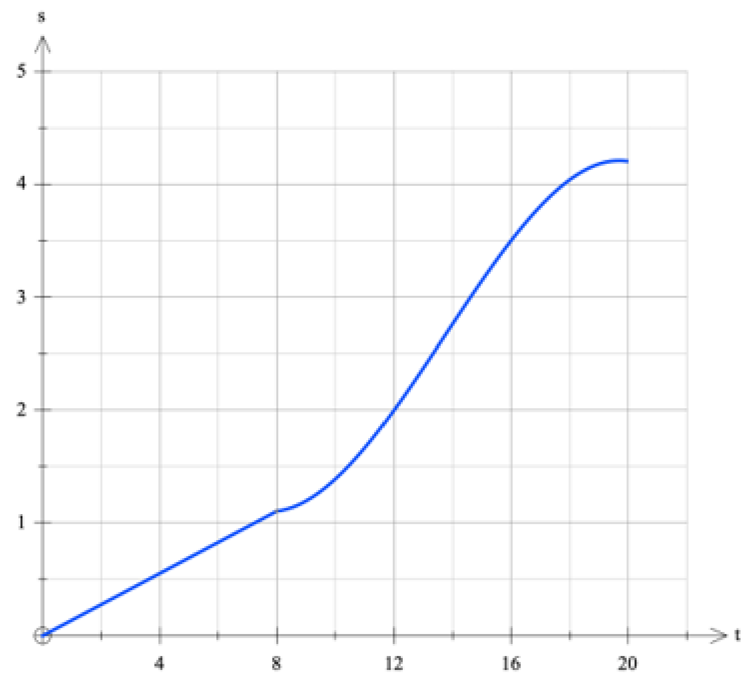
# Modelling AI HL part 2 [245 marks]

**1.** [Maximum mark: 13] EXM.2.AHL.TZ0.13

Beth goes for a run. She uses a fitness app to record her distance, $s$ km, and time, $t$ minutes. A graph of her distance against time is shown.




Beth runs at a constant speed of 2.3 ms$^{-1}$ for the first 8 minutes.

(a) Calculate her distance after 8 minutes. Give your answer in km, correct to 3 decimal places. [2]

Markscheme

$\frac{2.3 \times 8 \times 60}{1000} = 1.104$ ***M1A1***

***[2 marks]***

Between 8 and 20 minutes, her distance can be modeled by a cubic function, $s = at^3 + bt^2 + ct + d$. She reads the following data from her app.

| $t$ (minutes) | 10 | 12 | 16 |
|---|---|---|---|
| $s$ (km) | 1.389 | 2.000 | 3.501 |

(b) Find the value of $a$, $b$, $c$ and $d$. [5]

Markscheme

either using a cubic regression or solving a system of 4 equations ***M1***

$a = -0.00364,\ b = 0.150,\ c = -1.67,\ d = 6.72$ ***A1A1A1A1***

***[5 marks]***

Hence find

(c.i) the distance she runs in 20 minutes. [2]

Markscheme

$s\left(20\right) = 4.21$ km (Note: Condone $s\left(20\right) = 4.2$ km obtained from using rounded values.) ***M1A1***

***[2 marks]***

(c.ii) her maximum speed, in $ms^{-1}$. [4]

Markscheme

EITHER finding maximum of $\frac{ds}{dt}$ OR solving $\frac{d^2s}{dt^2} = 0$ ***M1***

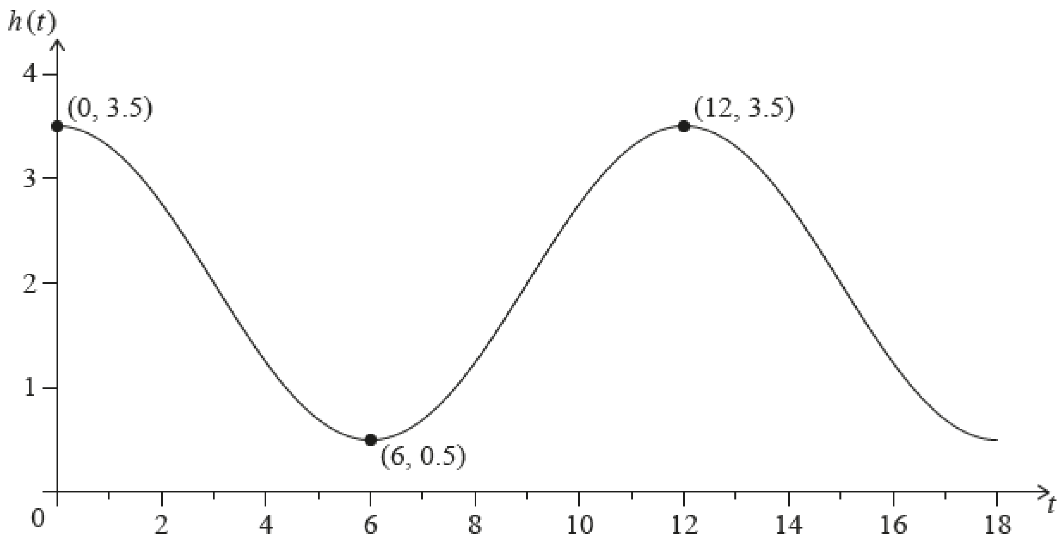
maximum speed = 0.390… km per minute ***A1***

maximum speed = 6.51 $ms^{-1}$ ***M1A1***

***[4 marks]***

**2.** [Maximum mark: 8] 23N.1.AHL.TZ0.9

Joon is a keen surfer and wants to model waves passing a particular point $\mathrm{P}$, which is off the shore of his favourite beach. Joon sets up a model of the waves in terms of $h(t)$, the height of the water in metres, and $t$, the time in seconds from when he begins recording the height of the water at point $\mathrm{P}$.




The function has the form $h(t) = p\cos\left(\frac{\pi}{6}t\right) + q, t \geq 0.$

(a) Find the values of $p$ and $q$. [2]

Markscheme

$p = 1.5; q = 2$ ***A1A1***

***[2 marks]***

(b) Find

Markscheme

attempt at using chain rule ***(M1)***

(b.i) $h\prime(t)$. [2]

Markscheme

$h\prime(t) = -\frac{\pi}{4}\sin\left(\frac{\pi}{6}t\right)$ $\left(= -0.785\sin\left(\frac{\pi}{6}t\right)\right)$ ***A1***

***[2 marks]***

(b.ii) $h\prime\prime(t)$. [1]

Markscheme

$h\prime\prime(t) = -\frac{\pi^2}{24}\cos\left(\frac{\pi}{6}t\right)$ $\left(= -0.411233\ldots\cos\left(\frac{\pi}{6}t\right)\right)$ ***A1***

***[1 mark]***

Joon will begin to surf the wave when the rate of change of $h$ with respect to $t$, at $\mathrm{P}$, is at its maximum. This will first occur when $t = k$.

(c.i) Find the value of $k$. [2]

Markscheme

attempt to locate points of inflexion or max value of $h\prime(t)$ ***(M1)***

$h\prime\prime(t) = -\frac{\pi^2}{24}\cos\left(\frac{\pi}{6}t\right) = 0$ **OR** sketch on graph **OR** $t = 3$ **OR** $\frac{\pi}{6}k = \frac{3\pi}{2}$

$(k =)\ 9$ ***A1***

***[2 marks]***

(c.ii) Find the height of the water at this time. [1]

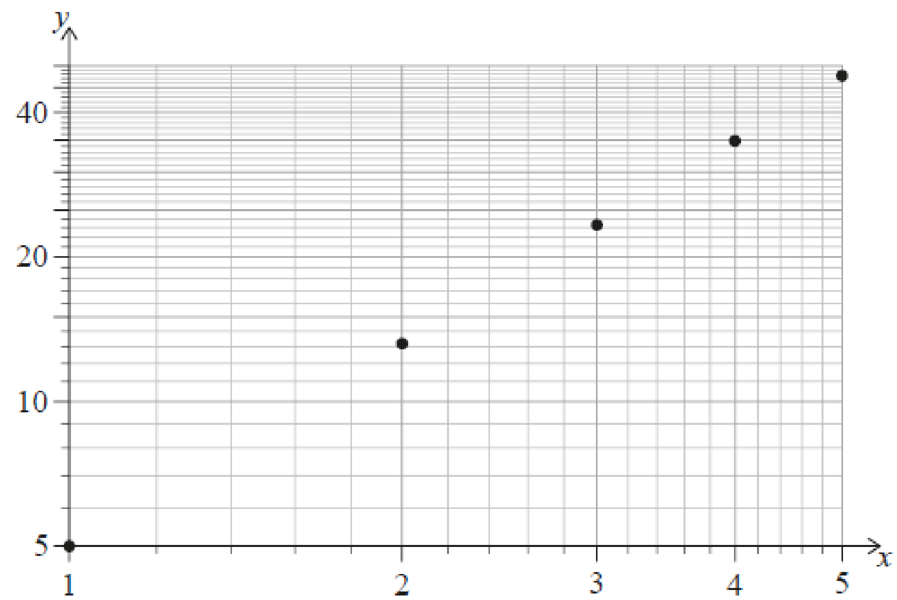
Markscheme

$(h(9) =)\ 2\,(\mathrm{m})$ ***A1***

***[1 mark]***

**3.** [Maximum mark: 6] 23M.1.AHL.TZ1.14

Petra examines two quantities, $x$ and $y$, and plots data points on a log-log graph.




She observes that on this graph the data points follow a perfect straight line. Given that the line passes through the points $(2,\ 13.1951)$ and $(4,\ 34.822)$, find the equation of the relationship connecting $x$ and $y$. Your final answer should not include logarithms. [6]

Markscheme

**METHOD 1 Analytical approach**

recognizing that the linear equation must be expressed in log form ***(M1)***

$\log y = m \log x + \log c$ (or $\log y = m \log x + C$)

**EITHER**

use of slope formula (must involve logs) ***(M1)***

$m = \frac{\log(34.822) - \log(13.1951)}{\log(4) - \log(2)} = 1.4$ ***A1*** attempt to substitute a value *(M1)*

$\log c = \log(13.1951) - 1.4 \log 2 \ (= 0.69897\ldots)$

$\Rightarrow c = 5$ ***A1***

**OR**

$y = c.x^m$ *(A1)*

attempt to set up two equations involving power functions *(M1)*

$13.1951 = c \times 2^m$ and $34.822 = c \times 4^m$

$2^m = \frac{34.822}{13.1951} = 2.639\ldots \Rightarrow m = \log_2 2.639\ldots = 1.4$ ***A1***

$c = \frac{13.1951}{2.639\ldots} = 5$ ***A1***

**THEN**

(so the equation is) $y = 5 \times x^{1.4}$ ***A1***

**METHOD 2 Regression analysis**

recognizing that a log-log graph results in a power function model *(M1)*

$y = a \times x^b$

attempt to find a power regression model using the given two points *(M1)*

$a = 5$ and $b = 1.4$ *(A1)(A1)*

(so the equation is) $y = 5 \times x^{1.4}$ ***A2***

***[6 marks]***

**4.** [Maximum mark: 17] SPM.2.SL.TZ0.5

The braking distance of a vehicle is defined as the distance travelled from where the brakes are applied to the point where the vehicle comes to a complete stop.

The speed, $s\ \mathrm{m\,s^{-1}}$, and braking distance, $d\ \mathrm{m}$, of a truck were recorded. This information is summarized in the following table.

| | | | |
|---|---|---|---|
| **Speed, $s\ \mathrm{m\,s^{-1}}$** | 0 | 6 | 10 |
| **Braking distance, $d\ \mathrm{m}$** | 0 | 12 | 60 |

This information was used to create Model A, where $d$ is a function of $s$, $s \geq 0$.

Model A: $d(s) = ps^2 + qs$, where $p, q \in \mathbb{Z}$

At a speed of $6\ \mathrm{m\,s^{-1}}$, Model A can be represented by the equation $6p + q = 2$.

(a.i) Write down a second equation to represent Model A, when the speed is $10\ \mathrm{m\,s^{-1}}$. [2]

Markscheme

$p(10)^2 + q(10) = 60$ ***M1***

$10p + q = 6\ (100p + 10q = 60)$ ***A1***

***[2 marks]***

(a.ii) Find the values of $p$ and $q$. [2]

Markscheme

$p = 1, q = -4$ ***A1A1***

**Note:** If $p$ and $q$ are both incorrect then award ***M1A0*** for an attempt to solve simultaneous equations.

***[2 marks]***

(b) Find the coordinates of the vertex of the graph of $y = d\left(s\right)$. [2]

Markscheme

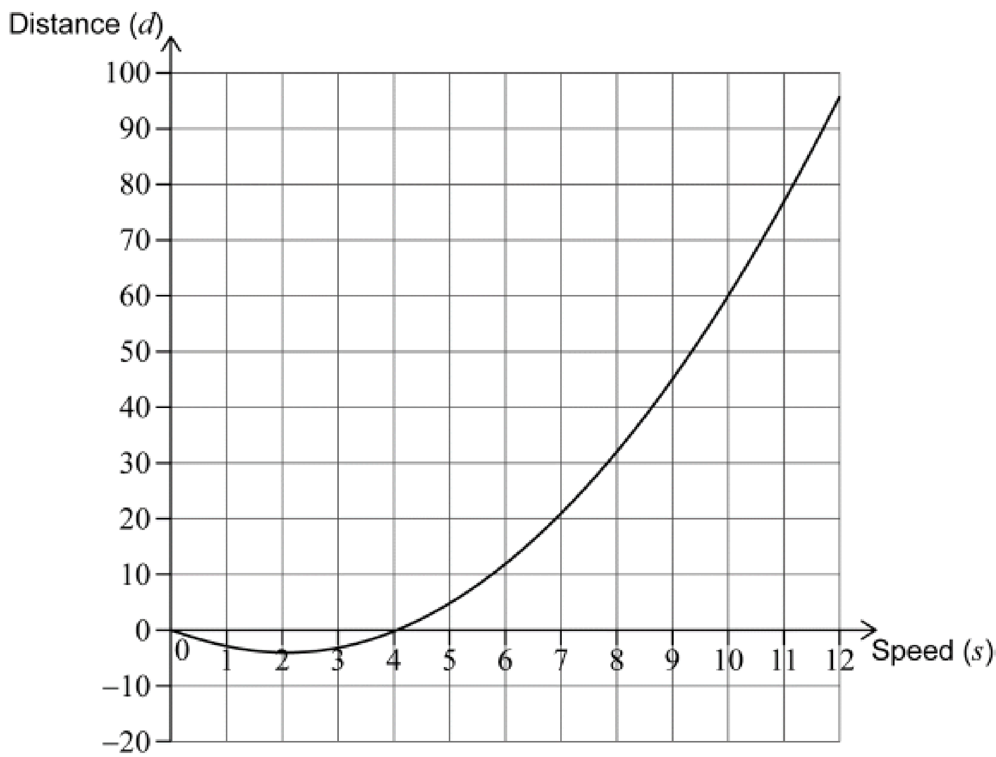
(2, −4) ***A1A1***

**Note:** Award ***A1*** for each correct coordinate.
Award ***A0A1*** if parentheses are missing.

***[2 marks]***

(c) Using the values in the table and your answer to part (b), sketch the graph of $y = d\left(s\right)$ for $0 \le s \le 10$ and $-10 \le d \le 60$, clearly showing the vertex. [3]

Markscheme




***A3***

**Note:** Award ***A1*** for smooth quadratic curve on labelled axes and within correct window.
Award ***A1*** for the curve passing through (0, 0) and (10, 60). Award ***A1*** for the curve passing through their vertex. Follow through from part (b).

***[3 marks]***

(d) Hence, identify why Model A may not be appropriate at lower speeds. [1]

Markscheme

the graph indicates there are negative stopping distances (for low speeds) ***R1***

**Note:** Award ***R1*** for identifying that a feature of their graph results in negative stopping distances (vertex, range of stopping distances…).

***[1 mark]***

Additional data was used to create Model B, **a revised model** for the braking distance of a truck.

Model B: $d\left(s\right)=0.95s^2-3.92s$

(e) Use Model B to calculate an estimate for the braking distance at a speed of $20\ \mathrm{m\,s^{-1}}$. [2]

Markscheme

$0.95\times 20^2-3.92\times 20$ ***(M1)***

$=302\ \ (\mathrm{m})\ \ (301.6\ldots)$ ***A1***

***[2 marks]***

The actual braking distance at $20\ \mathrm{m\,s^{-1}}$ is $320\ \mathrm{m}$.

(f) Calculate the percentage error in the estimate in part (e). [2]

Markscheme

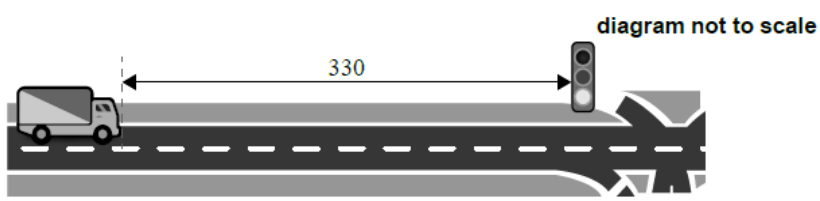
$\left|\frac{301.6-320}{320}\right| \times 100$ ***M1***

$= 5.75\ (\%)$ ***A1***

***[2 marks]***

(g) It is found that once a driver realizes the need to stop their vehicle, 1.6 seconds will elapse, on average, before the brakes are engaged. During this reaction time, the vehicle will continue to travel at its original speed.

A truck approaches an intersection with speed $s\ \mathrm{m\,s^{-1}}$. The driver notices the intersection's traffic lights are red and they must stop the vehicle within a distance of $330\ \mathrm{m}$.




Using model B and taking reaction time into account, calculate the maximum possible speed of the truck if it is to stop before the intersection. [3]

Markscheme

$330 = 1.6 \times s + 0.95 \times s^2 - 3.92 \times s$ ***M1A1***

**Note:** Award ***M1*** for an attempt to find an expression including stopping distance (model B) and reaction distance, equated to 330. Award ***A1*** for a completely correct equation.

$19.9\left(\mathrm{m\,s^{-1}}\right)\ \ (19.8988\ldots)$ ***A1***

***[3 marks]***

**5.** [Maximum mark: 5] EXN.1.SL.TZ0.2

A factory produces engraved gold disks. The cost $C$ of the disks is directly proportional to the cube of the radius $r$ of the disk.

A disk with a radius of $0.8$ cm costs $375$ US dollars (USD).

(a) Find an equation which links $C$ and $r$. [3]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$C = kr^3$ **(M1)**

$375 = k \times 0.8^3 \Rightarrow k = 732\ (732.421\ldots)$ **(M1)**

$C = 732r^3$ **A1**

**[3 marks]**

(b) Find, to the nearest USD, the cost of disk that has a radius of $1.1$ cm. [2]

Markscheme

$C = 732.42\ldots \times 1.1^3$ **(M1)**

$C = \$975\ (974.853\ldots)$ **A1**

**Note:** accept $\$974$ from use of $C = 732r^3$.

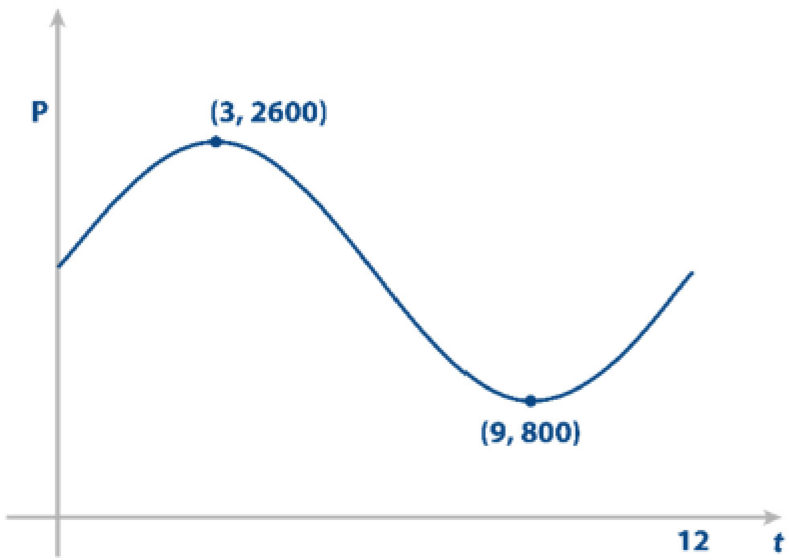
**[2 marks]**

**6.** [Maximum mark: 7] EXN.1.SL.TZ0.6

The size of the population $(P)$ of migrating birds in a particular town can be approximately modelled by the equation $P = a\,\sin(bt) + c,\ \ a,\,b,\,c \in \mathbb{R}^{+}$, where $t$ is measured in months from the time of the initial measurements.

In a $12$ month period the maximum population is $2600$ and occurs when $t = 3$ and the minimum population is $800$ and occurs when $t = 9$.

This information is shown on the graph below.




(a.i) Find the value of $a$. [2]

Markscheme

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$\frac{2600-800}{2} = 900$ **(M1)A1**

**[2 marks]**

(a.ii) Find the value of $b$. [2]

Markscheme

$\frac{360}{12} = 30$ **(M1)A1**

**Note:** Accept $\frac{2\pi}{12} = 0.524\ (0.523598\ldots)$.

**[2 marks]**

(a.iii) Find the value of $c$. [1]

Markscheme

$\frac{2600+800}{2} = 1700$ **A1**

**[1 mark]**

(b) Find the value of $t$ at which the population first reaches $2200$. [2]

Markscheme

Solve $900\sin(30t) + 1700 = 2200$ **(M1)**

$t = 1.12\ (1.12496\ldots)$ **A1**

**[2 marks]**

**7.** [Maximum mark: 13] EXM.2.SL.TZ0.3

Urvashi wants to model the height of a moving object. She collects the following data showing the height, $h$ metres, of the object at time $t$ seconds.

| $t$ (seconds) | 2 | 5 | 7 |
|---|---|---|---|
| $h$ (metres) | 34 | 38 | 24 |

She believes the height can be modeled by a quadratic function, $h(t) = at^2 + bt + c$, where $a,\ b,\ c \in \mathbb{R}$.

(a) Show that $4a + 2b + c = 34$. [1]

Markscheme

$t = 2,\ h = 34 \Rightarrow\ 34 = a2^2 + 2b + c$ ***M1***

$\Rightarrow\ 34 = 4a + 2b + c$ ***AG***

***[1 mark]***

(b) Write down two more equations for $a$, $b$ and $c$. [3]

Markscheme

attempt to substitute either (5, 38) or (7, 24) ***M1***

$25a + 5b + c = 38$ ***A1***

$49a + 7b + c = 24$ ***A1***

***[3 marks]***

(c) Solve this system of three equations to find the value of $a$, $b$ and $c$. [4]

Markscheme

$a = -\frac{5}{3},\ b = 13,\ c = \frac{44}{3}$ ***M1A1A1A1***

***[3 marks]***

Hence find

(d.i) when the height of the object is zero. [3]

Markscheme

$-\frac{5}{3}t^2 + 13t + \frac{44}{3} = 0$ ***M1***

$t = 8.8$ seconds ***M1A1***

***[3 marks]***

(d.ii) the maximum height of the object. [2]

Markscheme

attempt to find maximum height, e.g. sketch of graph ***M1***

$h = 40.0$ metres ***A1***

***[2 marks]***

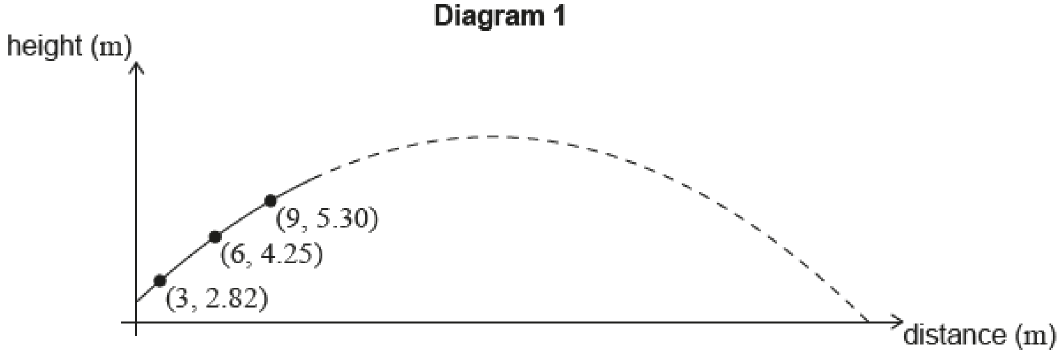
**8.** [Maximum mark: 9] 23N.1.AHL.TZ0.5

A sports player on a horizontal athletic field hits a ball. The height of the ball above the field, in metres, after it is hit can be modelled using a quadratic function of the form $f(x) = ax^2 + bx + c$, where $x$ represents the horizontal distance, in metres, that the ball has travelled from the player.

A specialized camera tracks the initial path of the ball after it is hit by the player. The camera records that the ball travels through the three points $(3,\ 2.36)$, $(10,\ 5)$ and $(17,\ 7.16)$, as shown in **Diagram 1**.

**diagram not to scale**

**Diagram 1**




(a) Use the coordinates $(3,\ 2.36)$ to write down an equation in terms of $a$, $b$, and $c$. [1]

Markscheme

$2.36 = a(3)^2 + b(3) + c$ **OR** $2.36 = 9a + 3b + c$ ***A1***

***[1 mark]***

(b) Use your answer to part (a) and two similar equations to find the equation of the quadratic model for the height of the ball. [3]

Markscheme

finding other equations to solve simultaneously ***(M1)***

$5 = a(10)^2 + b(10) + c$ **AND** $7.16 = a(17)^2 + b(17) + c$
**OR** $5 = 100a + 10b + c$ **AND** $7.16 = 289a + 17b + c$

any one coefficient in equation correct ***(A1)***

$f(x) = -0.00490x^2 + 0.441x + 1.08$ ***A1***

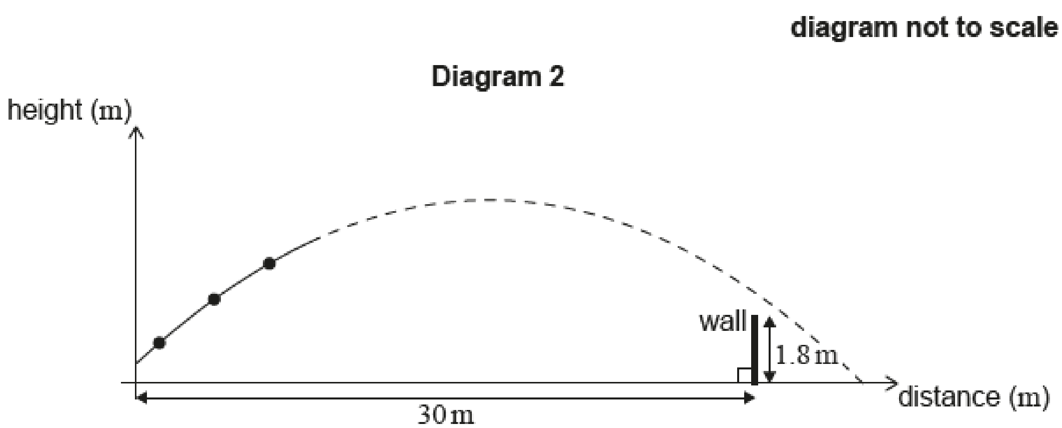
$\left(f(x) = -0.00489795\ldots x^2 + 0.440816\ldots x + 1.08163\ldots\right)$

$\left(f(x) = -\frac{6}{1255}x^2 + \frac{108}{245}x + \frac{53}{49}\right)$

**Note:** Award at most ***(M1)(A1)A0*** if answer is not expressed as an equation.

***[3 marks]***

A 4-metre-high fence is 80 metres from where the player hit the ball, as shown in **Diagram 2**.




(c) Show that the model predicts that the ball will go over the fence. [3]

Markscheme

attempt to substitute 80 into their equation ***(M1)***

$(f(80) = )5$ ***A1***

$5 > 4$ **OR** therefore the ball will go over the fence ***R1***

**Note:** Do not award ***A0R1***; their value must be seen to credit a correct conclusion.

***[3 marks]***

(d) Find the horizontal distance that the ball will travel, from the player until it first hits the ground, according to this model. [2]

Markscheme

setting their equation equal to zero, graph ***(M1)***

$0 = -0.00489795\ldots x^2 + 0.440816\ldots x + 1.08163\ldots$ **OR** $f(x) = 0$

$92.4 \ (92.3902\ldots) \ (\mathrm{m})$ ***A1***

***[2 marks]***

**9.** [Maximum mark: 5] 23M.1.SL.TZ1.6

When the brakes of a car are fully applied the car will continue to travel some distance before it completely stops. This stopping distance, $d$, in metres is directly proportional to the square of the speed of the car, $v$, in kilometres per hour ( $\text{km h}^{-1}$).

When a car is travelling at a speed of $50\ \text{km h}^{-1}$ it will travel $12.3\,\text{m}$ after the brakes are fully applied before it completely stops.

(a) Determine an equation for $d$ in terms of $v$. [2]

Markscheme

attempt to set up a direct variation equation that includes a constant, $k$, or the calculation of a constant using $12.3$ and $50$ ***(M1)***

e.g., $d = kv^2$ **OR** $12.3 = k \times 50^2$

$(k =)\ 0.00492\ \left(\frac{1}{203.252\ldots}\right)$

$d = 0.00492v^2$ **OR** $d = \frac{v^2}{203}$ ***A1***

***[2 marks]***

The police can use this equation to estimate if cars are exceeding the speed limit.

A car is found to have travelled $33\,\text{m}$, after fully applying its brakes, before it completely stopped.

(b) Use your equation from part (a) to estimate the speed at which this car was travelling before the brakes were applied. [2]

Markscheme

substituting $33$ for $d$ in their part (a) ***(A1)***

$33 = 0.00492 \times v^2$ **OR** $33 = \frac{v^2}{203.252\ldots}$

$(v =)\ 81.9\ (\text{km h}^{-1})\ \ (81.8982\ldots\ (\text{km h}^{-1}))$ ***A1***

***[2 marks]***

(c) After the brakes have been fully applied, identify one other variable besides speed that could affect stopping distance. [1]

Markscheme

Award ***R1*** for a reasonable variable that exists after the brakes are applied such as:

- road material
- weather conditions
- condition/type of brakes
- weight/type of vehicle
- gradient/incline of road
- traction
- wind resistance
- friction

***R1***

**Note:** Do not accept a variable that refers to the timing of the brakes being applied such as:

- slow reaction time
- inexperienced driver

*[1 mark]*

**10.** [Maximum mark: 7] 23M.1.SL.TZ1.8

"Password entropy" is a measure of the predictability of a computer password. The higher the entropy, the more difficult it is to guess the password.

The relationship between the password entropy, $p$, (measured in bits) and the number of guesses, $G$, required to decode the password is given by $0.301p = \log_{10} G$.

(a) Calculate the value of $p$ for a password that takes $5000$ guesses to decode. [2]

Markscheme

attempt to substitute $5000$ for $G$ ***(M1)***

$0.301p = \log_{10} 5000$

$(p =)\ 12.3$ (bits) $(12.2889\ldots)$ ***A1***

***[2 marks]***

(b) Write down $G$ as a function of $p$. [1]

Markscheme

$(G =) 10^{0.301p}$ **OR** $2^p$ ***A1***

***[1 mark]***

(c) Find the number of guesses required to decode a password that has an entropy of $28$ bits. Write your answer in the form $a \times 10^k$, where $1 \le a < 10, k \in \mathbb{Z}$. [3]

Markscheme

attempt to substitute $28$ for $p$ in given equation or $G(p)$ ***(M1)***

$0.301 \times 28 = \log_{10} G$ **OR** $(G =)\ 10^{0.301 \times 28}$

$(G =)\ 2.68 \times 10^{8}\ (2.67916\ldots \times 10^{8})$ ***A1A1***

**Note:** Award ***A1*** for $2.68$, ***A1*** for $10^{8}$. Award ***M1A1A0*** for a correct final answer not written in scientific notation or written incorrectly in scientific notation (e.g., $268\,000\,000$ or $26.8 \times 10^{7}$ or $2.68\text{E}08$).

***[3 marks]***

There is a point on the graph of the function $G(p)$ with coordinates $(0,\ 1)$.

(d) Explain what these coordinate values mean in the context of computer passwords. [1]

Markscheme

If a password has an **entropy of $0$** (bits), then the password can be **guessed in one try** / then the **password is known** ***R1***

**Note:** Reference must be made to both entropy and number of guesses/password known for ***R1*** to be awarded.

Do not accept "no password" as this contradicts the context.

*[1 mark]*

**11.** [Maximum mark: 8] 23M.1.SL.TZ2.10

A player throws a basketball. The height of the basketball is modelled by

$$h(t) = -4.75t^2 + 8.75t + 1.5,\ t \geq 0,$$

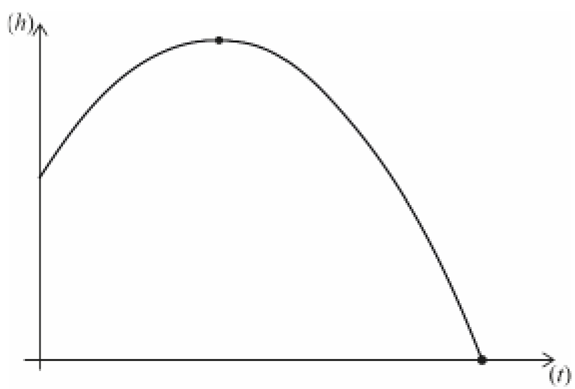
where $h$ is the height of the basketball above the ground, in metres, and $t$ is the time, in seconds, after it was thrown.

(a) Find how long it takes for the basketball to reach its maximum height. [2]

Markscheme

**METHOD 1**

correct sketch with some indication of maximum point ***(M1)***




$0.921$ (seconds) $\left(0.921052\ldots,\ \frac{35}{38}\right)$ ***A1***

**METHOD 2**

correct substitution into equation for line of symmetry ***(M1)***

$(t =)\ -\frac{8.75}{2\times -4.75}$

$0.921$ (seconds) $\left(0.921052\ldots,\ \frac{35}{38}\right)$ ***A1***

**METHOD 3**

equating the correct derivative to $0$ ***(M1)***

$-9.5t + 8.75 = 0$

$0.921$ (seconds) $\left(0.921052\ldots, \ \frac{35}{38}\right)$ ***A1***

**Note:** Award ***M1A0*** for a final answer of $0.92$ seen with no working.

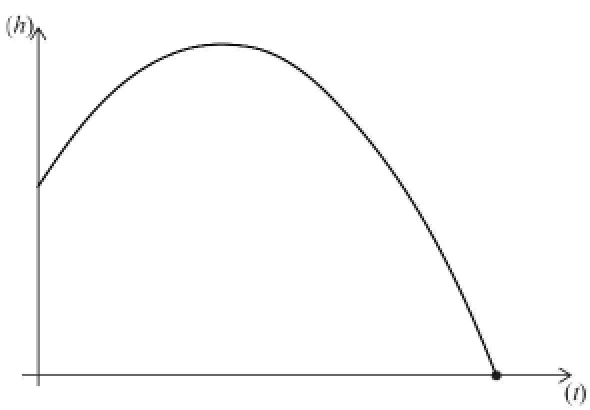
***[2 marks]***

(b) Assuming that no player catches the basketball, find how long it would take for the basketball to hit the ground. [2]

Markscheme

**METHOD 1**

correct sketch with some indication of $x$-intercept




***(M1)***

**Note:** May be seen in part (a).

$2$ (seconds) ***A1***

**METHOD 2**

setting the equation to zero ***(M1)***

$-4.75t^2 + 8.75t + 1.5 = 0$

$2$ (seconds) ***A1***

**Note:** If both roots are given, with or without working, award ***(M1)A0***.

***[2 marks]***

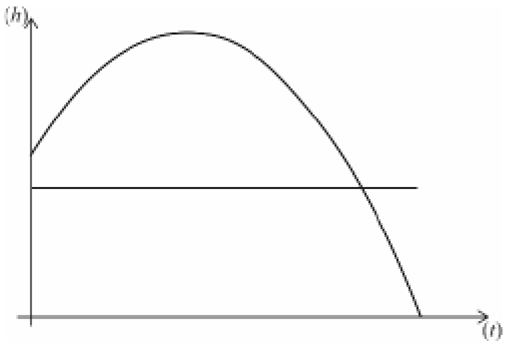
Another player catches the basketball when it is at a height of $1.2$ metres.

(c) Find the value of $t$ when this player catches the basketball. [2]

Markscheme

**METHOD 1**

correct sketch of quadratic function and a straight line in approximate correct position ***(M1)***




$1.88$ (seconds) ($1.87577\ldots$ (seconds)) ***A1***

**METHOD 2**

setting the equation equal to 1.2 ***(M1)***

$-4.75t^2 + 8.75t + 1.5 = 1.2$

$1.88$ (seconds) ($1.87577\ldots$ (seconds)) ***A1***

**Note:** Award ***(M1)A0*** if $-0.0336702\ldots$ seen as (part of) a final answer.

Award ***M1A0*** for answer of $1.9$ seen without working.

***[2 marks]***

(d) Write down two limitations of using $h(t)$ to model the height of the basketball. [2]

Markscheme

*Award* ***R1*** *for a sensible reason in the context of the question:* ***R1R1***
*e.g.*

The model ignores air resistance (or wind)

The model treats the ball as a point

The model assumes gravity is constant

The model assumes that the ball continues to follow the trajectory even after hitting the ground

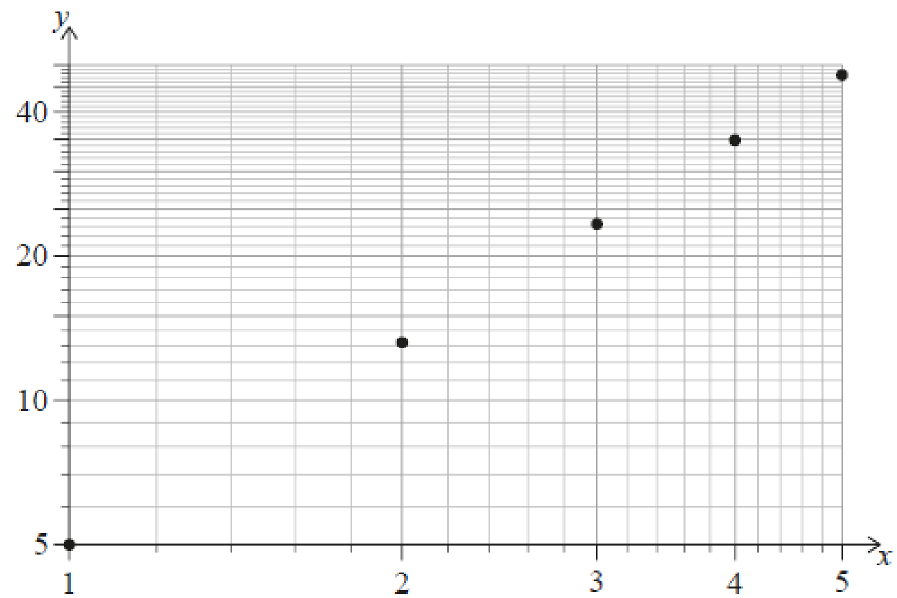
This model ignores the bouncing back of the ball after hitting the ground

**Note:** Do not accept generic criticisms of any mathematical model, such as:
There are assumptions being made
Models are never accurate / It is only a model

***[2 marks]***

**12.** [Maximum mark: 6] 23M.1.AHL.TZ1.14

Petra examines two quantities, $x$ and $y$, and plots data points on a log-log graph.




She observes that on this graph the data points follow a perfect straight line. Given that the line passes through the points $(2,\ 13.1951)$ and $(4,\ 34.822)$, find the equation of the relationship connecting $x$ and $y$. Your final answer should not include logarithms. [6]

Markscheme

**METHOD 1 Analytical approach**

recognizing that the linear equation must be expressed in log form ***(M1)***

$\log\, y = m\, \log\, x + \log\, c$ (or $\log\, y = m\, \log\, x + C$)

**EITHER**

use of slope formula (must involve logs) ***(M1)***

$m = \frac{\log(34.822) - \log(13.1951)}{\log(4) - \log(2)} = 1.4$ ***A1*** attempt to substitute a value *(M1)*

$\log c = \log(13.1951) - 1.4 \log 2 \ (= 0.69897\ldots)$

$\Rightarrow c = 5$ ***A1***

**OR**

$y = c.x^m$ *(A1)*

attempt to set up two equations involving power functions *(M1)*

$13.1951 = c \times 2^m$ and $34.822 = c \times 4^m$

$2^m = \frac{34.822}{13.1951} = 2.639\ldots \Rightarrow m = \log_2 2.639\ldots = 1.4$ ***A1***

$c = \frac{13.1951}{2.639\ldots} = 5$ ***A1***

**THEN**

(so the equation is) $y = 5 \times x^{1.4}$ ***A1***

**METHOD 2 Regression analysis**

recognizing that a log-log graph results in a power function model *(M1)*

$y = a \times x^b$

attempt to find a power regression model using the given two points *(M1)*

$a = 5$ and $b = 1.4$ *(A1)(A1)*

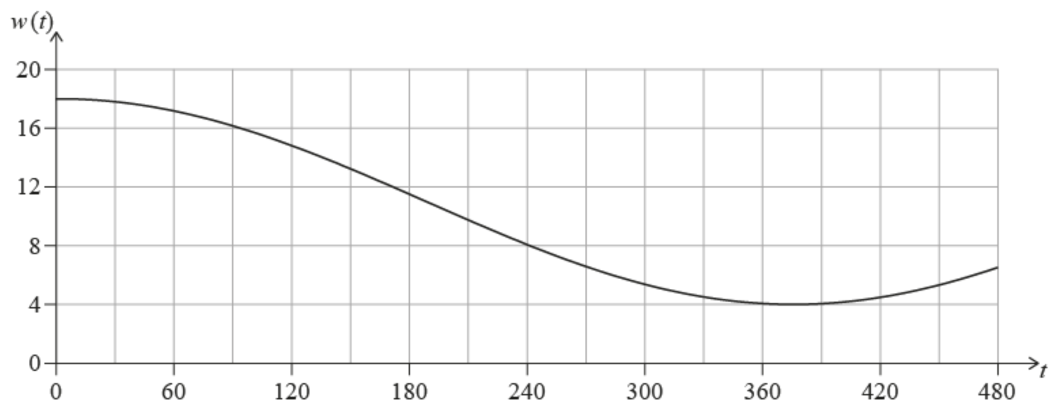
(so the equation is) $y = 5 \times x^{1.4}$ ***A2***

***[6 marks]***

**13.** [Maximum mark: 15] 23M.2.SL.TZ1.3

The depth of water, $w$ metres, in a particular harbour can be modelled by the function $w(t) = a\cos(bt^{\circ}) + d$ where $t$ is the length of time, in minutes, after $06:00$.

On $20$ January, the first high tide occurs at $06:00$, at which time the depth of water is $18\,\text{m}$. The following low tide occurs at $12:15$ when the depth of water is $4\,\text{m}$. This is shown in the diagram.




(a) Find the value of $a$. [2]

Markscheme

$\frac{18-4}{2}$ ***(M1)***

$(a =)\ 7$ ***A1***

***[2 marks]***

(b) Find the value of $d$. [2]

Markscheme

$\frac{18+4}{2}$ **OR** $18-7$ **OR** $4+7$ ***(M1)***

$(d =)\ 11$ ***A1***

***[2 marks]***

(c) Find the period of the function in minutes. [3]

Markscheme

(time between high and low tide is) $6\text{h}15\text{m}$ **OR** $375$ minutes ***(A1)***

multiplying by $2$ ***(M1)***

$750$ minutes ***A1***

***[3 marks]***

(d) Find the value of $b$. [2]

Markscheme

**EITHER**

$\frac{360^\circ}{b} = 750$ ***(A1)***

**OR**

$7\cos(b \times 375) + 11 = 4$ ***(A1)***

**THEN**

$(b =)\ 0.48$ ***A1***

**Note:** Award ***A1A0*** for an answer of $\frac{2\pi}{750}\ \left(= \frac{\pi}{375} = 0.00837758\ldots\right)$.

***[2 marks]***

Naomi is sailing to the harbour on the morning of $20$ January. Boats can enter or leave the harbour only when the depth of water is at least $6\,\mathrm{m}$.

(e) Find the latest time before $12:00$, to the nearest minute, that Naomi can enter the harbour. [4]

Markscheme

equating their cos function to $6$ **OR** graphing their cos function and $6$ ***(M1)***

$7\cos(0.48t) + 11 = 6$

$\Rightarrow t = 282.468\ldots$ (minutes) ***(A1)***

$= 4.70780\ldots$ (hr) **OR** $4$hr $42$ mins ($4$hr $42.4681\ldots$ mins) ***(A1)***

so the time is $10:42$ ***A1***

***[4 marks]***

(f) Find the length of time (in minutes) between $06:00$ and $15:00$ on $20$ January during which Naomi **cannot** enter or leave the harbour. [2]

Markscheme

next solution is $t = 467.531\ldots$ **(A1)**

$467.531\ldots - 282.468\ldots$

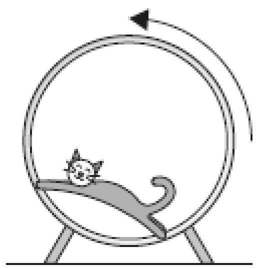
$185$ (mins) $(185.063\ldots)$ ***A1***

**Note**: Accept an (unsupported) answer of $186$ (from correct $3$ sf values for $t$ )

***[2 marks]***

**14.** [Maximum mark: 6] 22N.1.SL.TZ0.12

A cat runs inside a circular exercise wheel, making the wheel spin at a constant rate in an anticlockwise direction. The height, $h$ cm, of a fixed point, P, on the wheel can be modelled by $h(t) = a\sin(bt) + c$ where $t$ is the time in seconds and $a,\ b,\ c \in \mathbb{R}^+$.



When $t = 0$, point P is at a height of $78$ cm.

(a) Write down the value of $c$. [1]

Markscheme

$78$ ***A1***

***[1 mark]***

When $t = 4$, point P first reaches its maximum height of $143$ cm.

(b.i) Find the value of $a$. [1]

Markscheme

$65$ ***A1***

***[1 mark]***

(b.ii) Find the value of $b$. [2]

Markscheme

**EITHER**

(period =) $16$ (could be seen on sketch) ***(M1)***

$b = \frac{2\pi}{16}$ **OR** $b = \frac{360^{\circ}}{16}$

$(b =)\ 0.393\ \left(0.392699\ldots,\ \frac{\pi}{8}\right)$ **OR** $(b =)\ 22.5^{\circ}$ ***A1***

**OR**

$143 = 65\ \sin(4b) + 78$ ***(M1)***

$(\sin(4b) = 1)$

$(4b = \frac{\pi}{2}$ **OR** $4b = 90^{\circ})$

$(b =)\ 0.393\ \left(0.392699\ldots,\ \frac{\pi}{8}\right)$ **OR** $(b =)\ 22.5^{\circ}$ ***A1***

***[2 marks]***

(c) Write down the minimum height of point $\mathrm{P}$. [1]

Markscheme

$13$ ***A1***

**Note:** Apply follow through marking only if their final answer is positive.

**[1 mark]**

Later, the cat is tired, and it takes twice as long for point $\mathrm{P}$ to complete one revolution at a new constant rate.

(d) Write down the new value of $b$. [1]

Markscheme

$(b =)\ 0.196\ \left(0.196349\ldots,\ \frac{\pi}{16}\right)$ **OR** $(b =) 11.3^{\circ}\ \ (11.25^{\circ})$

***A1***

***[1 mark]***

**15.** [Maximum mark: 7] 22N.1.AHL.TZ0.10

Stars are classified by their brightness. The brightest stars in the sky have a magnitude of $1$. The magnitude, $m$, of another star can be modelled as a function of its brightness, $b$, relative to a star of magnitude $1$, as shown by the following equation.

$$m = 1 - 2.5 \log_{10}(b)$$

The star called Acubens has a brightness of $0.0525$.

(a) Find the magnitude of Acubens. [2]

Markscheme

$m = 1 - 2.5 \log_{10}(0.0525)$ ***(M1)***

$= 4.20 \ \ (4.19960\ldots)$ ***A1***

***[2 marks]***

Ceres has a magnitude of $7$ and is the least bright star visible without magnification.

(b) Find the brightness of Ceres. [2]

Markscheme

attempt to solve $7 = 1 - 2.5 \log_{10}(b)$ ***(M1)***

**Note:** Accept a sketch from their GDC as an attempt to solve $7 = 1 - 2.5 \log_{10}(b)$.

$b = 0.00398 \ \ (0.00398107\ldots)$ ***A1***

***[2 marks]***

The star Proxima Centauri has a greater magnitude than the planet Neptune. The difference in their magnitudes is $3.2$.

(c) Find how many times brighter Neptune is compared to Proxima Centauri. [3]

Markscheme

$-3.2=(1-2.5\log_{10}(b_n))-(1-2.5\log_{10}(b_p))$ ***M1***

$-3.2=-2.5\log_{10}\left(\frac{b_n}{b_p}\right)$ ***A1***

$\frac{b_n}{b_p}=19.1\ \ (19.0546\ldots)$ ***A1***

***[3 marks]***

**16.** [Maximum mark: 8] 22M.1.SL.TZ1.11

The strength of earthquakes is measured on the Richter magnitude scale, with values typically between $0$ and $8$ where $8$ is the most severe.

The Gutenberg–Richter equation gives the average number of earthquakes per year, $N$, which have a magnitude of at least $M$. For a particular region the equation is

$\log_{10} N = a - M$, for some $a \in \mathbb{R}$.

This region has an average of $100$ earthquakes per year with a magnitude of at least $3$.

(a) Find the value of $a$. [2]

Markscheme

$\log_{10} 100 = a - 3$ ***(M1)***

$a = 5$ ***A1***

***[2 marks]***

The equation for this region can also be written as $N = \frac{b}{10^M}$.

(b) Find the value of $b$. [2]

Markscheme

**EITHER**

$N = 10^{5-M}$ ***(M1)***

$= \frac{10^5}{10^M} \left(= \frac{100000}{10^M}\right)$

**OR**

$100 = \frac{b}{10^3}$ ***(M1)***

**THEN**

$b = 100000 \ \left(= 10^5\right)$ ***A1***

***[2 marks]***

(c) Given $0 < M < 8$, find the range for $N$. [2]

Markscheme

$0.001 < N < 100000 \ \left(10^{-3} < N < 10^5\right)$ ***A1A1***

**Note:** Award ***A1*** for correct endpoints and ***A1*** for correct inequalities/interval notation.

***[2 marks]***

The expected length of time, in years, between earthquakes with a magnitude of at least $M$ is $\frac{1}{N}$.

Within this region the most severe earthquake recorded had a magnitude of $7.2$.

(d) Find the expected length of time between this earthquake and the next earthquake of at least this magnitude. Give your answer to the nearest year. [2]

Markscheme

$N=\frac{10^5}{10^{7.2}}\ \ (=0.0063095\ldots)$ ***(M1)***

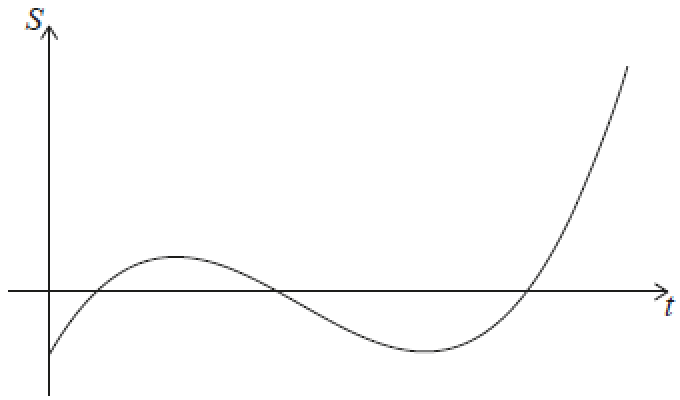
length of time $=\frac{1}{0.0063095\ldots}=10^{2.2}$

$=158$ years ***A1***

***[2 marks]***

**17.** [Maximum mark: 8] 22M.1.SL.TZ2.9

The graph below shows the average savings, $S$ thousand dollars, of a group of university graduates as a function of $t$, the number of years after graduating from university.




(a) Write down one feature of this graph which suggests a cubic function might be appropriate to model this scenario. [1]

Markscheme

*Accept any one of the following (or equivalent):*

one minimum and one maximum point
three $x$-intercepts or three roots (or zeroes)
one point of inflexion ***R1***

**Note:** Do not accept "S shape" as a justification.

***[1 mark]***

The equation of the model can be expressed in the form $S = at^3 + bt^2 + ct + d$, where $a,\ b,\ c$ and $d$ are real constants.

The graph of the model must pass through the following four points.

| $t$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $S$ | –5 | 3 | –1 | –5 |

(b.i) Write down the value of $d$. [1]

Markscheme

$(d =) - 5$ ***A1***

***[1 mark]***

(b.ii) Write down three simultaneous equations for $a,\ b$ and $c$. [2]

Markscheme

$8 = a + b + c$

$4 = 8a + 4b + 2c$

$0 = 27a + 9b + 3c$ ***A2***

**Note:** Award ***A2*** if all three equations are correct.
Award ***A1*** if at least one is correct. Award ***A1*** for three correct equations that include the letter "$d$".

***[2 marks]***

(b.iii) Hence, or otherwise, find the values of $a,\ b$ and $c$. [1]

Markscheme

$a = 2,\ b = -12,\ c = 18$ ***A1***

***[1 mark]***

A negative value of $S$ indicates that a graduate is expected to be in debt.

(c) Use the model to determine the total length of time, in years, for which a graduate is expected to be in debt after graduating from university. [3]

Markscheme

equating found expression to zero ***(M1)***

$0 = 2t^3 - 12t^2 + 18t - 5$

$t = 0.358216\ldots,\ 1.83174\ldots,\ 3.81003\ldots$ ***(A1)***

(so total time in debt is $3.81003\ldots - 1.83174\ldots + 0.358216 \approx$)

$2.34\ \ (2.33650\ldots)$ years ***A1***

***[3 marks]***

**18.** [Maximum mark: 8] 22M.1.AHL.TZ1.12

The strength of earthquakes is measured on the Richter magnitude scale, with values typically between $0$ and $8$ where $8$ is the most severe.

The Gutenberg–Richter equation gives the average number of earthquakes per year, $N$, which have a magnitude of at least $M$. For a particular region the equation is

$\log_{10} N = a - M$, for some $a \in \mathbb{R}$.

This region has an average of $100$ earthquakes per year with a magnitude of at least $3$.

(a) Find the value of $a$. [2]

Markscheme

$\log_{10} 100 = a - 3$ ***(M1)***

$a = 5$ ***A1***

***[2 marks]***

The equation for this region can also be written as $N = \frac{b}{10^M}$.

(b) Find the value of $b$. [2]

Markscheme

**EITHER**

$N = 10^{5-M}$ ***(M1)***

$= \frac{10^5}{10^M} \left( = \frac{100000}{10^M} \right)$

**OR**

$100 = \frac{b}{10^3}$ ***(M1)***

**THEN**

$b = 100000 \ \left(= 10^5\right)$ ***A1***

***[2 marks]***

Within this region the most severe earthquake recorded had a magnitude of $7.2$
.

(c) Find the average number of earthquakes in a year with a magnitude of at least $7.2$. [1]

Markscheme

$N = \frac{10^5}{10^{7.2}} = 0.00631 \quad (0.0063095\ldots)$ ***A1***

**Note:** Do not accept an answer of $10^{-2.2}$.

***[1 mark]***

The number of earthquakes in a given year with a magnitude of at least $7.2$ can be modelled by a Poisson distribution, with mean $N$.The number of earthquakes in one year is independent of the number of earthquakes in any other year.

Let $Y$ be the number of years between the earthquake of magnitude $7.2$ and the next earthquake of at least this magnitude.

(d) Find $\mathrm{P}(Y > 100)$. [3]

Markscheme

**METHOD 1**

$Y > 100 \Rightarrow$ no earthquakes in the first $100$ years ***(M1)***

**EITHER**

let $X$ be the number of earthquakes of at least magnitude $7.2$ in a year

$X \sim \mathrm{Po}(0.0063095\ldots)$

$(\mathrm{P}(X = 0))^{100}$ ***(M1)***

**OR**

let $X$ be the number of earthquakes in $100$ years

$X \sim \mathrm{Po}(0.0063095\ldots \times 100)$ ***(M1)***

$\mathrm{P}(X = 0)$

**THEN**

$0.532 \ \ (0.532082\ldots)$ ***A1***

**METHOD 2**

$Y > 100 \Rightarrow$ no earthquakes in the first $100$ years ***(M1)***

let $X$ be the number of earthquakes in $100$ years since $n$ is large and $p$ is small

$X$~$\mathrm{B}(100,\ 0.0063095\ldots)$ ***(M1)***

$\mathrm{P}(X=0)$

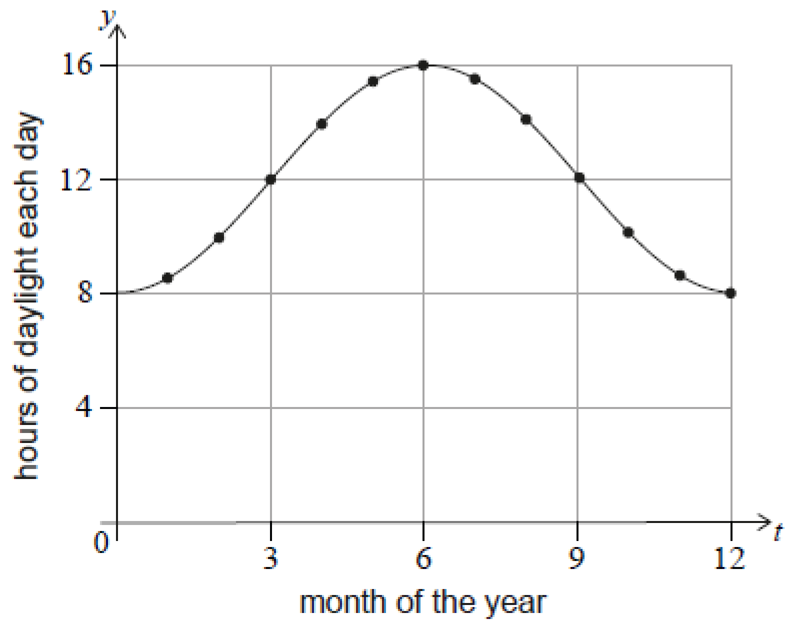
$0.531\ \ (0.531019\ldots)$ ***A1***

***[3 marks]***

**19.** [Maximum mark: 15] 22M.2.SL.TZ1.1

Boris recorded the number of daylight hours on the first day of each month in a northern hemisphere town.

This data was plotted onto a scatter diagram. The points were then joined by a smooth curve, with minimum point $(0,\ 8)$ and maximum point $(6,\ 16)$ as shown in the following diagram.




Let the curve in the diagram be $y = f(t)$, where $t$ is the time, measured in months, since Boris first recorded these values.

Boris thinks that $f(t)$ might be modelled by a quadratic function.

(a) Write down one reason why a quadratic function would not be a good model for the number of hours of daylight per day, across a number of years. [1]

Markscheme

**EITHER**

annual cycle for daylight length ***R1***

**OR**

there is a minimum length for daylight (cannot be negative) ***R1***

**OR**

a quadratic could not have a maximum and a minimum or equivalent ***R1***

**Note:** Do not accept "Paula's model is better".

***[1 mark]***

Paula thinks that a better model is $f(t) = a\cos(bt) + d, t \geq 0$, for specific values of $a,\ b$ and $d$.

For Paula's model, use the diagram to write down

(b.i) the amplitude. [1]

Markscheme

$4$ ***A1***

***[1 mark]***

(b.ii) the period. [1]

Markscheme

$12$ ***A1***

***[1 mark]***

(b.iii) the equation of the principal axis. [2]

Markscheme

$y = 12$ ***A1A1***

**Note:** Award ***A1*** "$y =$ (a constant)" and ***A1*** for that constant being $12$.

***[2 marks]***

(c) Hence or otherwise find the equation of this model in the form:

$f(t) = a\ \cos(bt) + d$ [3]

Markscheme

$f(t) = -4\ \cos(30t) + 12$ **OR** $f(t) = -4\ \cos(-30t) + 12$ ***A1A1A1***

**Note:** Award ***A1*** for $b = 30$ (or $b = -30$), ***A1*** for $a = -4$, and ***A1*** for $d = 12$. Award at most ***A1A1A0*** if extra terms are seen or form is incorrect. Award at most ***A1A1A0*** if $x$ is used instead of $t$.

***[3 marks]***

(d) For the first year of the model, find the length of time when there are more than $10$ hours and $30$ minutes of daylight per day. [4]

Markscheme

$10.5 = -4\ \cos(30t) + 12$ ***(M1)***

**EITHER**

$t_1 = 2.26585\ldots,\ \ t_2 = 9.73414\ldots$ ***(A1)(A1)***

**OR**

$t_1 = \frac{1}{30}\cos^{-1}\frac{3}{8}$ ***(A1)***

$t_2 = 12 - t_1$ ***(A1)***

**THEN**

$9.73414\ldots - 2.26585\ldots$

$7.47 \quad (7.46828\ldots)$ months $(0.622356\ldots$ years) ***A1***

**Note:** Award ***M1A1A1A0*** for an unsupported answer of $7.46$. If there is only one intersection point, award ***M1A1A0A0***.

***[4 marks]***

The true maximum number of daylight hours was $16$ hours and $14$ minutes.

(e) Calculate the percentage error in the maximum number of daylight hours Boris recorded in the diagram. [3]

Markscheme

$\left|\frac{16-\left(16+\frac{14}{60}\right)}{16+\frac{14}{60}}\right| \times 100\%$ ***(M1)(M1)***

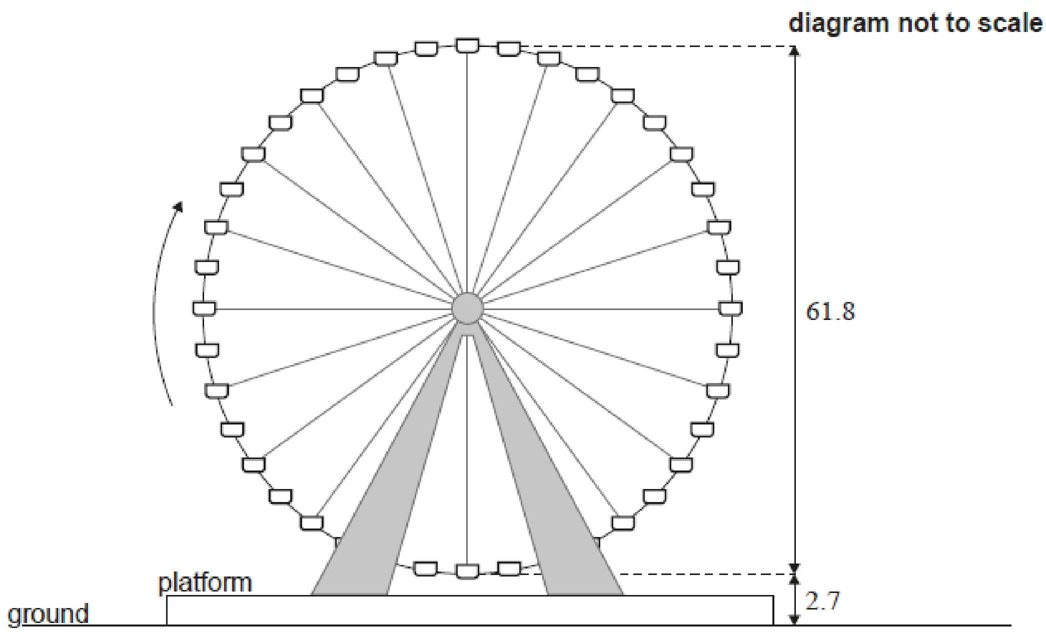
**Note:** Award ***M1*** for correct values and absolute value signs, ***M1*** for $\times 100$.

$= 1.44\% \quad (1.43737\ldots\%)$ ***A1***

***[3 marks]***

**20.** [Maximum mark: 17] 22M.2.SL.TZ2.4

The Texas Star is a Ferris wheel at the state fair in Dallas. The Ferris wheel has a diameter of $61.8\,\text{m}$. To begin the ride, a passenger gets into a chair at the lowest point on the wheel, which is $2.7\,\text{m}$ above the ground, as shown in the following diagram. A ride consists of multiple revolutions, and the Ferris wheel makes $1.5$ revolutions per minute.




The height of a chair above the ground, $h$, measured in metres, during a ride on the Ferris wheel can be modelled by the function $h(t) = -a\cos(bt) + d$, where $t$ is the time, in seconds, since a passenger began their ride.

Calculate the value of

(a.i) $a$. [2]

Markscheme

an attempt to find the amplitude ***(M1)***

$\frac{61.8}{2}$ **OR** $\frac{64.5-2.7}{2}$

$(a =)\ 30.9\,\text{m}$ ***A1***

**Note:** Accept an answer of $(a =) \; -30.9\,\text{m}$.

***[2 marks]***

(a.ii) $b$. [2]

Markscheme

(period $= \frac{60}{1.5} =$) $40\,(\text{s})$ ***(A1)***

($(b =)$ $\frac{360^\circ}{40}$)

$(b =)$ 9 ***A1***

**Note:** Accept an answer of $(b =) \; -9$.

***[2 marks]***

(a.iii) $d$. [2]

Markscheme

attempt to find $d$ ***(M1)***

$(d =)$ $30.9 + 2.7$ **OR** $\frac{64.5+2.7}{2}$

$(d =)$ $33.6\,\text{m}$ ***A1***

***[2 marks]***

A ride on the Ferris wheel lasts for $12$ minutes in total.

(b) Calculate the number of revolutions of the Ferris wheel per ride. [2]

Markscheme

$12 \times 1.5$ **OR** $\frac{12\times60}{40}$ ***(M1)***

$18$ (revolutions per ride) ***A1***

***[2 marks]***

For exactly one ride on the Ferris wheel, suggest

(c.i) an appropriate domain for $h(t)$. [1]

Markscheme

$0 \leq t \leq 720$ ***A1***

***[1 mark]***

(c.ii) an appropriate range for $h(t)$. [2]

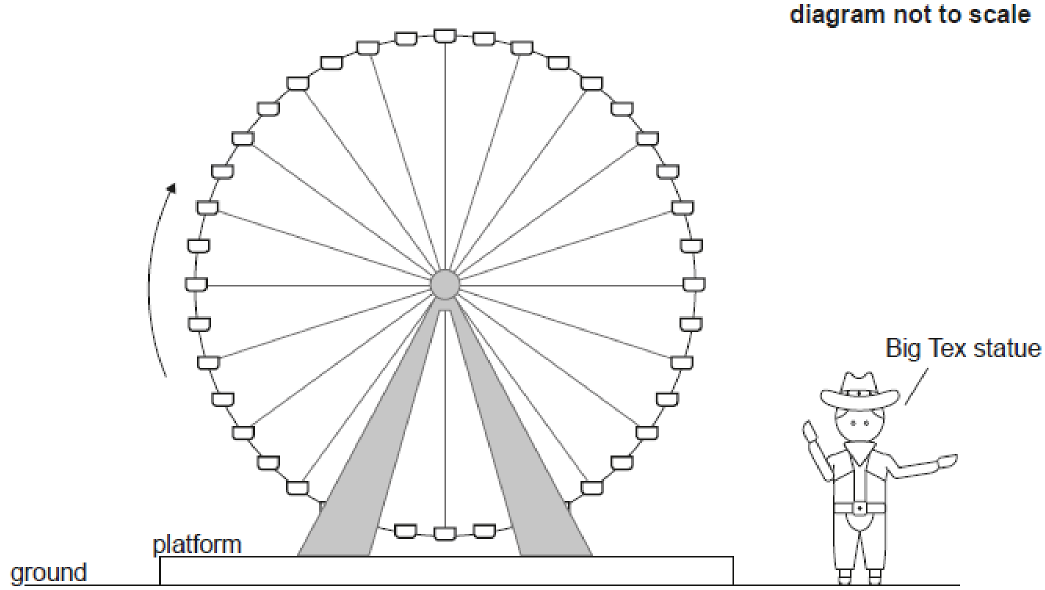
Markscheme

$2.7 \leq h \leq 64.5$ ***A1A1***

**Note:** Award ***A1*** for correct endpoints of domain and ***A1*** for correct endpoints of range. Award ***A1*** for correct direction of both inequalities.

***[2 marks]***

Big Tex is a $16.7$ metre-tall cowboy statue that stands on the horizontal ground next to the Ferris wheel.




[Source: Aline Escobar., n.d. Cowboy. [image online] Available at: https://thenounproject.com/search/?q=cowboy&i=1080130

 [Accessed 13/05/2021]. Source adapted.]

(d) By considering the graph of $h(t)$, determine the length of time during one revolution of the Ferris wheel for which the chair is higher than the cowboy statue. [3]

Markscheme

graph of $h(t)$ and $y = 16.7$ **OR** $h(t) = 16.7$ ***(M1)***

$6.31596\ldots$ and $33.6840\ldots$ ***(A1)***

$27.4\,(\mathrm{s}) \quad (27.3680\ldots)$ ***A1***

***[3 marks]***

There is a plan to relocate the Texas Star Ferris wheel onto a taller platform which will increase the maximum height of the Ferris wheel to $65.2\,\mathrm{m}$. This will change the value of one parameter, $a$, $b$ or $d$, found in part (a).

(e.i) Identify which parameter will change. [1]

Markscheme

$d$ ***A1***

***[1 mark]***

(e.ii) Find the new value of the parameter identified in part (e)(i). [2]

Markscheme

**EITHER**

$d + 30.9 = 65.2$ ***(A1)***

**OR**

$65.2 - (61.8 + 2.7) = 0.7$ ***(A1)***

**OR**

$3.4$ (new platform height) ***(A1)***

**THEN**

$(d =)\ 34.3\,\mathrm{m}$ ***A1***

***[2 marks]***

**21.** [Maximum mark: 4] 21N.1.SL.TZ0.3

Natasha carries out an experiment on the growth of mould. She believes that the growth can be modelled by an exponential function

$P(t) = Ae^{kt}$,

where $P$ is the area covered by mould in $\mathrm{mm}^2$, $t$ is the time in days since the start of the experiment and $A$ and $k$ are constants.

The area covered by mould is $112\,\mathrm{mm}^2$ at the start of the experiment and $360\,\mathrm{mm}^2$ after $5$ days.

(a) Write down the value of $A$. [1]

Markscheme

$(A =)\ 112$ ***A1***

***[1 mark]***

(b) Find the value of $k$. [3]

Markscheme

$112e^{5k} = 360$ ***(M1)***

**Note:** Award ***(M1)*** for their correct equation.

**EITHER**

graph of $y = 112e^{5k}$ and $y = 360$ with indication of point of intersection ***(M1)***

**OR**

$(k =)\ \frac{1}{5}\ln\left(\frac{360}{112}\right)$ ***(M1)***

**Note:** Award ***(M1)*** for correct rearranging and use of $\log$.

**THEN**

$(k =)\ 0.234\ \ (0.233521\ldots)$ ***A1***

**Note:** Award ***(M1)(M1)(A0)*** for $0.233$.

***[3 marks]***

**22.** [Maximum mark: 5] 21N.1.AHL.TZ0.12

The following table shows the time, in days, from December $1$st and the percentage of Christmas trees in stock at a shop on the beginning of that day.

| Days since December 1st ($d$) | 1 | 3 | 6 | 9 | 12 | 15 | 18 |
|---|---|---|---|---|---|---|---|
| Percentage of Christmas trees left in stock ($x$) | 100 | 51 | 29 | 21 | 18 | 16 | 14 |

The following table shows the natural logarithm of both $d$ and $x$ on these days to $2$ decimal places.

| ln ($d$) | 0 | 1.10 | 1.79 | 2.20 | 2.48 | 2.71 | 2.89 |
|---|---|---|---|---|---|---|---|
| ln ($x$) | 4.61 | 3.93 | 3.37 | 3.04 | 2.89 | 2.77 | 2.64 |

(a) Use the data in the second table to find the value of $m$ and the value of $b$ for the regression line, $\ln x = m(\ln d) + b$.

[2]

Markscheme

$m = -0.695 \ (-0.695383\ldots); \ b = 4.63 \ (4.62974\ldots)$ ***A1A1***

***[2 marks]***

(b) Assuming that the model found in part (a) remains valid, estimate the percentage of trees in stock when $d = 25$. [3]

Markscheme

$\ln x = -0.695(\ln 25) + 4.63$ ***M1***

$\ln x = 2.39288\ldots$ ***(A1)***

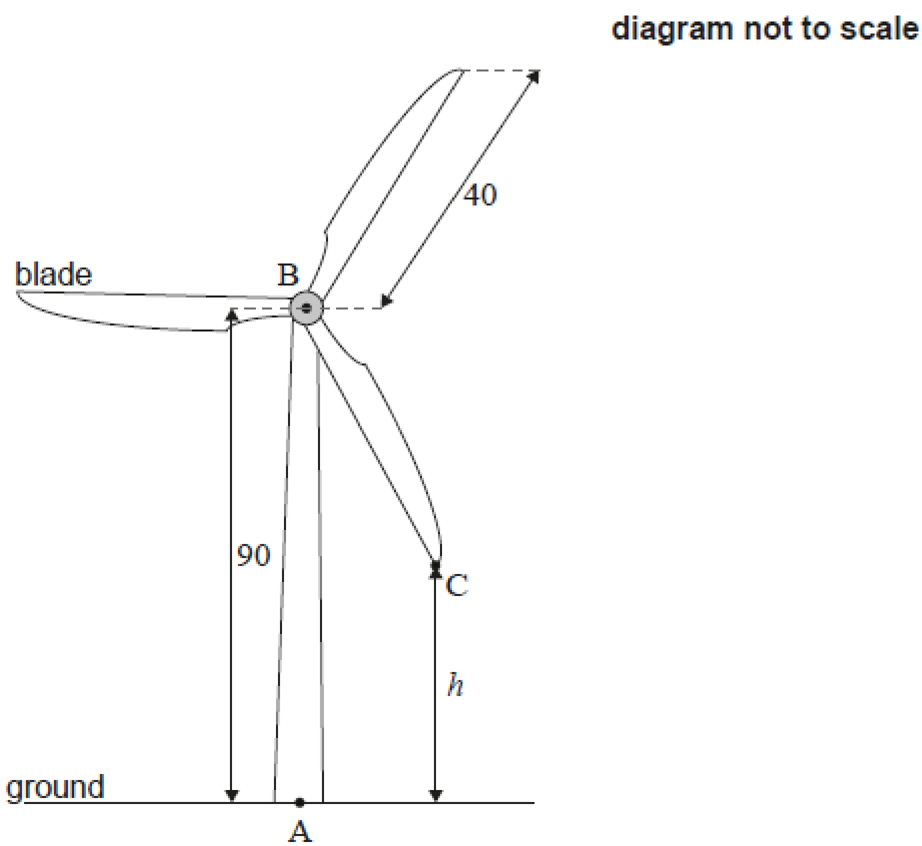
$x = 10.9\%$ ***A1***

***[3 marks]***

**23.** [Maximum mark: 20] 21N.2.AHL.TZ0.2

A wind turbine is designed so that the rotation of the blades generates electricity. The turbine is built on horizontal ground and is made up of a vertical tower and three blades.

The point $\mathrm{A}$ is on the base of the tower directly below point $\mathrm{B}$ at the top of the tower. The height of the tower, $\mathrm{AB}$, is $90\,\mathrm{m}$. The blades of the turbine are centred at $\mathrm{B}$ and are each of length $40\,\mathrm{m}$. This is shown in the following diagram.




The end of one of the blades of the turbine is represented by point $\mathrm{C}$ on the diagram. Let $h$ be the height of $\mathrm{C}$ above the ground, measured in metres, where $h$ varies as the blade rotates.

Find the

(a.i) maximum value of $h$. [1]

Markscheme

maximum $h = 130$ metres ***A1***

***[1 mark]***

(a.ii) minimum value of $h$. [1]

Markscheme

minimum $h = 50$ metres ***A1***

***[1 mark]***

The blades of the turbine complete $12$ rotations per minute under normal conditions, moving at a constant rate.

(b.i) Find the time, in seconds, it takes for the blade $[\mathrm{BC}]$ to make one complete rotation under these conditions. [1]

Markscheme

$(60 \div 12 =)\ 5 \text{ seconds}$ ***A1***

***[1 mark]***

(b.ii) Calculate the angle, in degrees, that the blade $[\mathrm{BC}]$ turns through in one second. [2]

Markscheme

$360 \div 5$ ***(M1)***

**Note:** Award ***(M1)*** for $360$ divided by their time for one revolution.

$= 72^{\circ}$ ***A1***

***[2 marks]***

The height, $h$, of point $\mathrm{C}$ can be modelled by the following function. Time, $t$, is measured from the instant when the blade $[\mathrm{BC}]$ first passes $[\mathrm{AB}]$ and is measured in seconds.

$h(t) = 90 - 40\cos(72t^{\circ}),\ t \geq 0$

(c.i) Write down the amplitude of the function. [1]

Markscheme

(amplitude =) $40$ ***A1***

***[1 mark]***

(c.ii) Find the period of the function. [1]

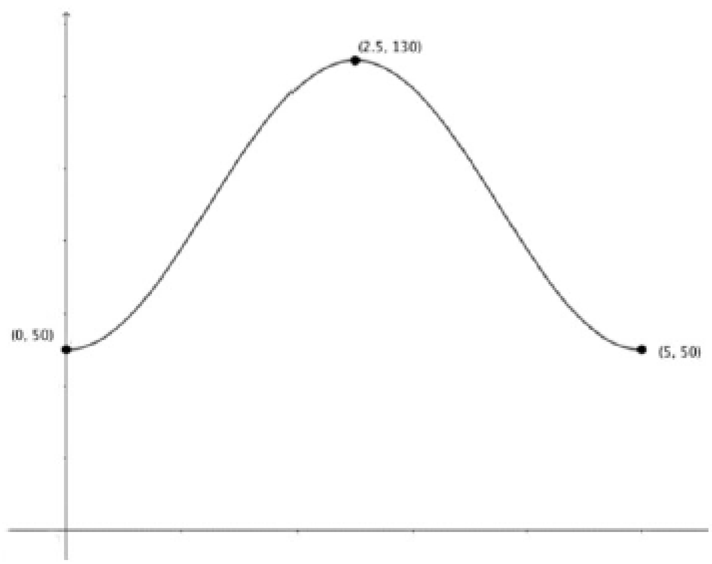
Markscheme

(period $= \frac{360}{72} =$) $5$ ***A1***

***[1 mark]***

(d) Sketch the function $h(t)$ for $0 \leq t \leq 5$, clearly labelling the coordinates of the maximum and minimum points. [3]

Markscheme




Maximum point labelled with correct coordinates. ***A1***

At least one minimum point labelled. Coordinates seen for any minimum points must be correct. ***A1***

Correct shape with an attempt at symmetry and “concave up" evident as it approaches the minimum points. Graph must be drawn in the given domain. ***A1***

***[3 marks]***

(e.i) Find the height of $\mathrm{C}$ above the ground when $t = 2$. [2]

Markscheme

$h = 90 - 40\cos(144^\circ)$ ***(M1)***

$(h =)\ 122\,(\mathrm{m})\ \ (122.3606\ldots)$ ***A1***

***[2 marks]***

(e.ii) Find the time, in seconds, that point $\mathrm{C}$ is above a height of $100\ \mathrm{m}$, during each complete rotation. [3]

Markscheme

evidence of $h = 100$ on graph **OR** $100 = 90 - 40\cos(72t)$ ***(M1)***

$t$ coordinates $3.55\ (3.54892\ldots)$ **OR** $1.45\ (1.45107\ldots)$ or equivalent ***(A1)***

**Note:** Award ***A1*** for either $t$-coordinate seen.

$= 2.10$ seconds $(2.09784\ldots)$ ***A1***

***[3 marks]***

(f) The wind speed increases and the blades rotate faster, but still at a constant rate.

Given that point $\mathrm{C}$ is now higher than $110\ \mathrm{m}$ for $1$ second during each complete rotation, find the time for one complete rotation. [5]

Markscheme

**METHOD 1**

$90 - 40\cos(at^\circ) = 110$ ***(M1)***

$\cos(at^\circ) = -0.5$

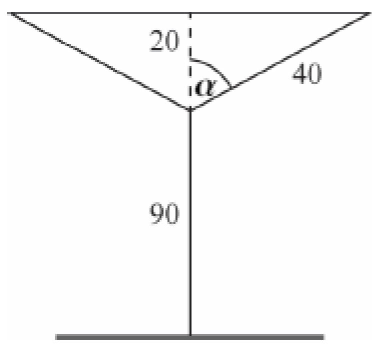
$at^\circ = 120,\ 240$ ***(A1)***

$1 = \frac{240}{a} - \frac{120}{a}$ ***(M1)***

$a = 120$ ***(A1)***

period $= \frac{360}{120} = 3$ seconds ***A1***

**METHOD 2**




attempt at diagram ***(M1)***

$\cos\alpha = \frac{20}{40}$ (or recognizing special triangle) ***(M1)***

angle made by $\mathrm{C}, 2\alpha = 120^\circ$ ***(A1)***

one third of a revolution in $1$ second ***(M1)***

hence one revolution $= 3$ seconds ***A1***

**METHOD 3**

considering $h(t) = 110$ on original function ***(M1)***

$t = \frac{5}{3}$ or $\frac{10}{3}$ ***(A1)***

$\frac{10}{3} - \frac{5}{3} = \frac{5}{3}$ ***(A1)***

**Note:** Accept $t = 1.67$ or equivalent.

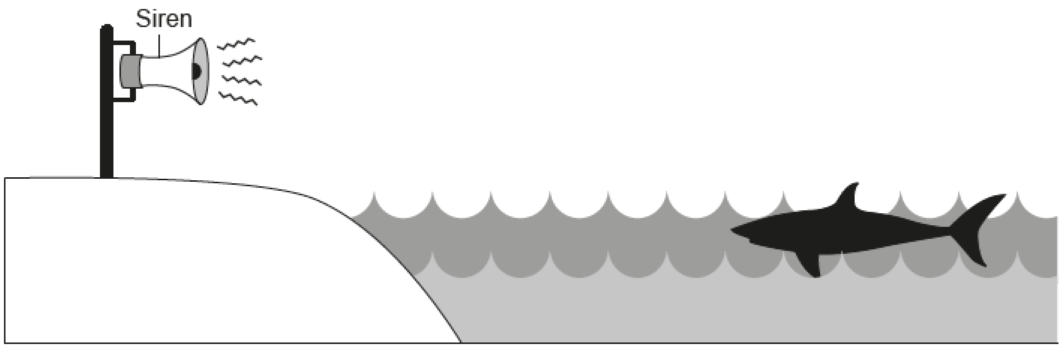
so period is $\frac{3}{5}$ of original period ***(R1)***

so new period is $3$ seconds ***A1***

***[5 marks]***

**24.** [Maximum mark: 6] 21M.1.SL.TZ1.11

If a shark is spotted near to Brighton beach, a lifeguard will activate a siren to warn swimmers.




The sound intensity, $I$, of the siren varies inversely with the square of the distance, $d$, from the siren, where $d > 0$.

It is known that at a distance of $1.5$ metres from the siren, the sound intensity is $4$ watts per square metre ($\mathrm{W\,m^{-2}}$).

(a) Show that $I = \frac{9}{d^2}$. [2]

Markscheme

$I = \frac{k}{d^2}$ ***(M1)***

$4 = \frac{k}{1.5^2}$ ***M1***

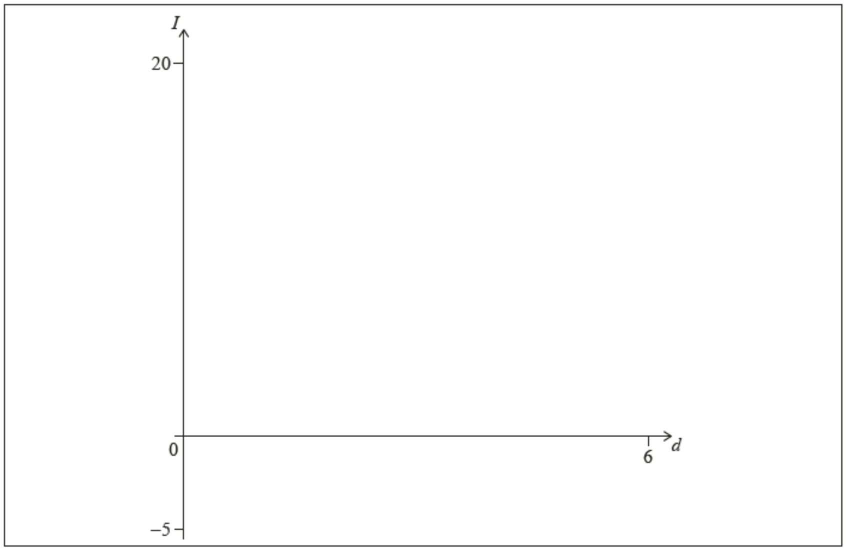
$I = \frac{9}{d^2}$ ***AG***

**Note:** The ***AG*** line must be seen for the second ***M1*** to be awarded.

Award no marks for substituting $1.5$ and $4$ into $I = \frac{9}{d^2}$ (i.e., working backwards).

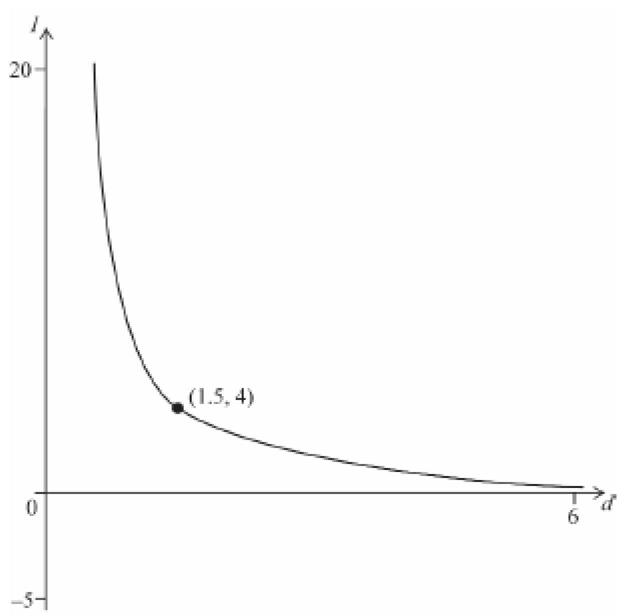
***[2 marks]***

(b) Sketch the curve of $I$ on the axes below showing clearly the point $(1.5,\ 4)$.




[2]

Markscheme




***A1A1***

**Note:** Award ***A1*** for correct general shape (concave up) with no $I$-intercept, passing through the marked point $(1.5,\ 4)$; the point must be labelled with either the coordinates or the values $1.5$ and $4$ on the $x$ and $y$ axes.

Award ***A1*** for the curve showing asymptotic behavior (i.e. $I$ tends to $0$, as $d$ tends to infinity), extending to at least $d = 6$; the curve must not cross nor veer away from the horizontal asymptote.

***[2 marks]***

(c) Whilst swimming, Scarlett can hear the siren only if the sound intensity at her location is greater than $1.5 \times 10^{-6}\,\mathrm{W\,m^{-2}}$.

Find the values of $d$ where Scarlett cannot hear the siren. [2]

Markscheme

$1.5 \times 10^{-6} \geq \frac{9}{d^2}$ ***(M1)***

**Note:** Award ***(M1)*** for a correct inequality.

$d \geq 2450\,(\mathrm{m})\ \ (2449.48\ldots)$ ***A1***

**Note:** Award ***A0*** for $d = 2450$.

***[2 marks]***

**25.** [Maximum mark: 5] 21M.1.SL.TZ2.1

The amount, in milligrams, of a medicinal drug in the body $t$ hours after it was injected is given by $D(t) = 23(0.85)^t,\ t \geq 0$. Before this injection, the amount of the drug in the body was zero.

Write down

(a.i) the initial dose of the drug. [1]

Markscheme

$23\,\text{mg}$ ***A1***

***[1 mark]***

(a.ii) the percentage of the drug that leaves the body each hour. [2]

Markscheme

$1 - 0.85$ **OR** $\frac{23-19.55}{23}$ **OR** $0.15$ ***(M1)***

$15\,(\%)$ ***A1***

***[2 marks]***

(b) Calculate the amount of the drug remaining in the body $10$ hours after the injection. [2]

Markscheme

$23(0.85)^{10}$ ***(M1)***

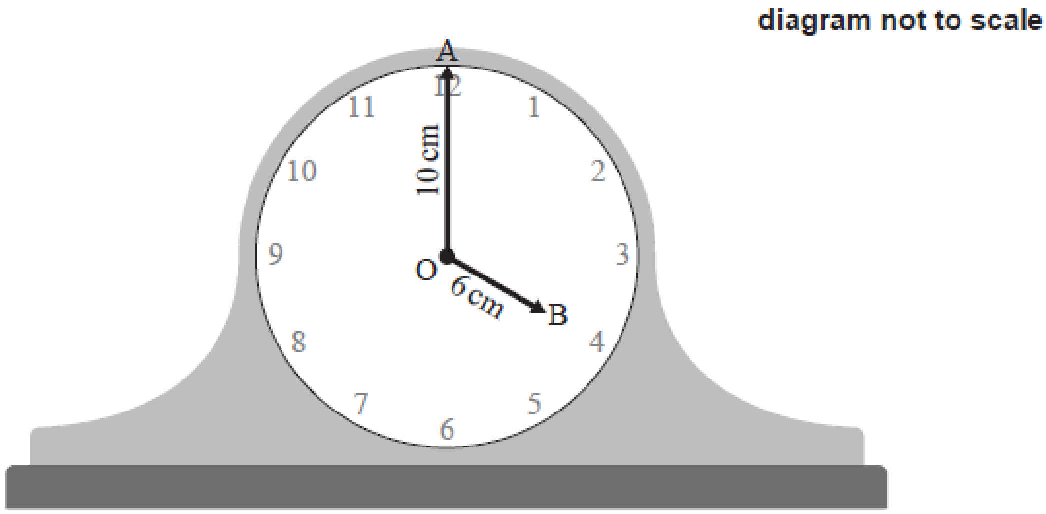
$4.53\,\text{mg}\ \ (4.52811\ldots)$ ***A1***

*[2 marks]*

**26.** [Maximum mark: 17] 21M.2.SL.TZ1.2

The diagram below shows a circular clockface with centre $\mathrm{O}$. The clock's minute hand has a length of $10\,\mathrm{cm}$. The clock's hour hand has a length of $6\,\mathrm{cm}$.

At $4:00$ pm the endpoint of the minute hand is at point $\mathrm{A}$ and the endpoint of the hour hand is at point $\mathrm{B}$.




(a) Find the size of angle $\mathrm{A\hat{O}B}$ in degrees. [2]

Markscheme

$4 \times \frac{360^\circ}{12}$ **OR** $4 \times 30^\circ$ ***(M1)***

$120^\circ$ ***A1***

***[2 marks]***

(b) Find the distance between points $\mathrm{A}$ and $\mathrm{B}$. [3]

Markscheme

substitution in cosine rule ***(M1)***

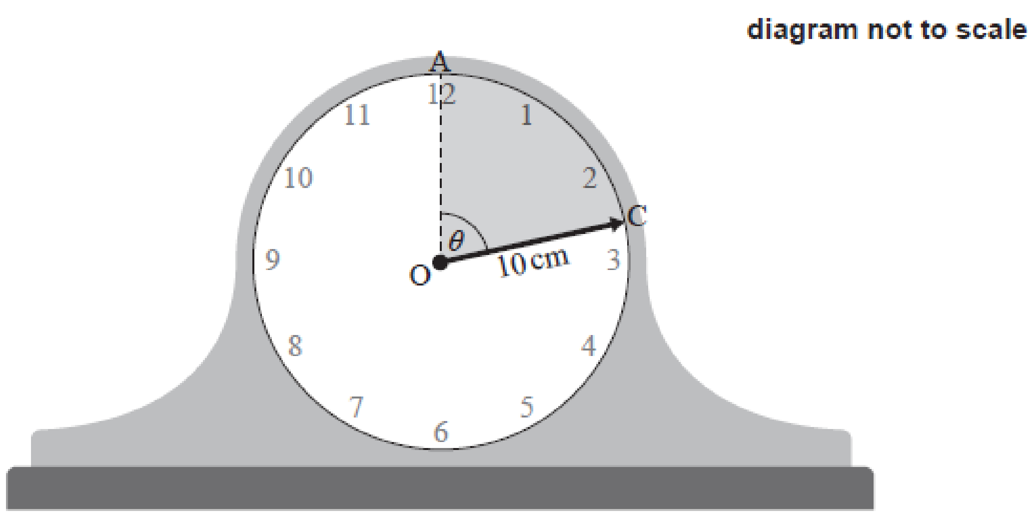
$\mathrm{AB}^2 = 10^2 + 6^2 - 2 \times 10 \times 6 \times \cos(120^\circ)$ ***(A1)***

$\mathrm{AB} = 14\,\mathrm{cm}$ ***A1***

**Note:** Follow through marks in part (b) are contingent on working seen.

***[3 marks]***

Between $4:00$ pm and $4:13$ pm, the endpoint of the **minute hand** rotates through an angle, $\theta$, from point $\mathrm{A}$ to point $\mathrm{C}$. This is illustrated in the diagram.




(c) Find the size of angle $\theta$ in degrees. [2]

Markscheme

$\theta = 13 \times 6$ ***(M1)***

$= 78^\circ$ ***A1***

***[2 marks]***

(d) Calculate the length of arc $\mathrm{AC}$. [2]

Markscheme

substitution into the formula for arc length ***(M1)***

$l = \frac{78}{360} \times 2 \times \pi \times 10$ **OR** $l = \frac{13\pi}{30} \times 10$

$= 13.6 \text{ cm} \left(13.6135\ldots,\ 4.33\pi,\ \frac{13\pi}{3}\right)$ ***A1***

***[2 marks]***

(e) Calculate the area of the shaded sector, $\mathrm{AOC}$. [2]

Markscheme

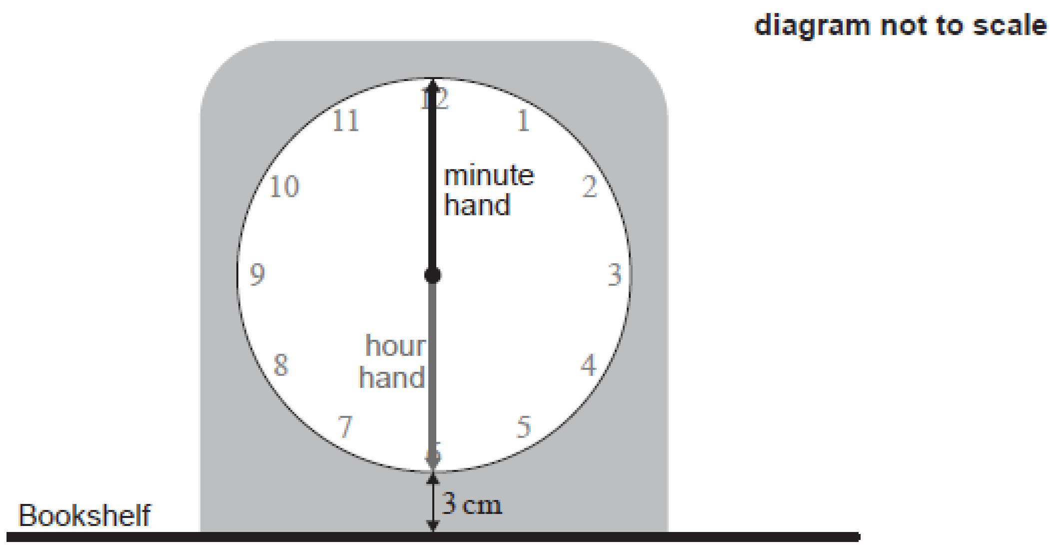
substitution into the area of a sector ***(M1)***

$A = \frac{78}{360} \times \pi \times 10^2$ **OR** $l = \frac{1}{2} \times \frac{13\pi}{30} \times 10^2$

$= 68.1 \text{ cm}^2 \left(68.0678\ldots,\ 21.7\pi,\ \frac{65\pi}{3}\right)$ ***A1***

***[2 marks]***

A **second** clock is illustrated in the diagram below. The clock face has radius $10 \text{ cm}$ with minute and hour hands both of length $10 \text{ cm}$. The time shown is $6:00$ am. The bottom of the clock face is located $3 \text{ cm}$ above a horizontal bookshelf.




(f) Write down the height of the endpoint of the minute hand above the bookshelf at $6:00$ am. [1]

Markscheme

$23$ ***A1***

***[1 mark]***

The height, $h$ centimetres, of the endpoint of the minute hand above the bookshelf is modelled by the function

$$h(\theta) = 10\cos\theta + 13,\ \theta \geq 0,$$

where $\theta$ is the angle rotated by the minute hand from $6:00$ am.

(g) Find the value of $h$ when $\theta = 160^\circ$. [2]

Markscheme

correct substitution ***(M1)***

$$h = 10\cos(160^\circ) + 13$$

$= 3.60 \text{ cm} \ \ (3.60307\ldots)$ ***A1***

***[2 marks]***

The height, $g$ centimetres, of the endpoint of the **hour hand** above the bookshelf is modelled by the function

$g(\theta) = -10 \cos\left(\frac{\theta}{12}\right) + 13, \ \theta \geq 0,$

where $\theta$ is the angle in degrees rotated by the minute hand from $6 : 00$ am.

(h) Write down the amplitude of $g(\theta)$. [1]

Markscheme

$10$ ***A1***

***[1 mark]***

(i) The endpoints of the minute hand and hour hand meet when $\theta = k$.

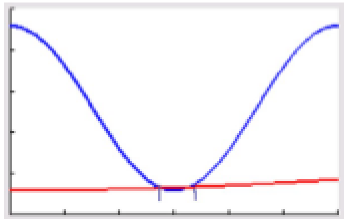
Find the smallest possible value of $k$. [2]

Markscheme

**EITHER**

$10 \times \cos(\theta) + 13 = -10 \times \cos\left(\frac{\theta}{12}\right) + 13$ ***(M1)***

**OR**



*(M1)*

**Note:** Award ***M1*** for equating the functions. Accept a sketch of $h(\theta)$ and $g(\theta)$ with point(s) of intersection marked.

**THEN**

$k = 196^{\circ} \quad (196.363\ldots)$ ***A1***

**Note:** The answer $166.153\ldots$ is incorrect but the correct method is implicit. Award ***(M1)A0***.

***[2 marks]***