Internal Assessment

Exploring Radii of Curvature with Functions and Real-life Curves.

How Can Radii of Curvature Indicate if a Curve is a Part of a Circle?

IB Mathematics Analysis and Approaches HL

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1. Introduction and Exploration Outline

From curved walls to differential calculus, curvature is present throughout our lives. This exploration will encompass both of these concepts to analyze radii of curvature of known curves and real-life curved surfaces. During an observation of our school's interior I noticed a curved wall with a painting on it. This peculiar interior design choice spurred an exploration of curvatures and how they can be described.

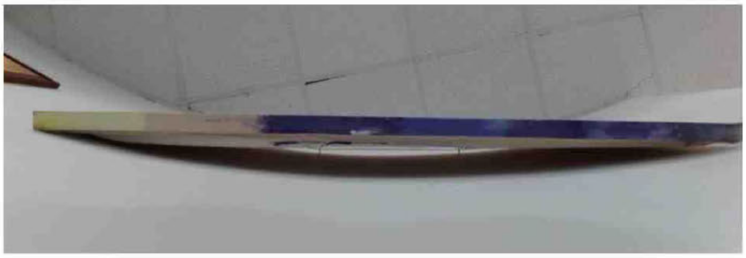

Figure 1: The curved wall.

One of the questions that arose during the exploration was whether the curve on the photograph (Figure 1) above can be described by a single circle. During the exploration of curvature, the concept of radii of curvature became most interesting in application to this question. Radii of curvature are a way to describe the radius of a circle needed to draw a curve at a single point on a continuous curve (Svirin). Its most practical uses are in physics, where it is applied to measure the stress in semiconductors (Gutenberg), and it can be used to find instantaneous angular velocities. However, in this investigation the concept is used to determine whether a curve is a part of a circle. Any additional information from the investigation was also analyzed to demonstrate what this concept can indicate about the nature of a curve.

The formula for the radii of curvature was derived via calculus and applied to various known functions and a circle to determine any relationships between the nature of the functions and their radii. If any conjectures were formed during the analysis, they were tested with an application to another function. The knowledge gained from these observations was then applied to the wall curve to determine whether it is a part of a circle or if a different function would make for a more accurate fit. However, since the wall could not be described with a known function, linearization had to be used to approximate the first and second derivatives. The main graphing software used in this investigation was GeoGebra and it gave point values to two decimal points, hence, all values in this experiment were kept to two decimal points for consistency. Additionally, for most applications to functions the change of x was constant for consistency and to simplify the comparisons of any patterns in the changes of derivatives. This methodology was tested with an application to a known circular object – a plant pot. The conclusions reached from these analyses indicated the criteria necessary to determine whether a curve could be a part of a circle.

1.1 Derivation of the Formula

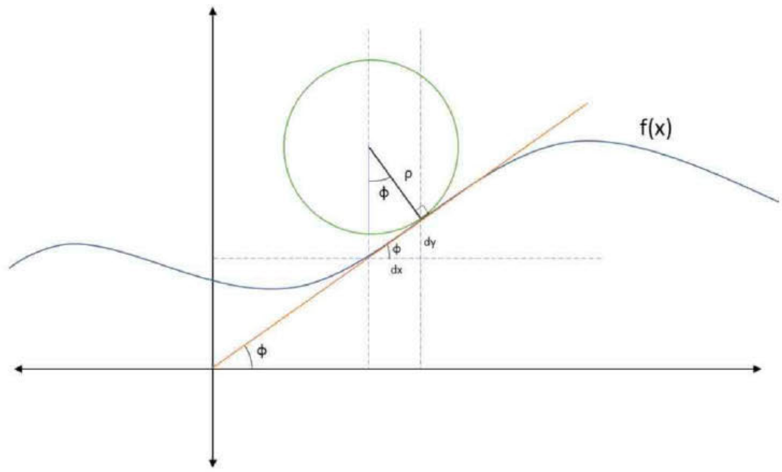

Figure 2: Example Diagram of the Radius of Curvature on a Function f(x), self-generated.

Assuming a continuous curve represented by the function $f(x)$ one can determine the radius of a circle needed to draw the curve at that point. As seen on the diagram (Figure 2) above, the radius ρ has to be perpendicular to the tangent at that point for the derivation of the formula to be successful. Furthermore, one more radius needs to be drawn at an angle φ to the first one. To determine the formula for the radii of curvature a combination of sources was used. The variables seen in the equations below are the conventional variables for this calculation since they were used in all sources.

One formula that can give us the radius is the arc length formula:

$$s = \rho\varphi \qquad Eq.1$$

Where s is the arc length, ρ is the radius, and φ is the angle between the two radii.

However, in the current investigation I do not know the angle measure and the arc length. Additionally, the radius of curvature has to be instantaneous since the tangent is also instantaneous. Therefore, one can consider the two radii moving closer together as the angle between them approaches zero

$$\lim_{\varphi \to 0} = d\varphi \qquad Eq.\ 2$$

Which means that the arc length at a single point can be expressed as:

$$ds = \rho\, d\varphi \qquad Eq.\ 3$$

Additionally, as the angle approaches 0 ds becomes more linear and closer to the gradient of the tangent line. Therefore, as seen on the diagram (Figure 2) its x and y components can be expressed in terms of the change in y and change in x. Taking the tangent of the angle φ can relate the x and y components of ds.

$$\tan\varphi = \frac{dy}{dx} \qquad Eq.\ 4$$

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Considering that ds becomes the hypotenuse of a right triangle with legs dy and dx, as seen on Figure 2, one can apply the Pythagorean theorem to find ds in terms of dy and dx.

$$ds^2 = dx^2 + dy^2 \qquad Eq.\ 5$$

If one factors out dx^2, Equation 5 can be expressed as:

$$ds = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad Eq.\ 6$$

If one equates the previously established equation for arc length (Eq. 3) with the one above (Eq. 6) the unknown arc length variable is eliminated:

$$\rho\, d\varphi = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \textit{Eq. 7}$$

To find ρ in terms of known variables, dy and dx, one needs to eliminate $d\varphi$. If one finds the derivative of the previous equation with tangent of the angle (Eq. 4) one can isolate the derivative of φ with respect to x (Svirin).

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\varphi)$$

$$= (sec^2\,\varphi)\left(\frac{d\varphi}{dx}\right) \qquad \textit{Eq. 8}$$

Additionally, one can substitute a trigonometric identity for secant squared of φ and equate it to the second derivative of y with respect to x (Bourne).

$$(1+tan^2\,\varphi)\frac{d\varphi}{dx} = \frac{d^2y}{dx} \qquad \textit{Eq. 9}$$

Through the substitution of the previous definition of the tangent of φ (Eq. 3) one can eliminate the angle φ and isolate $\frac{d\varphi}{dx}$ by manipulating the following equation:

$$\left(1+\left(\frac{dy}{dx}\right)^2\right)\frac{d\varphi}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{d\varphi}{dx} = \frac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)} \qquad \textit{Eq. 10}$$

By moving the dx to the left side of Equation 7 and substituting the new expression (in Eq. 10) for $\frac{d\varphi}{dx}$ one can eliminate all unknown variables except for the radius, ρ.

$$\rho\frac{\frac{d^2y}{dx}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad \textit{Eq. 11}$$

The final step is to express ρ in terms of $\frac{dy}{dx}$.

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} \qquad \textit{Eq. 12}$$

To avoid a negative radius of curvature the absolute value of the second derivative is taken since the second derivative can change signs depending on the concavity of the curve (Bourne).

2. Application to a Circle

The formula was applied to a circle equation to determine its behavior in an ideal situation. Considering that a circle can be drawn with a center and one radius, I conjectured that the radii of curvature for a circle would be equivalent to the radius of the circle.

First, an initial function was chosen:

$$x^2 + y^2 = 81 \qquad \textit{Eq. 13}$$

When taking the derivatives of the equation (Eq. 13) above, the value for the radius on :he right-hand side becomes zero. Thus, the initial value does not affect further calculations. Hence, any known radius could be used. Its only role was to provide a comparison to the resultant radii of curvature and determine whether they would be equivalent to it.

Furthermore, the equation was rearranged for finding the first and second derivatives. The steps to find the derivatives are shown below:

$$y^2 = 81 - x^2$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y} \qquad \textit{Eq. 14}$$

Then the quotient rule was used to determine the second derivative (Formula Booklet).

$$\frac{d^2y}{dx^2} = \frac{x\frac{dy}{dx} - y}{y^2} \qquad \textit{Eq. 15}$$

The values of $0 \le x < 5$ at the interval of $\Delta x = 0.06$ were used since they were all in close proximity to each other and had the same concavity – similarly to the points on the wall curve[1].

[1] Further justification and application of these x-values will be shown in later sections.

The values for y were found with the initial equation (Eq.13). Using these values and equations (Eq.14 and Eq.15) the radii of curvature were calculated.

X-Values	Y-Values	First Derivative Values	Second Derivative Values	Radii of Curvature
0.00	9.00	0.00	-0.11	9.00
0.60	8.98	-0.07	-0.11	9.00
1.20	8.92	-0.13	-0.11	9.00
1.80	8.82	-0.20	-0.12	9.00
2.40	8.67	-0.28	-0.12	9.00
3.00	8.49	-0.35	-0.13	9.00
3.60	8.25	-0.44	-0.14	9.00
4.20	7.96	-0.53	-0.16	9.00
4.80	7.61	-0.63	-0.18	9.00

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Table 1: Data for the application of the radius of curvature formula to a circle.

Table 1 demonstrates how the radii of curvature of a circle were always 9.00 – the known radius of the circle. Thus, if the radii of curvature are consistent the curve is likely to be part of a circle.

3. Application to Known Functions

The exact function which would describe the curved wall is unknown. However, some functions can be approximated and tested for their validity.

Figure 3: The Curved Wall

The requirements for fitting a function were determined visually. The functions require a minimum at the origin, symmetry about the y-axis, and a positive concavity throughout. Any functions that changed concavity were stretched horizontally to fit the requirements.

For all functions, the coordinates of the points of the wall curve were used. The radii of curvature were observed for their variations, for instance a non-constant increase. The standard deviation of the radii of a circle was zero, therefore the standard deviations of other curves were examined to determine their consistency.

$$\sigma = \sqrt{\frac{\sum_{i=1}^{k}(x_i-\mu)^2}{n}} \qquad \textit{Eq. 16}$$

In Eq. 16 above the standard deviation σ, or how much the data deviates from the mean (Wazir), is expressed in terms of the sum of the square of the difference of data points x_i and the average μ over the number of data points n. The standard deviation was determined through Microsoft Excel functions. If the standard deviation is reasonable – close to zero – then the mean radius can be assumed to approximate the radius of curvature.

3.1 Cosine

The first function analyzed was negative cosine of $\frac{x}{4}$ shifted up one unit, with the angle measurements in radians.

$$f(x) = -\cos\left(\frac{x}{4}\right) + 1 \qquad \textit{Eq. 17}$$

The function was negative and shifted up to place the minimum at the origin, as seen in Figure 4. Additionally, it was horizontally stretched by a factor of 4 to account for the change in concavity at $\frac{\pi}{4}$.

The first and second derivatives were:

$$\frac{dy}{dx} = \frac{1}{4}\sin\left(\frac{x}{4}\right) \qquad \textit{Eq. 18}$$

$$\frac{d^2y}{dx^2} = \frac{1}{16}\cos\left(\frac{x}{4}\right) \qquad \textit{Eq. 19}$$

X Values	First Derivative Values	Second Derivative Values	Radii of Curvature
0.00	0.00	0.06	16.00
0.60	0.04	0.06	16.22
1.20	0.07	0.06	16.89
1.80	0.11	0.06	18.09
2.40	0.14	0.05	19.97
3.00	0.17	0.05	22.83
3.60	0.20	0.04	27.23
4.20	0.22	0.03	34.45
4.80	0.23	0.02	47.80

Table 2: The radii of Curvature of $f(x)$.

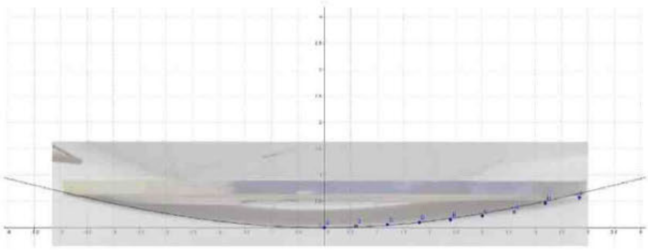

Figure 4: The curved wall with f(x), self-generated.

As seen in Figure 4, *f(x)* (the black curve) matches the seemingly symmetrical nature of the curved wall. The x-values used for *f(x)* are also those of the points used for later calculations with the curved wall. Even though this function fulfills the stated requirements, the radii of curvature are not consistent. Additionally, the standard deviation is 10.68, far from the desired zero. An important observation is that as its first derivative is increasing and the second derivative is decreasing the radii of curvature increase. Therefore, I conjecture that, as the function approaches its inflexion point the radii of curvature will increase. Furthermore, it appears that the radii at points closer to the minimum are smaller than the radii at points further away.

3.2 Testing the Conjecture

To test this conjecture a quick analysis was conducted.

$$g(x) = 2x^3 - x + 3 \qquad \text{Eq. 20}$$

This function was chosen because it switches concavities at $x=0$, and it has minima and maxima in proximity to its inflexion point.

To keep this analysis concise the x-values ranged from -2 to 2 at non-constant intervals. The x-interval was decreased between the turning points to analyze the pattern as *g(x)* approaches the inflexion point. Without the decrease it may appear that the radii were consistently small in that region. Meanwhile, few points further away from the origin were sufficient to see the pattern. Additionally, the absolute value around the second derivative was removed. Negative radii of curvature values indicated that the radius is below the curve and vice versa.

X Values	Y Values	First Derivative Values	Second Derivative Values	Radii of Curvature
-2.00	**-11.00**	**23.00**	**-24.00**	**-508.40**
-1.00	2.00	5.00	-12.00	-11.05
-0.50	3.25	0.50	-6.00	-0.23
-0.41	**3.27**	**0.00**	**-4.92**	**-0.20**
-0.30	3.25	-0.46	-3.60	-0.37
-0.20	3.18	-0.76	-2.40	-0.83
-0.10	3.10	-0.94	-1.20	-2.15
0.00	3.00	-1.00	0.00	Undefined
0.10	**2.90**	**-0.94**	**1.20**	**2.15**
0.20	2.82	-0.76	2.40	0.83
0.30	2.75	-0.46	3.60	0.37
0.41	**2.73**	**0.00**	**4.92**	**0.20**
0.50	2.75	0.50	6.00	0.23
1.00	4.00	5.00	12.00	11.05
2.00	**17.00**	**23.00**	**24.00**	**508.40**

Table 3: Data for g(x).

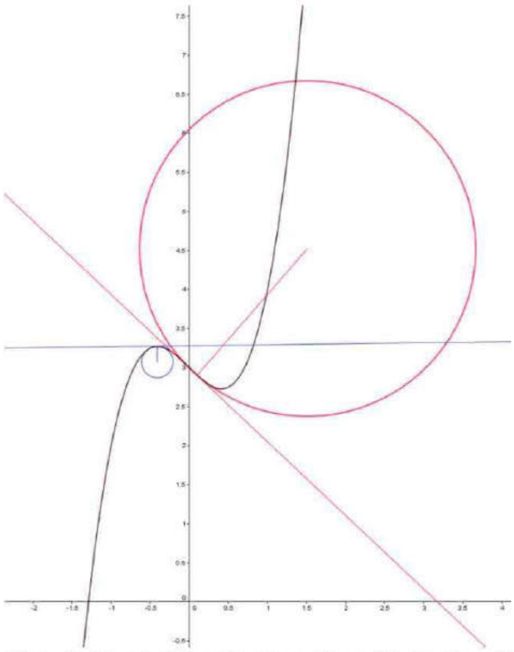

Figure 5: A graph of the radius at a maximum (blue) and a radius closer to the inflexion point (red) on g(x), self generated.

As seen on the data (Table 3) above, the conjecture is correct. The radii of curvature are smallest at the curve's maximum (-0.41, 3.27) and minimum (0.41, 2.73) and their magnitude increases as the second derivative values approach 0. This is demonstrated in Figure 5. The red lines represent the radius of curvature, corresponding circle, and a tangent line to the curve close to its inflexion point (0.10, 2.90). The radius at an inflexion point is undefined since the second derivative value is on the denominator of the radius of curvature formula (Eq. 12). The blue lines represent the radius of curvature and its corresponding aspects at the curve's maximum (-0.41, 2.73).

However, it must be noted that even though the radii of curvature increase from the minima and maxima to the inflexion point, the values in proximity to the inflexion point are not the greatest. The greatest values are at the portions of the curve with a lesser curvature, at x-values of -2 and 2. This can be explained by the fact that, as the curvature of the function decreases the curve at that instant approaches the tangent line. At a less curved section of the curve there are smaller changes in the first derivatives, hence, smaller second derivatives. Due to the nature of the radii of curvature formula (Eq.12) this results in large radii of curvature. This explains why the radii increase closer to the inflexion point since inflexion points occur at less curved portions of the function. Moreover, as seen in Table 3, the radii are below the curve when it is concave down and above the curve when it is concave up.

3.3 Quadratic

Considering the confirmed conjecture, the second function examined was a polynomial curve fitted using GeoGebra to the points on the curved wall.

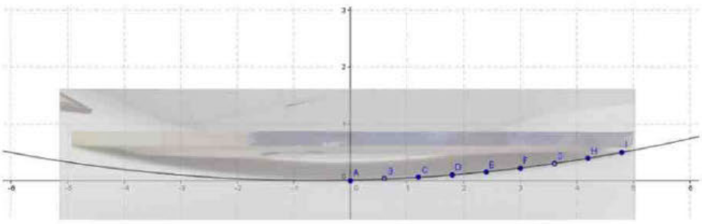

Figure 6: A quadratic fitted to points on the curved wall, self-generated.

The equation for the fitted polynomial was:

$$h(x) = 0.02x^2 + 0.02x + 0.01 \qquad \textit{Eq. 21}$$

The first and second derivatives were:

$$\frac{dy}{dx} = 0.04x + 0.02 \qquad \textit{Eq. 22}$$

$$\frac{d^2y}{2x^2} = 0.04 \qquad \textit{Eq. 23}$$

This curve showed the least variation among the radii of curvature, most likely due to the constant second derivative.

X-Values	0.00	0.60	1.20	1.80	2.40	3.00	3.60	4.20	4.80
Radii of Curvature	25.02	25.07	25.17	25.32	25.51	25.74	26.02	26.34	26.70

Table 4: Radii of Curvature for h(x)

The standard deviation is only 0.59. However, what must be noted about this curve is that it was only fitted to the points used for all other analyses since using more data for this approximation but not the others would skew the perception of the accuracy of the prediction. Thus, the fitted function accurately fits only the points on the positive side and deviates from the points on the negative side. Additionally, the radii are increasing which indicates that the curve is becoming steeper. It is known that this function does not have an inflexion point, therefore, it would be safe to extrapolate that as the x-values continue to increase so would the radii, and they would further deviate from the initial values.

One can approximate this section of the fitted function with a circle of radius 25.65 units, the average radius, however, it would not be the most accurate approximation since the radii vary even for these points. Additionally, the parabola only approximates a curve that could fit these points, the wall's curve is not necessarily a parabolic function itself. This is evident in Figure 6, as the parabolic function does not fit the wall's curve in the second quadrant.

4. Application to the Wall Curve

The painting was approximated to be 1 meter in length. An exact measurement was impossible since soon after this photograph was taken, our school was moved online, and the wall was rebuilt. The photograph was scaled down to the ratio of 1 unit on the x-axis to 10 centimeters to avoid having small coefficients for the fitted polynomial curve and small changes in y. The points were plotted along the line of the shadow on the wall using GeoGebra.

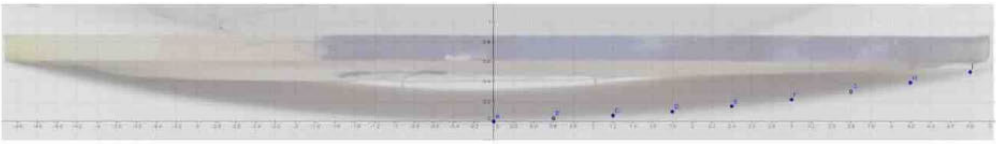

Figure 7: The photograph of the wall and painting in GeoGebra, self-generated.

The points were plotted only in the first quadrant to confirm whether the curve is symmetric about the y-axis, as this is paramount if the wall is parabolic. As evident from the previous section it was not parabolic. Additionally, if the radii of curvature were consistent or showed a distinct pattern then only one side of the curve would be sufficient, and any superfluous points would be avoided.

The coordinates of the points were:

X-Values	0.00	0.60	1.20	1.80	2.40	3.00	3.60	4.20	4.80
Y-Values	0.00	0.03	0.06	0.10	0.15	0.22	0.30	0.39	0.50

Table 5: x and y coordinates of the points on the curve.

The first and second derivatives were found through linearization:

$$f'(x) = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$ *Eq. 24 (Kahn 247-248)*

Linearization reduces the curvature, subsequently increasing the required radius of curvature, as evident from section 3.2. For this curve, taking the first derivatives at small x-intervals decreases the difference between them. Thus, an interval of 0.60 was chosen for the x-values since it was not minute enough to create small second derivatives and inaccurately large radii of curvature.

However, the methodology and intervals were frequently adjusted until the most reasonable values were found. Therefore, it was most useful to create a simple Python program to calculate the radii of curvature.

```python
i = 0
while i < 8:
    i +=1

    list2 = []

    R = ((1 + (list1[i][0])**2)**(3/2))/abs(list1[i][1])

    P = round(R, 3)
    list2.append(P)
    print(list2)
```

Figure 8: The Python program for calculating radii of curvature from a list (table 6) of first and second derivatives, self-generated.

Even though the derivatives were found through simple operations in Microsoft Excel, the complex formula for the radius of curvature posed many risks. The ease of making typing errors or creating an incorrect order of operations without noticing the results required frequent manual checks with a calculator. The Python program saved time from entering values into a calculator and proved to be more accurate and less prone to order of operations and typing errors than Microsoft Excel. Additionally, it allowed for quick adjustments when adjusting the x-values' intervals. The radius at (0.00, 0.00) could not be determined since these calculations would require a new point from the second quadrant. Nevertheless, the radii varied significantly.

X-Values	First Derivative	Second Derivative	Radii with 1:10 scale
0.00	Undetermined	Undetermined	Undetermined
0.60	0.05	0.08	12.39
1.80	0.07	0.03	32.48
2.40	0.08	0.02	40.41
3.00	0.12	0.06	18.23
3.60	0.13	0.03	36.67
4.20	0.15	0.03	36.93
4.80	0.18	0.06	18.76

Table 6: First and Second Derivatives and the Radii of Curvature of the Wall Curve

The point at $x = 1.20$ was removed as it was an outlier with a radius of 334.58 units. This was most likely caused by the second derivative of 0.003 at that point. As previously discussed, most of very small second derivative values were avoided with increasing the x-interval.

The standard deviation is 11.21, relatively close the standard deviation of the negative cosine function, which was 10.68 (Eq. 17). However, there are no similarities in the patterns of the radii-size with any of the previously examined curves. These radii are not continuously increasing or decreasing. Their variations can indicate a frequent change of the curve's steepness. The radius of curvature closest to the origin, at point (0.60, 0.03) is the smallest, corresponding with the previously noted pattern that the radius is at the minimum at turning points.

Some radii were graphed to demonstrate their relationship to the curve.

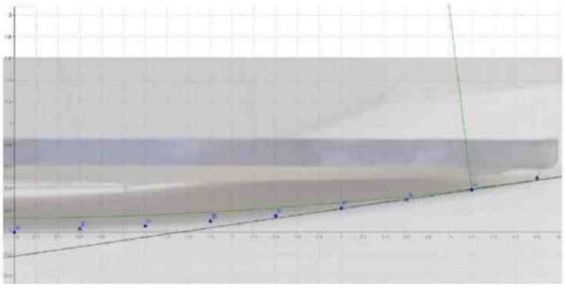

Figure 9: The Radius of Curvature at 4.20, self-generated.

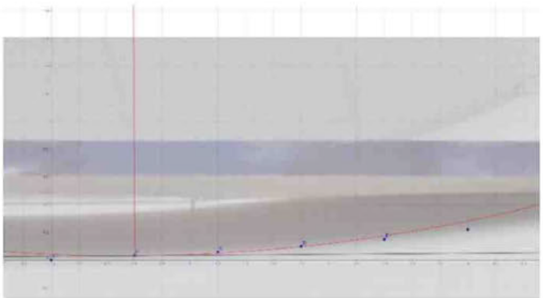

Figure 10: The Radius of Curvature at 0.60, self-generated.

The green line in Figure 9 demonstrates the radius of curvature at $x = 4.20$, and is 36.39 units long. The black line perpendicular to the radius is the point's tangent, and the green curve is a segment of the circle. The red line in Figure 10 demonstrates the radius of curvature at $x = 0.60$, and is 12.39 units long. Through visual and mathematical comparisons of Figures 9 and 10, it becomes apparent how the radii vary greatly from point to point. While the radius of Figure 10 relatively accurately describes the curvature around point (0.60, 0.03), it deviates from points further away from it. The same applies to the radius at (4.20, 0.39).

The graphs visually demonstrate how this curve most likely does not have a consistent radius of curvature. Since a constant radius would encompass all points on the curve. The major limitations of this approach are the imprecisions in point plotting, as the photograph did not have a clear outline, and the approximations made by linearization.

5. Application to a Known Circular Shape

Considering the limitation in point plotting precision, the methodology was applied to an object with a clearer outline. The object was a plant pot with a circular top with a radius of approximately 4 centimeters. A picture was taken and analyzed with GeoGebra through precise point plotting. The major difference between this object and the curved wall was the clear line of curvature unlike the unprecise shadow.

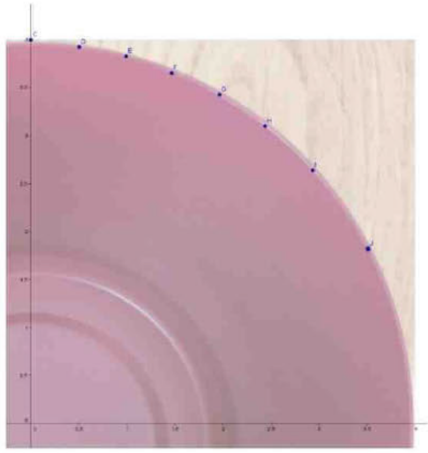

Figure 11: Plant Pot Point, self-generated.

X Values	Y Values	First Derivative Values	Second Derivative Values	Radii of Curvature
0.50	3.96	-0.08	-0.16	6.31
1.00	3.87	-0.18	-0.20	5.24
1.50	3.71	-0.32	-0.28	4.13
2.00	3.46	-0.50	-0.36	3.88
2.50	3.12	-0.68	-0.36	4.91
3.00	2.65	-0.94	-0.52	4.97
3.50	1.92	-1.46	-1.04	5.33

Table 7: Data used to analyze the plant pot's curvature.

The same method of linearization was used to determine first and second derivatives to examine whether the point plotting was truly a major source of error. Furthermore, the same issue with small x-intervals resulting in inaccurately large radii occurred. Thus, an x-interval of 0.50 units was chosen.

As seen in Table 7, precise measurement allows for more accurate results. While the radii still vary, the standard deviation is only 0.80. This deviation is greater than that of the fitted parabola, which is 0.59 (Eq. 21), but there is a vital distinction – these radii are not changing in a uniform manner. There is a slight decrease as points approach (2.00, 3.46) and increase as they move away. Furthermore, most change occurred in the x direction up to this point and in the y direction after it. Visually, there is no critical point at those coordinates, which is confirmed by the non-zero values of the first and second derivatives. Thus, this variation is most likely caused by imprecisions in linearization approximations.

After the point (2.00, 3.46), I examined greater portions of the curve using the same x-interval. Considering that linearization approximates the movement from one point to the next in a straight line, this implies that those portions of the curve were approximated to have lesser curvature. As established in section 3.2, lesser curvature results in greater radii of curvature, hence, these approximations in linearization can justify the increase in radii after the point (2.00, 3.46).

Considering that the variations in radii are slight and can be explained, it is reasonable to assume that if the radii of curvature vary slightly and do not consistently increase or decrease then the curve is likely a part of a circle. The curved wall showed no consistent increase in the radii and matched the pattern of the circular plant pot curve in this regard. However, while the minimal deviation in the plant pot data supports the conclusion of it being a part of a circle, the standard deviation of the curved wall data is significantly larger.

5.1 Analysis of the Standard Normal Distributions

As stated previously, the main differentiating factor between the radii of the wall and the plant pot are the standard deviations. However, one must consider the scales of the radii as well, the calculated radii for the wall's curve were significantly larger than those of the plant pot. To create a more accurate comparison of the standard deviations the values of both real-life curves were standardized using standard normal distribution. This approach was not used for the known functions since it is evident from the data that if the radii are increasing then the deviation from the mean will also increase.

$$z = \frac{x-\mu}{\sigma} \qquad \textit{Eq. 24}$$

The formula for z-score, which indicates how far a point is from the mean on a standard normal distribution curve (Glen), was applied in Excel to set the values to the same scale. In Equation 24, x is the radius of curvature, μ is the mean radius of curvature, and σ is the standard deviation. The resultant z-score was plotted on a normal distribution curve using the Desmos graphic calculator. To graph the standard normal distribution the following function was used (Galarnyk):

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad \textit{Eq. 25}$$

This function is a simplification of the function used to describe the normal distribution since in the standard normal distribution the mean is 0 and the standard deviation is 1. This function was graphed in Desmos and the z-scores for the wall curve and the plant pot were plotted as points. The points on the graph have different colors for visual clarity. The black lines represent one standard deviation from the mean.

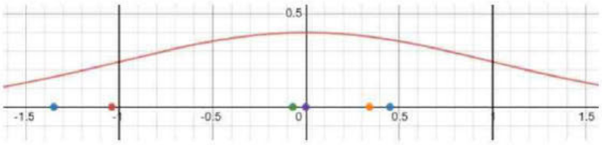

Figure 12: Standard normal distribution of the linearized plant pot curve, self-generated.

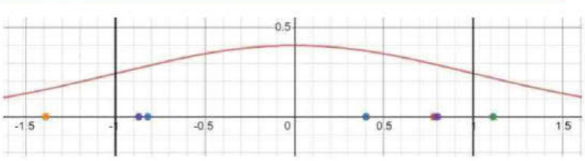

Figure 13: Standard normal distribution of the linearized wall curve, self-generated.

As seen in Figures 12 and 13, the radii of curvature of both curves stayed within one and a half standard deviations. There are two points on the curve in Figure 12 which lie more than 1 standard deviation to the left of the mean, and correspond to the smallest radii on the curve. Considering the justification in the previous section for the decrease in radii, one can conlude that these two points are also caused by the imprecisions in linearization. The rest of the points lie close to the mean value and do not deviate by more than half of a standard deviation. Thus, the unadjusted standard deviation and the standard normal distribution of the plant pot curve support that it is a circle. In Figure 13, there are also two radii which lie more than one standard deviation away from the mean. However, the rest of the points are also more scattered. These observations undermine the certainty of the wall curve being a part of a circle. Nevertheless, the effect of imprecisions in the methodology must also be considered when making the final conclusions.

6. Conclusion and Evaluation

The exploration of the radii of curvature in application to known functions and real-life curves lead to the conclusion of several criteria necessary to determine whether a curve is likely described by a circle. Consistent radii imply a perfect circle, and the closer the standard deviation of the radii is to zero the more likely it is that the radii are consistent throughout. This can be further demonstrated by the proximity of the z-score values to the mean on a standard normal distribution curve, the radii cannot be continuously increasing or decreasing at a rate that would allow for extrapolation of further deviation from the average. Some further observations were noted as well, if the second derivative is negative, the radius will be under the curve and if it is positive it will be above the curve. Moreover, if the radii of curvature are increasing for a portion of a curve it implies that the curve is either approaching its inflexion point or that this is a steeper segment of the curve. If the radii of curvature are decreasing, it implies that the curve is approaching a maximum or a minimum. A portion with a relatively less curvature would be closer to the tangent line and would require a larger radius of curvature.

The answer to the research question, *how can radii of curvature indicate if a curve is a part of a circle*, is that they can only disprove that a curve is a part of a circle if it fails to mean the established criteria. It would be almost impossible to prove whether an unknown curve is a part of a circle as this would necessitate taking continuous derivatives at every point on the curve to justify that they do not vary. Additionally, the use of linearization does not allow for fully consistent radii and some variation will be present, undermining the certainty of the curve being a part of a circle.

The curved wall at school has varying radii, which does not match the pattern of the radii in any of the approximated functions. Furthermore, its z-score values are within one and a half standard deviations. Hence, the best approximation would be that it is most likely a circle, however, one cannot state this with certainty. Even though the z-score values are within one and a half standard deviations, they vary from the mean, and its unstandardized standard deviation is significantly greater than that of a known circular object.

The main limitation of this exploration were the imprecisions in point-plotting and the approximations that arose due to linearization. As seen in the second real-life example, to plot precise points one needs a curve with a clear outline. Any distortion caused by the angle of the picture or the vagueness of the outline severely skews the results, as seen in the first example. Additionally, while linearization was the only method of finding derivatives available for this approach at this level of mathematics, it caused additional imprecisions in the radii. Linearization reduces the curvature of the curve, resulting in an overestimation of the radii of curvature.

✓D

Furthermore, one might think that a curve approximation would be a solution to imprecisions in linearization, however, due to the nature of non-circular functions, it would prove itself as non-circular. The radii of curvature vary for functions and stay constant for circles. If a function had been used as a curve approximation, doing calculations with its first and second derivatives would show variation in radii of curvature and it would indicate that the curve cannot be a part of a circle. Thus, it would prove itself since it is already known that a function cannot be accurately described by a circular equation. This is another reason why function approximations were not used for the conclusive analysis of the nature of the wall.

Another limitation ɔf this investigation is how a constant change of x results in changing arc lengths. The constant change in x simplified the examination of any changes in the derivatives – there were no unprecedented changes in the derivatives or y-values which could be caused by varying x-values. However, for the constant x intervals the arc lengths varied. This resulted in examining and using linearization for different portions of the curve, which caused some imprecisions, as seen in section 5. If this investigation was expanded upon, the portions of the curve which would be analyzed would be approximated to be of the same arc length. ✓D

If this exploration was conducted at a larger scope one would use more precise measuring equipment which would allow for more and more precise data points. Additionally, the formula would be applied to other real-life curves to test whether one can approximate the curve given the pattern in radii of curvature in known functions and real-life curves.

Some of the strengths of this methodology were that the conjectures which were formed during the exploration were tested for more conclusive results. The curves and some radii were graphed to confirm the conclusions visually. Multiple known curves and an ideal, circular, situation was examined to determine any patterns and relationships between the natures of the curves and radii of curvature. This was done before the concept was applied to real life curves where linearization and a constant x-value created imprecisions. Furthermore, the standard normal distribution analysis placed the points on the plant pot and school wall curves on the same scale and facilitated the comparisons between them.

While more conclusive results would be favorable, the exploration deepened my knowledge on curvature and radii of curvature, which will be highly applicable when I am studying engineering in university.

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