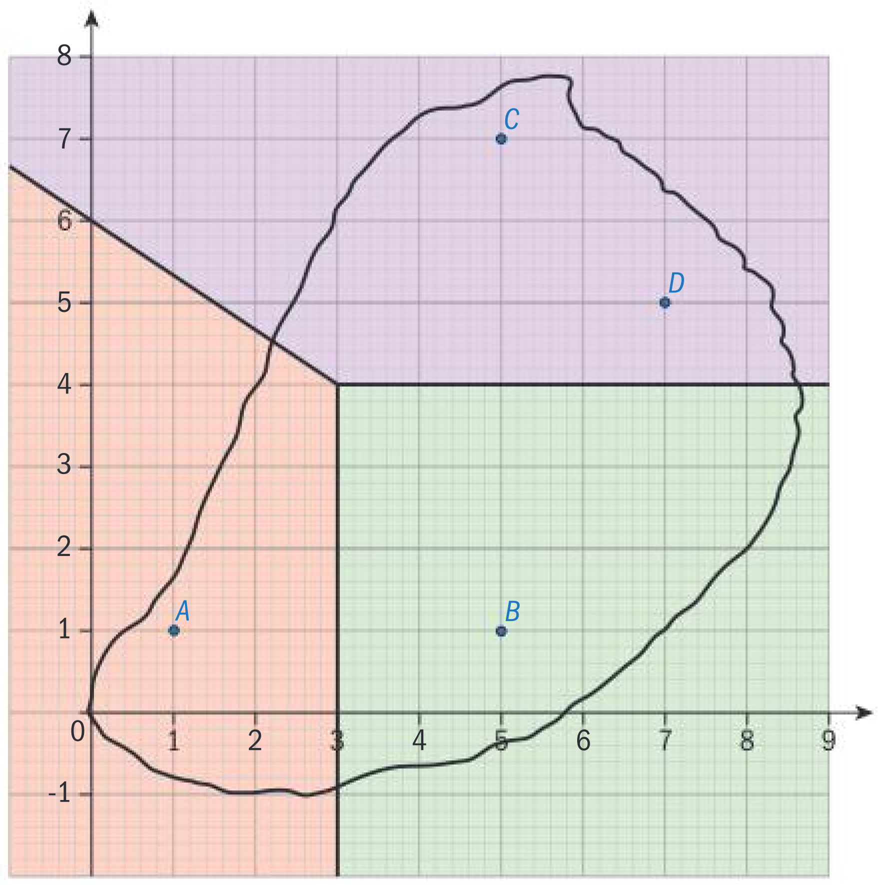
A heavy ball with a radius of 2.9 cm is dropped into the container. As a result, the height of the water increases to h cm, as shown in the diagram.



b. Find the value of h.

PAPER 2

1. An island has four schools which are shown on a coordinate grid at the points $A(1, 1)$, $B(5, 1)$, $C(5, 7)$ and $D(7, 5)$. Also shown is the Voronoi diagram for the schools A, B and C.



a. Find:

i. the midpoint of $[BD]$

ii. the gradient of $[BD]$

b. Hence show that the equation of the perpendicular bisector of $[BD]$ is $y = -\frac{1}{2}x + 6$

c. Find the equation of the perpendicular bisector of $[CD]$

d. Complete the Voronoi diagram for the schools A, B, C and D

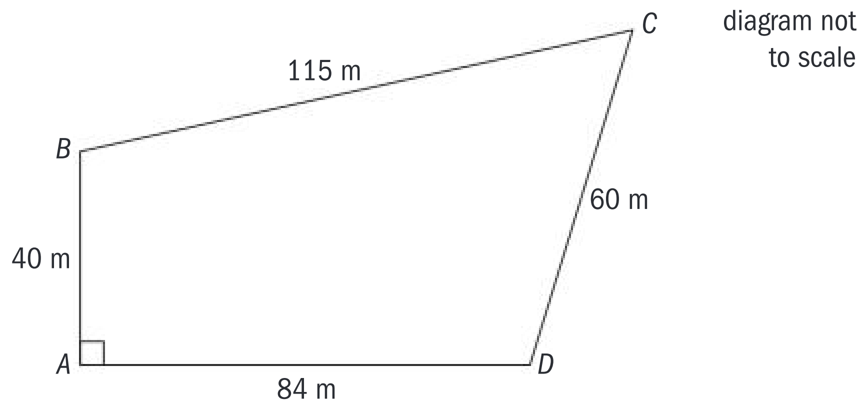
(Note: in an exam, you would have a copy of the grid on which to draw the Voronoi diagram.)

A new school is to be built on the island which needs to be an equal distance from schools A, B and C.

e. Write down the coordinates of the point where the new school should be built.

f. Give the straight line distance of the new school from each of the other schools.

2. Abdallah owns a plot of land near the River Nile, in the form of a quadrilateral $ABCD$. The lengths of the sides are $AB = 40$ m, $BC = 115$ m, $CD = 60$ m, $AD = 84$ m, and $B\hat{A}D = 90°$. This information is shown on the diagram.



a. Show that $BD = 93$ m, correct to the nearest metre.

b. Calculate the size of angle $B\hat{C}D$.

c. Find the area of $ABCD$.

The formula that the ancient Egyptians used to estimate the area of a quadrilateral $ABCD$ is:

$$\text{Area} = \frac{(AB + CD)(AD + BC)}{4}$$

Abdallah uses this formula to estimate the area of his plot of land.

d. i. Calculate Abdallah's estimate for the area.

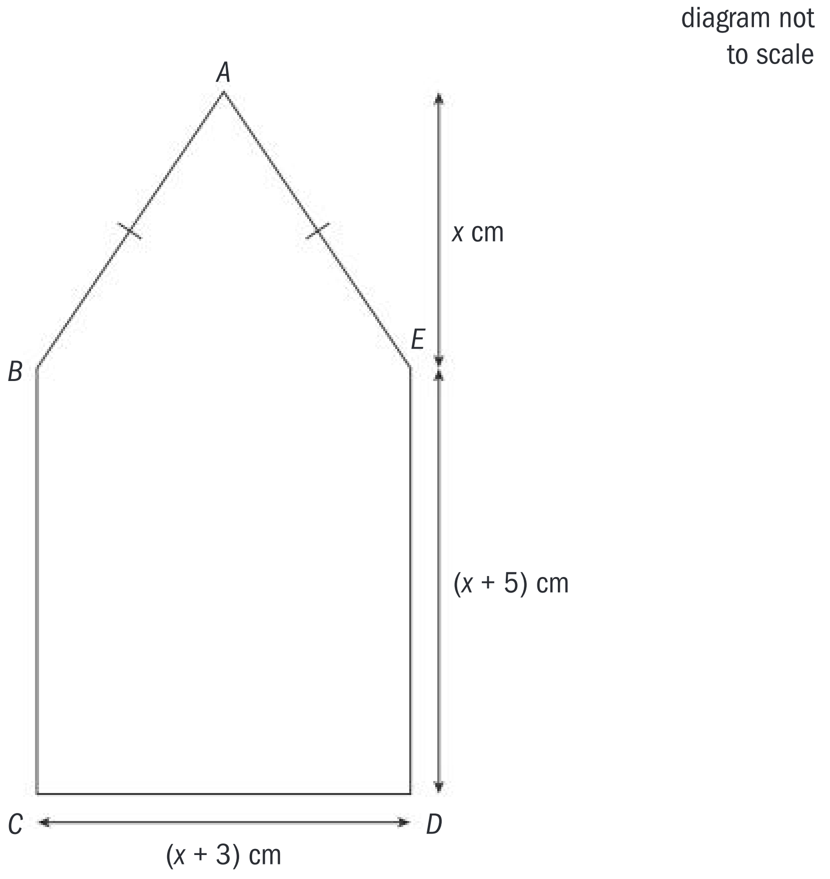
ii. Find the percentage error in Abdallah's estimate.

3. The base of an electric iron can be modelled as a pentagon $ABCDE$, where:

$BCDE$ is a rectangle with sides of length $(x + 3)$ cm and $(x + 5)$ cm

ABE is an isosceles triangle with $AB = AE$, and a height of x cm

The area of $ABCDE$ is 222 cm^2.



a. **i.** Write down an **equation** for the area of *ABCDE*, using the information given.

ii. Show that the equation in part **i** simplifies to $3x^2 + 19x - 414 = 0$

b. Find the length of *CD*.

c. Show that $B\hat{A}E = 67.4°$, correct to 1 decimal place.

Insulation tape is wrapped around the perimeter of the base of the iron *ABCDE*.

d. Find the length of the perimeter of *ABCDE*.

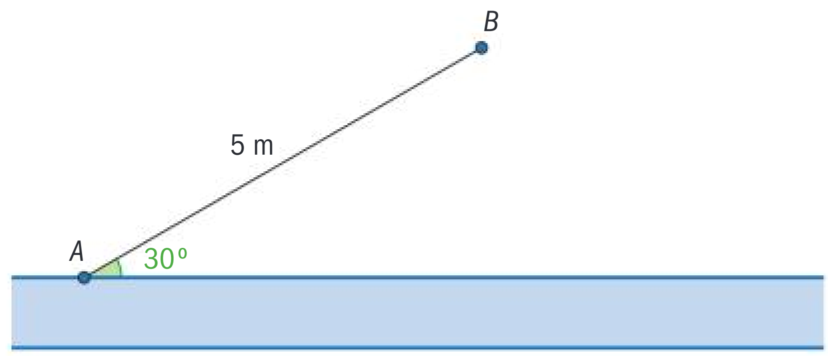
F is the point on *AB* such that $BF = 8$ cm. A heating element in the iron runs in a straight line from *C* to *F*.

e. Calculate the length of *CF*.

HL PRACTICE QUESTIONS

PAPER 1, GROUP 1

1. A wall 5 m long makes an angle of 30° with a straight section of river, as shown in the diagram. The owner of the field on the one side of the river wants to make a triangular pen with the wall *AB* as one of the sides and the river as a second side. For the third side, he will use a metal pole 3.5 m long which he does not wish to cut. He will attach one end of the pole to the end of the wall at point *B*.



a. Show that there are two possible positions for the other end of the pole.

b. Find the area of the smaller of the two pens.

2. The transformation matrix $\mathbf{T} = \begin{pmatrix} 2 & 1 \\ a & -1 \end{pmatrix}$ maps the point (4, 3) onto (*b*, 3). Find:

a. the value of *a* and the value of *b*

b. the point that is mapped onto (0, –7)

3. An aircraft is flying such that, at *t* hours after 11 am, its position relative to its destination airport is given by the equation:

$$\boldsymbol{r} = \begin{pmatrix} 105 \\ 226 \\ 12 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The *x* and *y* directions are east and north respectively, and the *z* direction gives the vertical height of the aircraft.

If the aircraft's velocity remains constant, it will land at the airport at 11:30 am.

a. Find the values of *a*, *b* and *c*.

b. Find the speed of the aircraft.

4. Three points in three-dimensional space have coordinates

A(1, 0, 5), *B*(2, 4, 3) and *C*(1, –2, 4)

a. Find the vector

i. $\overrightarrow{AB}$ **ii.** $\overrightarrow{AC}$

b. Hence or otherwise, find the area of triangle *ABC*.

5. A particle has acceleration $\ddot{x}$ given by $\ddot{x} = \begin{pmatrix} 6t \\ 2 \end{pmatrix}$

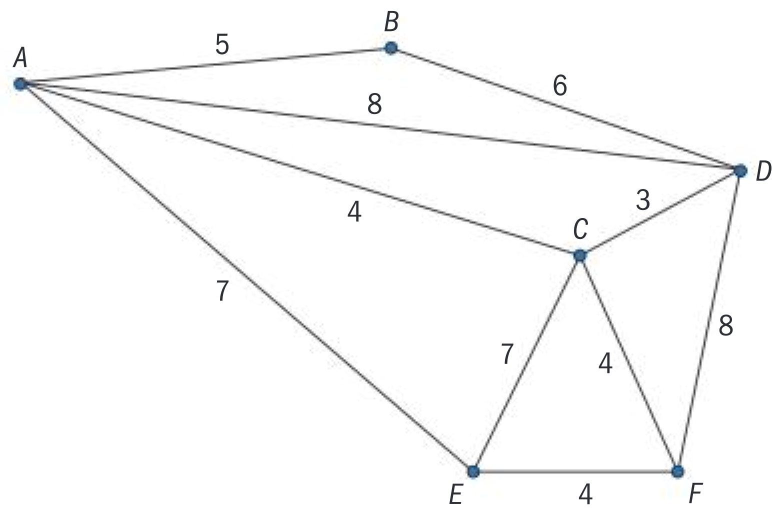
Given at $t = 0$ the particle is at (0, 0) and has velocity $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, find:

a. an expression for its velocity at time *t*

b. an expression for its displacement from (0, 0) at time *t*

c. its distance from (0, 0) at $t = 2$

6. Sameer is trying to design a road system to connect six towns, A, B, C, D, E and F. The possible roads and the costs of building them are shown in the graph. Each vertex represents a town. Each edge represents a road and the weight of each edge is the cost of building that road. Sameer needs to design the lowest cost road system that will connect the six towns.



   a. Name an algorithm that will allow Sameer to find the lowest cost road system.

   b. Find the lowest cost road system and state the cost of building it. Clearly show the steps of the algorithm.

GROUP 2

7. In triangle ABC, $AB = 5$ cm, $BC = 4$ cm and $B\hat{A}C = \frac{\pi}{7}$

   a. Use the cosine rule to find the two possible values for AC.

   b. Hence, find the area of the larger of the two possible triangles.

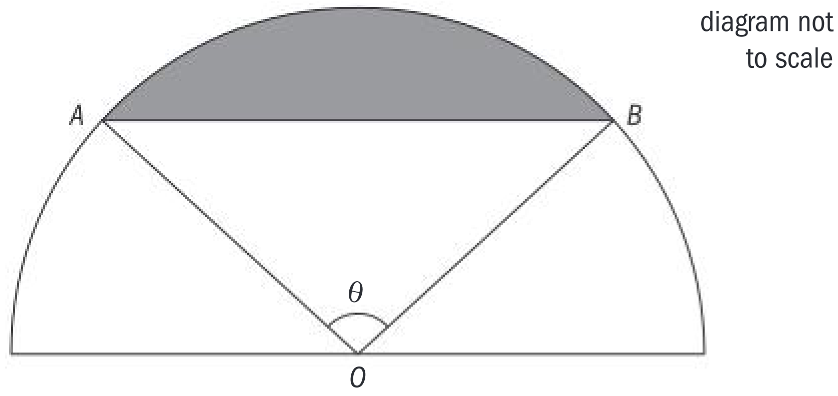
8. Triangle T has coordinates (2, 3), (2, 5) and (6, 4).

   a. Find the area of T.

   A transformation **M** is represented by the matrix $\begin{pmatrix} 2\sin\theta & -2\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$

   b. Show that the area of the image of T under this transformation is independent of θ.

9. The diagram shows a semi-circle of diameter 20 cm, centred at O, with two points A and B on the circumference such that $A\hat{O}B = \theta$, where θ is measured in radians.

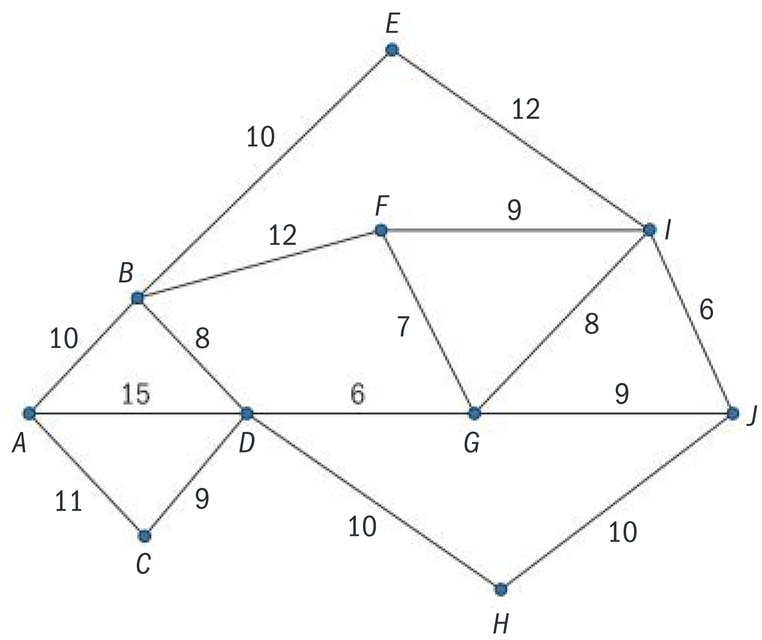


   a. Show that the shaded area can be expressed as $50\theta - 50\sin\theta$

   b. Find the value of θ for which the shaded area is equal to half that of the unshaded area, giving your answer correct to four significant figures.

10. Let $\boldsymbol{u} = -3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = m\boldsymbol{j} + n\boldsymbol{k}$. Given that $\boldsymbol{v}$ is a unit vector perpendicular to $\boldsymbol{u}$, find the possible values for m and n.

11. The graph shows the costs in USD to travel by bus between 10 towns.



The total cost for travelling each of the routes exactly once is 152 USD.

   a. Explain why the graph has no Euler circuit.

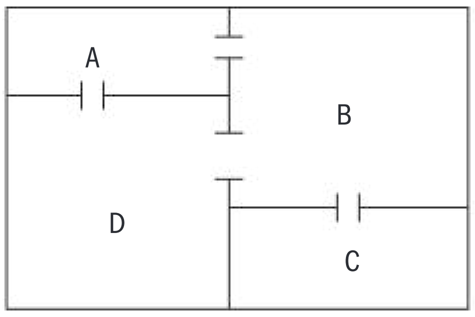
A tourist would like to travel along all the bus routes for the least possible cost, and to start and finish in the same town.

   b. Use the Chinese postman algorithm to find the least possible cost and state the bus routes which need to be repeated.

The tourist realizes that she can reduce her cost and the number of repeated routes if she starts and finishes at different towns.

   c. State which towns she should choose to start and finish at, and how much money she would save on bus travel.

12. A security guard patrols four rooms in a museum. The rooms and all the connecting doors are shown in the floor plan below.



While patrolling, the security guard decides he will leave each room by a randomly selected door.

a. Show the floor plan as a graph with the vertices as the rooms and the edges as the connecting doors.

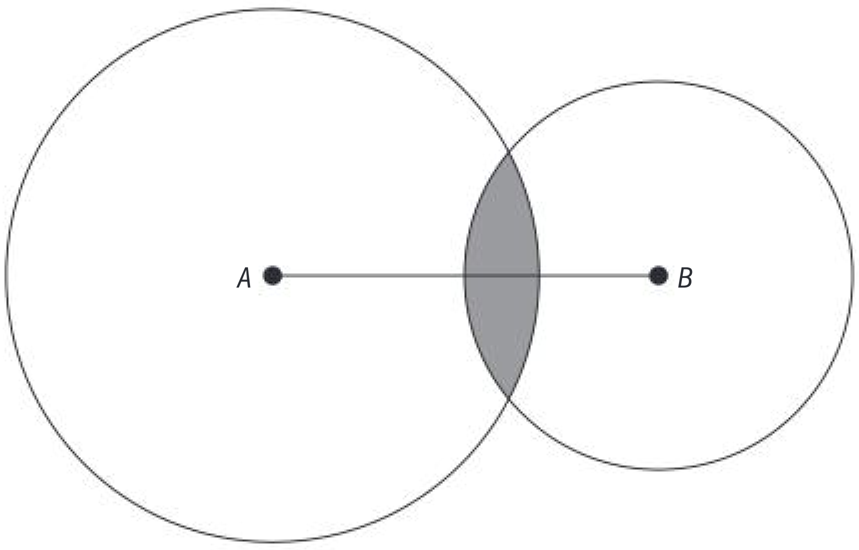
b. Find the transition matrix for the graph.

c. Find the steady state vector for this matrix.

d. Use your answer to part **c** to say which room the security guard spends most time in and which room he spends least time in. State an assumption that you are making.

GROUP 3

13. Boat *A* is situated 10 km away from boat *B*, and each boat has a marine radio transmitter on board. The range of the transmitter on boat *A* is 7 km, and on boat *B* it is 5 km. The region in which both transmitters can be detected is represented by the shaded region in the diagram. Find the area of this region.



14. The matrix **S** represents an enlargement scale factor 1.1, centred at (0, 0), and the matrix **R** represents a reflection in the line $y = x$.

a. Find:

i. **S**

ii. **R**

Let **T** = **SR**

b. Find an expression for the matrix $\mathbf{T}^n$, when:

i. n is odd

ii. n is even

Let $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{T}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and let the distance of (x_n, y_n) from the origin be d_n

N is the smallest value of n such that $d_n > 20$

c. Find (x_N, y_N)

15. At 13:00, a boat is sailing directly to a port with a velocity of $5.2\boldsymbol{i} + 3.9\boldsymbol{j}$ kmh^{-1} where $\boldsymbol{i}$ is due east and $\boldsymbol{j}$ is due north.

a. Find the speed of the boat.

b. Find the bearing on which the boat is sailing.

The boat is 12 km from the port. Assume that the velocity of the boat remains constant.

c. Find the time, to the nearest minute, when the boat reaches the port.

d. Find an expression for the displacement of the boat from the port in terms of t, where t is the number of hours after 13:00

16. Two students are standing 20 m apart and are throwing beanbags towards each other.

Student *A* throws his beanbag towards student *B* from a height of 1.5 m with an initial velocity of 5 ms^{-1} and at an angle of 30°

After release, the acceleration of the beanbag is $\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$ ms^{-2}

a. Find the initial velocity of the beanbag as a column vector.

b. Find the displacement from *A* of the beanbag at time t, where t is the number of seconds after it is released.

Student *B* throws her beanbag towards student *A* at $t = 0$. The initial speed of the beanbag is 6 ms^{-1} and it is released from a height of 1 m.

c. Given that the beanbags collide, find the angle of release of student *B*'s beanbag.

17. Using suitable units, the displacement of a planet from a star centred at (0, 0) can modelled by the equation

$$\boldsymbol{r} = \begin{pmatrix} 2\sin\left(\frac{\pi t}{2}\right) \\ 3\cos\left(\frac{\pi t}{2}\right) \end{pmatrix}$$

a. Sketch the position of the planet at times:

i. $t = 0$

ii. $t = 1$

iii. $t = 2$

iv. $t = 3$

b. Find an expression for the velocity, v, of the planet at time t.

c. Hence, show that the speed of the planet at time t is $|\boldsymbol{v}| = \frac{\pi}{2}\sqrt{4 + 5\sin^2\left(\frac{\pi t}{2}\right)}$

d. Hence, write down the first two positive values of t at which the speed is maximum.

e. How does your answer to part **d** relate to the distance of the planet from the star?

18. A graph G has adjacency matrix **M** given by:

	A	B	C	D	E	F
A	0	1	0	0	0	1
B	1	0	1	0	1	0
C	0	1	0	1	0	0
D	0	0	1	0	1	0
E	0	1	0	1	0	1
F	1	0	0	0	1	0

a. Draw the graph G.

b. i. Find the number of walks of length 6 starting and ending at A.

ii. Write down how many of these walks will pass through **all** the vertices.

c. List all the walks of length 3 starting at A and ending at D.

d. Find the probability that a random walk of length 3 beginning at A will finish at D.

PAPER 2

1. *Note: distances are in metres and time is in seconds in this question.*

Two particles P_1 and P_2 both leave from a point A along two different straight lines.

t seconds after leaving A, the position of P_1 is given by $\boldsymbol{r} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

a. Write down the coordinates of A.

Five seconds after leaving A, P_1 is at point B.

b. Find:

i. $\overrightarrow{AB}$

ii. $|\overrightarrow{AB}|$

P_2 leaves A three seconds after P_1. Two seconds after it leaves A, P_2 is at the point C, where $\overrightarrow{AC} = \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix}$

c. Write down the velocity vector for P_2

d. Hence write down an equation for the displacement of P_2 in terms of t, for $t \geq 3$

e. Find $\cos B\hat{A}C$

f. Hence or otherwise, find the distance between P_1 and P_2 when $t = 5$

2. a. Use Prim's algorithm, beginning at vertex A, to find the length of the minimum spanning tree for the graph represented by the weighted adjacency table below. Write down the order in which the edges are selected.

	A	B	C	D
A		12	15	9
B	12		8	11
C	15	8		18
D	9	11	18	

The weighted adjacency table shown above is a table of least distances, in kilometres, for the journeys between four hospitals A, B, C and D.

Rainer needs to deliver medicine to these four hospitals and uses the table to help him decide his route.

On one occasion, another hospital E is added to the list.

Rainer knows that he can reach E only by first going to either A or B and the distance from A to E is 5 km and from B to E is 8 km.

b. Copy and complete the table of least distances shown below so it includes hospital E.

	A	B	C	D	E
A		12	15	9	5
B	12		8	11	8
C	15	8		18	
D	9	11	18		
E	5	8			

Rainer needs to begin at hospital A.

c. Beginning at A, use the nearest neighbour algorithm to find an upper bound for Rainer's journey.

d. By first deleting hospital E, use the deleted vertex algorithm and your answer to part **a** to find a lower bound for Rainer's journey.

e. Find a different lower bound by deleting vertex B.

f. Hence, write down the smallest interval containing the least length of the route Rainer needs to take.

PAPER 3

1. George is working on an animation that requires him to rotate points about a centre other than (0, 0). He has been told that this is possible using affine transformations, so he is investigating how these can be constructed.

a. Describe fully the transformation represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

An affine transformation, **T**, is defined as

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ for } n \geq 1$$

Let (x_0, y_0) be (4, 0)

b. Find:

i. $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

ii. $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

iii. $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$

c. Plot these points on a set of coordinate axes.

d. From your diagram, show that the centre of rotation is (–0.5, 1.5)

e. Verify that this point is invariant under the affine transformation.

George decides he wants to rotate his points 180° about the point (–4, 2) and wishes to find the affine transformation, **A**, that will do this.

f. Write down the matrix representing a rotation of 180° about (0, 0)

g. By considering the invariant point under **A**, or otherwise, find **A** in the form

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{M}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} + \boldsymbol{b}$$

In order to do his calculations more quickly, George needs an expression for $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ in terms of x_0 and y_0

h. Verify that for **A**, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

i. Explain geometrically why $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, whenever n is even.

j. Find an expression for $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ in terms of x_0 and y_0, when n is odd.

k. Find an expression in terms of θ, a and b, for the affine transformation that rotates a point θ anticlockwise about the point (a, b).

SL PRACTICE QUESTIONS

PAPER 1, GROUP 1

1. Sheldon is comparing the cost of identical soft drinks in Germany and Italy. As he travels through the two countries, he collects the following data in euros.

Germany	Italy
1.80	1.50
2.00	1.70
1.70	3.00
1.80	1.90
2.00	1.60
2.10	1.70
2.20	1.80
1.60	1.50
1.80	

a. Find the sample means for the costs in the two countries.

Sheldon decides to use a two-sample pooled t-test at the 5% significance level to test the hypothesis that drinks are cheaper on average in Italy than in Germany.

b. State whether this is a one-tailed or a two-tailed test

c. i. State the null and alternative hypotheses.

ii. Find the p-value for the test.

iii. Write down the conclusion of the test, justifying your answer.

2. Student lockers in a corridor of Wiles Academy are numbered with consecutive integers starting at 21 and ending at 53. At the end of term, a locker is chosen at random. Find the probability that the locker number is:

a. 49 or higher

b. 26 or less

c. a multiple of 9

d. not a multiple of 9

e. a factor of 120

3. A survey in a large city is conducted in order to find out more about the lives of the people living there. The number of beds in a randomly chosen home, T, is modelled by the discrete probability distribution represented in the table.

t	1	2	3	4	5	6	7
$P(T = t)$	0.47	$3a$	0.17	0.09	a	0.02	0.01

a. Find the value of a.

b. Find the probability that a randomly selected home has exactly 4 beds, given that it has more than 2.

4. A restaurant has four choices of main course. Lisa thinks that the choice of main course will be independent of gender, whereas Michelle believes that it is not. They decide to do a test and record which course is chosen by 60 randomly selected customers.

The results are shown in the table.

	Male	Female	Total
Beef	a	8	20
Fish	5	9	14
Vegetarian	8	b	15
Seafood pasta	3	8	11
Total	28	32	60

a. Write down the values of a and b.

Lisa and Michelle decide to perform a χ^2 test for independence at the 5% significance level.

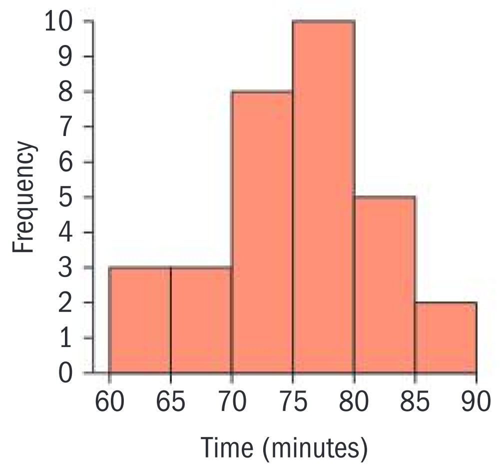
b. i. Write down the null and alternative hypotheses for the test.

ii. Find the χ^2 statistic for the test.

The critical value for a 5% significance level is 7.81

iii. Use your answer to part **ii** to write down the conclusion of the test, clearly justifying your answer.

5. A teacher asks the students in a class to record how many minutes they spent working on an extended task, designed to take between one and two hours. The results are shown in the diagram.



a. Find the number of students in the class.

b. Find estimates for the mean and standard deviation of the time taken by the students.

In addition to the time spent at home on the task, the whole class also spent 30 minutes working on it during their lesson.

c. Write down the mean and standard deviation for the total time spent on the task.

6. A fair tetrahedral die numbered 2, 3, 5 and 8 is thrown, and a fair cubical die numbered 1, 1, 2, 3, 5 and 8 is thrown. T is the total of the two numbers thrown.

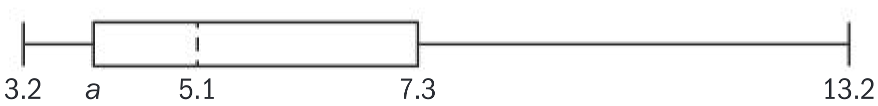
a. Find the probability that T is a prime number.

b. Find the probability that T is a prime number or a factor of 10.

c. Find the probability that T is a prime number or a multiple of 4.

GROUP 2

7. Carole recorded the distances in kilometres that she ran over the course of 6 weeks and showed the results in this box and whisker plot.



The interquartile range for the lengths of the runs is 3.3 km.

a. Find the value of a.

b. Find whether or not Carole should have shown the maximum length as an outlier, fully justifying your answer.

8. In a class of 30 students, 10 study Economics, 18 study Art and 3 students study neither Economics nor Art.

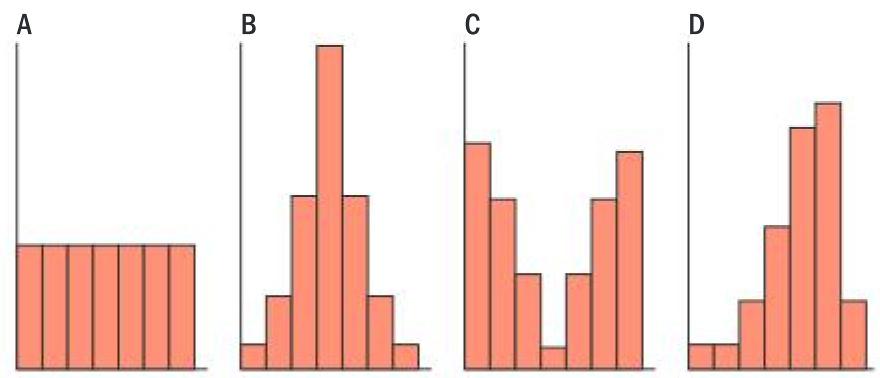
a. Find the probability that a randomly selected student from this class studies both Economics and Art.

b. Given that a randomly selected student studies Economics, find the probability that the student also studies Art.

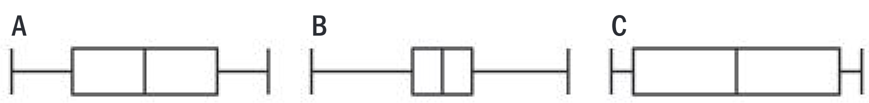
c. Determine if the events *studies Economics* and *studies Art* when choosing a student at random are independent. Justify your reasoning.

9. The four populations A, B, C and D are the same size and have the same range.

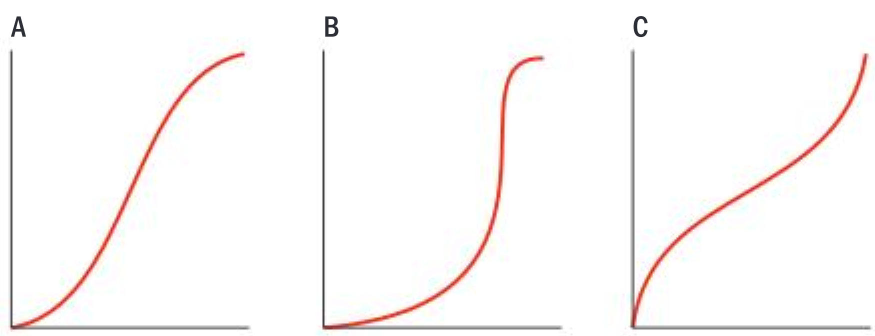
Frequency histograms for the four populations are shown here:



a. Each of the three box and whisker plots below corresponds to one of the four populations. Match each box and whisker plot with the correct frequency diagram.



b. Each of the three cumulative frequency diagrams below corresponds to one of the four populations. Match one with the correct frequency diagram.



10. A machine refills 500 ml washing liquid bottles in order to reduce waste. The volume refilled by the machine is normally distributed with mean 499.3 ml and standard deviation 3.7 ml.

In a quality control procedure, 4% of the bottles are rejected for containing not enough washing liquid. Find the minimum volume, to the nearest ml, that a bottle must contain in order to be accepted.

11. Natalia wanted to perform a *t*-test to see if the mean distance from school for students in grade 10 was greater than the mean distance from school for the students in grade 9.

To do so, she decided to collect a sample of 50 students from grades 9 and 10 and to ask them how far they lived from school.

She obtained a list of the students from the principal, arranged alphabetically. Grade 9 had 85 students and grade 10 had 115 students.

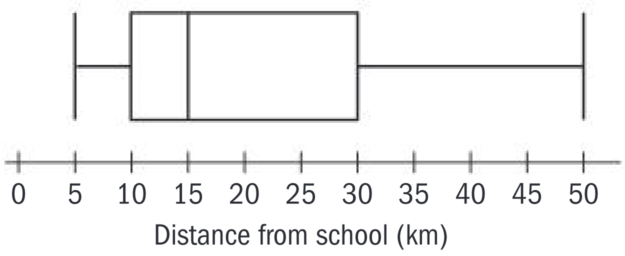
Natalia decided to use stratified sampling to find the number to be selected from each year group.

a. Find how many students she should select from:

i. grade 9 **ii.** grade 10

b. Describe how she could use systematic sampling to select a random sample from grade 9.

Before carrying out her *t*-test, Natalia plotted her values from grade 10 as a box and whisker plot, shown here.



c. Explain why the box and whisker plot indicates that the *t*-test might not be valid.

12. In Ted's class, the number of vowels in each student's name is shown in this table.

Number of vowels	1	2	3	4	5
Frequency	1	6	4	1	1

Each student's name is written on a card and put in a bag. In an experiment, one card is chosen from the bag.

a. The experiment is carried out once. Find the probability that there are at least 3 vowels in the student's name.

b. The experiment is repeated 25 times. Each time a card is chosen it is put back in the bag. Find the expected number of times a student with at least 3 vowels in his/her name is chosen.

c. The experiment is repeated twice, but without replacing the card after the first selection. Find the probability that exactly one of the two names chosen has at least 3 vowels.

GROUP 3

13. Helen is very keen to improve the mean score in her tests. The tests are out of 50 and she scored 8 out of 50 on her first test. She aims to improve her score by 2 marks in every test until she reaches full marks.

Assuming she succeeds in her aim,

a. write down an expression for her score in the nth test she takes

b. find the number of tests she must take to have mean score over 20.

14. The number of points scored in a game Y is modelled by the discrete probability distribution:

y	4	5	6	7	8
$P(Y = y)$	0.2	p	q	0.1	0.3

It is known that $E(Y) = 6.2$

a. Find the values of p and of q.

b. If each point scored gains a cash prize of y USD, find the price of a ticket which would make this a fair game.

c. The game is played 10 times. Find the probability that at least 7 USD is won in exactly four of the games.

15. Frances grows eight sets of plants in controlled conditions, adding different amounts of nutrients to each set. The data in the table shows the amount of nutrient (m) in grams given to each plant and the average height (h), in centimetres, of each set of plants after six weeks.

	Mass (m)	Height (h)
A	0	5.2
B	5	9.2
C	10	14.4
D	15	18.5
E	20	21.2
F	25	21.0
G	30	21.0
H	35	21.6

a. Calculate Spearman's rank correlation coefficient and comment on your result.

Frances decides that it would be better to model the data as a piecewise function.

In this model, the height increases linearly for $0 \le m \le 20$ and is approximately constant for $m > 20$

b. Find the line of best fit for $0 \le m \le 20$ in the form $h = am + b$

c. Interpret the meaning of a.

For $m > 20$, the equation is $h = c$

d. Given the function is continuous at $m = 20$, find the value of c.

16. David is designing computer games involving two experiments on his laptop. The outcomes of experiment A and of experiment B are shown in the tables below, with the probability of each outcome.

Outcome for A	1	3	6	10	15
Probability	0.1	0.1	0.15	0.5	0.15

If the outcome of experiment A is a multiple of 3, the player scores a point. All other outcomes score zero.

Outcome for B	2	3	5	7	11
Probability	0.03	0.02	0.05	0.4	0.5

If the outcome of experiment B is a factor of 10, the player scores a point. All other outcomes score zero.

X is the number of points scored in 20 trials of A.
Y is the number of points scored in 100 trials of B.

a. Write down the distributions of X and of Y stating any assumptions you make.

b. Hence show that the expected values of X and of Y are both equal to 8.

c. Find the variances of the distributions of X and of Y.

d. Hence compare and contrast the distribution of X with that of Y.

17. Brian always turns up to the stadium of his football team exactly 15 minutes before the game kicks off. He knows that the time taken, T, for him to queue, gain entry to the stadium and then find his seat is distributed normally with mean 10 minutes and standard deviation 2.5 minutes.

a. Find the probability that Brian will find his seat before the game kicks off.

b. In a season of 20 games, find the probability that Brian is not in his seat at the start of the game on:

i. at least two occasions

ii. exactly two occasions.

State any assumptions that you make.

18. On her way to and from work, Julia passes through a set of traffic lights.

For a period of 60 days, she records the number of times each day she has to stop at traffic lights (X). Her results are shown in the table.

Number of times stopped (X)	0	1	2
Frequency	12	24	24

She decides to use this data to test the hypothesis that X is distributed binomially, with the probability of being stopped equal to 0.5

a. Write down the null and alternative hypotheses for Julia's test.

b. Find the expected values for the number of times she has to stop if the null hypothesis is true.

c. Perform the test and state why the null hypothesis should be rejected at the 5% level.

d. Give one reason why the null hypothesis might not be true.

PAPER 2 QUESTIONS

1. Adam is a beekeeper who collected data about monthly honey production in his beehives. The data for six of his hives is shown in this table.

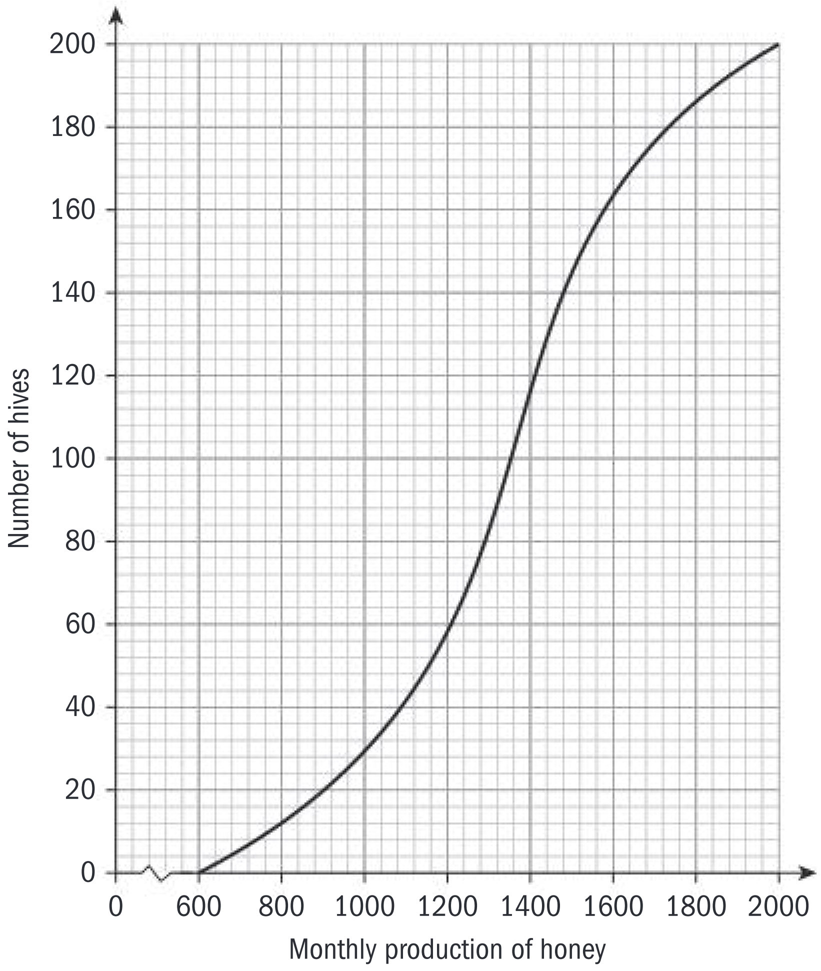
Number of bees (I)	Monthly honey production, in grams (P)
190	900
220	1100
250	1200
285	1500
305	1700
320	1800

The relationship between the variables is modelled by the regression line with equation $P = aN + b$.

a. Write down the value of a and of b.

b. Use this regression line to estimate the monthly honey production from a hive that has 270 bees.

Adam has 200 hives in total. He collects data on the monthly honey production of all the hives. This data is shown in the cumulative frequency graph.



Adam's hives are labelled as low, regular or high production, as defined in this table.

Type of hive	Monthly honey production, in grams (P)
low	$P < 1080$
regular	$1080 < P \leq k$
high	$P > k$

c. Write down the number of low production hives.

Adam knows that 128 of his hives have a regular production.

d. Find:

i. the value of k

ii. the number of hives that have a high production.

e. Adam decides to increase the number of bees in each low production hive. Research suggests that there is a probability of 0.75 that a low production hive becomes a regular production hive. Calculate the probability that 30 low production hives become regular production hives.

2. A health survey is carried out in a large city. The heights of 16 to 18-year-old students are measured. The heights of the females are normally distributed with mean 163 cm and standard deviation 10 cm. The heights of the males are normally distributed with a mean of 170 cm and a standard deviation of 11 cm.

a. Find the probability that a randomly selected male is taller than 175 cm.

b. Given that 15% of the females are shorter than t cm, find t.

c. Find the inter-quartile range of the heights of the males.

The students surveyed are 51% female and 49% male. If the person is female, the probability they are taller than 175 cm is 0.115.

A student is selected at random.

d. Find the probability that the student is taller than 175 cm.

e. Given that the student is taller than 175 cm, find the probability that the student is male.

HL PRACTICE QUESTIONS

PAPER 1, GROUP 1

1. A sample is taken from a population which is normally distributed with a mean equal to μ. The values in the sample are given in the table.

12.1	14.2	10.7	9.8	15.6	10.6	11.8	17.2	9.4	10.5

a. State the distribution of the sample mean.

b. Find the 95% confidence interval for μ.

2. **a.** Given that $A \sim \text{Po}(3.1)$ and $B \sim \text{Po}(2.7)$, find $\text{E}(A - 2B)$ and $\text{Var}(A - 2B)$.

b. Hence explain why $A - 2B$ does not follow a Poisson distribution.

3. For his Science Internal Assessment, Antoine has collected 40 samples and he records how far his results are from the expected results. He wants to test that the difference is due to random error with a mean of 0.

Let $\bar{X}$ be the the sample mean of Antoine's data.

a. State why $\bar{X}$ can be assumed to be normally distributed.

The sample mean for Antoine's data is –0.54 and the standard deviation is 1.2

b. Test, at the 5% signifcance level, whether or not Antoine can assume his errors have a mean of 0.

c. State one implication of your result in the context of the question.

4. Given that $X \sim \text{Po}(4.9)$, $Y \sim \text{B}(7, 0.35)$ and $Z \sim \text{N}(61, 25)$ are independent random variables, find:

a. $\text{E}(2Y+7)$

b. $\text{Var}(8-2X)$

c. $\text{Var}(X+2Y-Z)$

5. Vedant wishes to test whether or not a coin is fair, or whether it is more likely to show Heads. He decides to flip it 50 times and to record the number of Heads that appear.

Let p be the probability of the coin showing Heads.

a. Write down the null and alternative hypotheses.

b. Find the critical region for a test at the 5% significance level.

c. Given that $p = 0.6$ find the probability of a type II error.

6. *Life drinks* produces one-litre bottles of orange juice. The company wants to determine the amount of vitamin C in milligrams in these bottles.

A random sample of ten bottles is analysed and the results are as follows:

243	251	237	252	257	254	248	250	239	252

Find unbiased estimates of the population mean and variance of the amount of vitamin C in the one-litre bottles.

GROUP 2

7. Ana believes that the number of goals scored by her favourite football team follows a Poisson distribution. To test her hypothesis, she records the number of goals they score in each game during a 38 game season and the results are shown in the table.

Number of goals	0	1	2	3	4	≥5
Frequency	9	14	12	2	1	0

Perform an appropriate test to see if the number of goals scored follows a Poisson distribution.

8. The number of admissions per hour on Saturday evenings to a hospital emergency room follows a Poisson distribution with mean 4.7. Vicky works a shift of 4 hours every Saturday evening.

a. Find the probability that on a given Saturday evening there will be at least 20 admissions to the emergency room during Vicky's shift, stating any assumptions that you make.

Vicky works on 5 consecutive Saturday evening shifts.

b. Find the most likely number of Saturday evening shifts on which there are at least 20 admissions to the emergency room on 5 consecutive Saturday evenings, stating any assumptions that you make.

9. In office A, the number of IT problems that occur in a week follows a Poisson distribution with a mean of 4.2

In office B, the number of IT problems follows a Poisson distribution with a mean of 3.9

a. Write down the distribution of the total number of weekly IT problems in the two offices, stating any assumptions you make.

The firm introduces new software which it is hoped will help reduce the problems.
In the next two weeks there are only 12 IT problems reported.

b. Test, at the 5% significance level, whether this is sufficient evidence that the number of IT problems has been reduced.

10. A fabric manufacturer produces high quality silk in sheets of area 10 square metres. During the manufacturing process, flaws in the fabric occur at the rate of 1.8 flaws per 10 square metres. It is assumed that the number of flaws per sheet is modelled by a Poisson distribution.

a. Find the probability that a randomly chosen sheet contains at least one flaw.

Silk sheets with no flaws earn a profit of €350. Silk sheets with one flaw are sold at a discount and earn a profit of €150. Any sheets with more than one flaw incur a loss of €100.

b. Find the expected profit gained by manufacturing 70 silk sheets, to the nearest €100

11. A market research company is conducting a survey to find the proportion of people in a town who would visit a new coffee shop. If the proportion is greater than 0.1 then the company would be likely to build it.

The research company visits the town centre and asks 10 people from each of five age ranges whether or not they would visit the coffee shop and the results are shown in the table below.

Age (x)	Would visit
$x < 20$	8
$20 \le x < 30$	4
$30 \le x < 40$	3
$40 \le x < 50$	2
$x \ge 50$	3

a. Name this method of sampling a population.

b. Find the proportion of the sample who said they would visit the coffee shop.

c. Test, at the 1% significance level, the hypothesis that the proportion, p, of people in the town who would go to the coffee shop is greater than 0.2

d. Give one criticism of the question asked in the survey.

12. The effects of two types of fertilizer on the growth of corn are being compared in a study. Corn plants fertilized with type A have a mean height of 253 cm and a standard deviation of 10 cm. Corn plants fertilized with type B have a mean height of 250 cm and a standard deviation of 12 cm.

a. Find the probability that a sample of 70 corn plants fertilized with type A has a mean greater than 255 cm.

b. Find the probability that a sample of 70 corn plants fertilized with type A has a mean greater than a sample of 80 corn plants fertilized with type B.

GROUP 3

13. A factory produces bags of flour labelled 1 kg. Over a long period time, it is known that the standard deviation of the weight of the bags is 50 g.

The bags are regularly checked to ensure that the mean weight is 1 kg rather than less than 1 kg.

In these checks, 15 bags are taken and the average weight measured. The null hypothesis for the test is that $\mu = 1000$ g and the alternative hypothesis is that $\mu < 1000$ g.

If the mean is less than a grams then the null hypothesis is rejected and the factory is fined.

a. Find the value of a if the probability of a type I error is 0.05

The owner of the factory has set the machine to produce bags with an average weight of 980 g.

b. Find the probability of a type II error.

c. Find the probability that the machine fails at least one of the next three tests.

14. Residents in an urban area have a choice of three mobile phone companies: Pi, Mu and Fi. Each year, Pi expects to retain 76% of its customers, losing 10% to Mu and the rest to Fi. Mu expects to retain 70% of its customers, losing 19% to Fi and the rest to Pi. Fi expects to lose 20% of its customers to Pi and 5% to Mu, retaining the rest.

a. Construct a transition matrix to show the probabilities of a customer transitioning from each company.

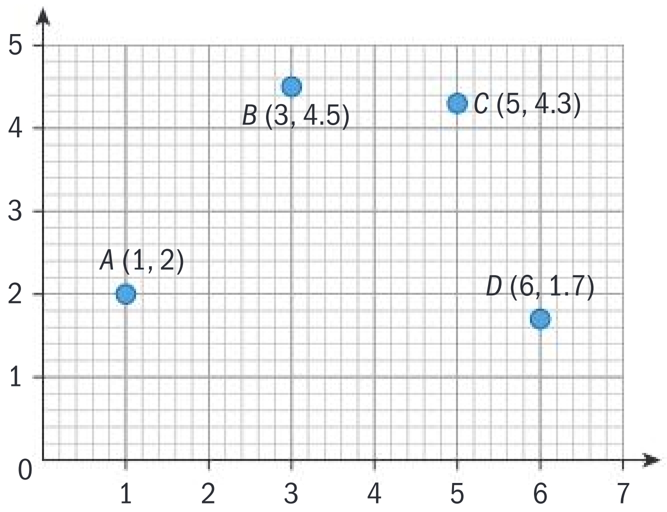
Currently, 32% of the residents are customers of Pi, 40% are customers of Mu and the rest are all with Fi. Fi predict that they will have the largest share of the market after three years.

b. Determine if Fi's claim is correct, stating any assumptions you make.

15. a. Give one reason why you might choose:

i. a quadratic curve **ii.** a cubic curve

to model the points shown in the diagram.



Theory suggests that the model will either be

$y = -0.4x^2 + 2.9x$ or $y = -0.155x^3 + 0.944x^2$

The four points have coordinates $A(1, 2)$, $B(3, 4.5)$, $C(5, 4.3)$ and $D(6, 1.7)$

b. Find the sum of the square residuals for each curve and state which model you would choose on the basis of this result. Justify your reasoning.

c. Find the least squares cubic regression curve for these four points.

d. State the R^2 value from your calculator and state what this indicates.

e. Given that the points represent the height, y, at a horizontal distance, x, from the initial position of an object projected from ground level, state one reason why either of the first two models might be preferred to the least squares model.

16. Callum grows strawberries and raspberries on his farm. The masses of Callum's strawberries are normally distributed with mean 12 g and standard deviation 2.7 g. The masses of Callum's raspberries are normally distributed with mean 4 g and standard deviation 0.5 g.

a. Find the probability that the total mass of five randomly chosen strawberries is more than 70 g.

b. Find that probability that the mass of a randomly chosen strawberry is more than four times the mass of a randomly chosen raspberry.

Strawberries are packaged in containers of n strawberries. The label on the container states that the average mass of a strawberry in the container is 12 g. If the average mass is above 15 g, the container is rejected as too heavy, and if it is below 11 g it is rejected as being too light.

c. If the probability of a package of n being rejected is 0.01, find n.

17. A group of scientists is assessing posters for a conference. Each poster is either accepted or rejected. Concerns are expressed that each decision is not independent of the previous decisions.

To address these concerns, the group looked at the 250 most recent posters and divided them into 25 groups of 5. The number of posters accepted in each group is shown in the table.

Number of posters accepted	0	1	2	3	4	5
Frequency	9	6	10	7	13	5

a. Find the proportion, p_0, of posters that are accepted.

b. Test to see whether the results above come from a binomial distribution, and state the likely meaning of the result in the context of the question.

18. Izzy measured the heights of 291 sunflowers in a botanical garden and found that the heights could be modelled by a normal distribution with mean 189.5 cm and a standard deviation of 15.3 cm.

a. The following season, the botanical garden grew 310 sunflowers. Predict the number of sunflowers taller than 195 cm, stating any assumptions that you make.

Izzy notices that the tape she had used to measure the heights was faulty. The scale started at 3 cm, not the zero mark.

b. What are the correct values of the mean and variance of the distribution of the heights of the sunflowers?

c. Hence find the true answer to part **a** to the nearest whole number.

PAPER 2

1. A manager is collecting data on how the firm's cafeteria is perceived, and she has five employees fill in a survey. Their mean scores out of 10 are recorded, where a higher score indicates greater satisfaction.

After assessing the results of the survey, changes are put in place and the same five employees are given the questionnaire again.

The results are shown in the table.

Employee	A	B	C	D	E
First survey	7.2	4.1	6.1	5.4	3.9
Second survey	7.3	5.2	6.3	6.7	4.2

a. Find the product moment correlation coefficient between the two sets of data and comment on the value obtained.

b. Perform a test to show that there is significant evidence at the 5% level that there is a correlation between the results of the first survey and the second. You may assume that all necessary conditions for the use of this test are satisfied.

The manager claims that this result shows her survey is a reliable means of collecting the data.

c. Give the name for this test of reliability.

The cafeteria manager claims that the data shows the cafeteria has improved between the two surveys.

d. Carry out an appropriate test of the claim of the cafeteria manager.

2. The masses of items of hand luggage carried on to an aircraft by the passengers are normally distributed with mean 9.4 kg and standard deviation 2.8 kg.

a. Find the probability that the mass of a randomly chosen item of hand luggage is between 9 kg and 12 kg.

The airline sets the hand luggage dimensions such that eight items of hand luggage will fit into each overhead locker. The maximum load each overhead locker can hold is 100 kg.

b. Find the probability that eight items of hand luggage will be too heavy for the overhead locker, stating any assumptions that you make.

Large cases must be carried in the hold of the aircraft. The masses of the large cases are normally distributed with mean 25 kg and standard deviation 7 kg.

c. Find the probability that the mass of a randomly chosen large case is more than three times that of a randomly chosen item of hand luggage.

d. The Customs Officers at the destination airport select a random sample of three large cases and two items of hand luggage. Find the probability that the total mass of the sample exceeds 105 kg.

PAPER 3

1. In Euclid city there are only two gym franchises, ClearGym and ResultsNow, who both sell annual membership packages starting on January 1. Each year, ClearGym retains 60% of its members and the rest move to ResultsNow. Each year, ResultsNow retains 70% of its customers and the rest move to ClearGym.

a. Explain why this context can be modelled as a Markov chain.

b. Sketch a transition diagram to represent this context.

c. Hence, write down and label a transition matrix **G** to represent the probabilities of customers changing membership between the two companies.

At the start of 2019, ClearGym had 17 530 customers and ResultsNow had 8956.

d. Predict the number of customers each gym has after one year, to the nearest 100 customers.

e. Find the eigenvalues and the associated eigenvectors of **G**.

f. Hence, express **G** in the form $\mathbf{G} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$

g. Hence, show that

$$\mathbf{G}^n = \frac{1}{7}\begin{bmatrix} 3+4(0.3)^n & 3-3(0.3)^n \\ 4-4(0.3)^n & 4+3(0.3)^n \end{bmatrix}$$

h. Use your result for part **g** to verify your answer to part **d**.

The management of ClearGym are concerned that in the long term they will lose customers to ResultsNow.

i. Explain if the management of ClearGym are justified in having this concern.