We looked at solving simultaneous equations by substitution in Section 4B.

Notice that there are two numbers which both have the same square; this is consistent with what you have already seen with real numbers.

Exercise 15A

1. State the imaginary part of each of the following complex numbers:

(a) (i) $-3+5i$ (ii) $8-2i$

(b) (i) $6+i$ (ii) $19-i$

(c) (i) $2i-8$ (ii) $7i-2$

(d) (i) 15 (ii) $-3i$

(e) (i) i^2 (ii) $(1+i)-i$

(f) (i) $1+ai+b-i, a,b\in\mathbb{R}$ (ii) $2-4i-(bi-a), a,b\in\mathbb{R}$

2. Evaluate, giving your answer in the form $x+iy$:

(a) (i) $2i+3i$ (ii) $i-9i$

(b) (i) $5i^2$ (ii) i^2

(c) (i) $(-3i)^2$ (ii) $(4i)^2$

(d) (i) $(4i+3)-(6i-2)$ (ii) $2(2i-1)-3(4-2i)$

3. Evaluate, giving your answer in the form $x+iy$:

(a) (i) $i(1+i)$ (ii) $3i(2-5i)$

(b) (i) $(2+i)(1+2i)$ (ii) $(5+2i)(4+3i)$

(c) (i) $(2+3i)(1-2i)$ (ii) $(3+i)(5-i)$

(d) (i) $(3+i)^2$ (ii) $(4-3i)^2$

(e) (i) $\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$ (ii) $(3+2i)(3-2i)$

4. Evaluate, giving your answer in the form $x+iy$:

(a) (i) $\frac{6+8i}{2}$ (ii) $\frac{9-3i}{3}$

(b) (i) $\frac{5+2i}{10}$ (ii) $\frac{i-4}{8}$

(c) (i) $\frac{3+i}{2}+i$ (ii) $9i-\frac{6-4i}{2}$

5. Evaluate, giving your answer in the form $x+iy$:

(a) (i) $\sqrt{-4}$ (ii) $\sqrt{-49}$

(b) (i) $\sqrt{-8}$ (ii) $\sqrt{-50}$

(c) (i) $\dfrac{4-\sqrt{-36}}{3}$ (ii) $\dfrac{-1+\sqrt{-25}}{3}$

(d) (i) $\dfrac{2+\sqrt{16-25}}{6}$ (ii) $\dfrac{5-2\sqrt{4-9}}{4}$

6. Solve the equations below, simplifying your answers:

(a) (i) $x^2+9=0$ (ii) $x^2+36=0$

(b) (i) $x^2=-10$ (ii) $x^2=-13$

(c) (i) $x^2-2x+5=0$ (ii) $x^2-x+10=0$

(d) (i) $3x^2+20=6x$ (ii) $6x+5=-5x^2$

7. Evaluate, simplifying your answers:

(a) (i) i^3 (ii) i^4

(b) (i) $(-2\mathrm{i})^4$ (ii) $(-5\mathrm{i})^3$

(c) (i) $\left(1-\sqrt{3}\,\mathrm{i}\right)^3$ (ii) $\left(\sqrt{3}+\mathrm{i}\right)^3$

(d) (i) $\left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\mathrm{i}\right)^3$ (ii) $\left(-\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\mathrm{i}\right)^3$

8. By expanding and comparing real and imaginary parts, find real numbers a and b such that:

(a) (i) $(a+b\mathrm{i})(3-2\mathrm{i})=5\mathrm{i}+1$ (ii) $(6+\mathrm{i})(a+b\mathrm{i})=2$

(b) (i) $(a+2\mathrm{i})(1+2\mathrm{i})=4-b\mathrm{i}$ (ii) $(1+a\mathrm{i})(1+\mathrm{i})=b+2\mathrm{i}$

(c) (i) $(a+b\mathrm{i})(2+\mathrm{i})=2a-(b-1)\mathrm{i}$ (ii) $\mathrm{i}(a+b\mathrm{i})=a-6\mathrm{i}$

9. By writing $z=x+y\mathrm{i}$, solve the following equations:

(a) (i) $z^2=-4\mathrm{i}$ (ii) $z^2=9\mathrm{i}$

(b) (i) $z^2=2+2\sqrt{3}\mathrm{i}$ (ii) $z^2=5+\mathrm{i}$

10. Find the exact values of $a, b\in\mathbb{R}$ such that $(3+a\mathrm{i})(b-\mathrm{i})=-4\mathrm{i}$. Give your answers in the form $k\sqrt{3}$. *[4 marks]*

11. Find the exact values of $a, b\in\mathbb{R}$ such that:

$$(1+a\mathrm{i})(1+b\mathrm{i})=b+9\mathrm{i}-a.$$ *[4 marks]*

12. Solve the equation $\mathrm{i}z+2=\mathrm{i}-3z$. *[4 marks]*

13. (a) Find values x and y such that $(x+\mathrm{i}y)(2+\mathrm{i})=-\mathrm{i}$.

(b) Evaluate $-\dfrac{3\mathrm{i}}{2+\mathrm{i}}$ in the form $x+\mathrm{i}y$. *[5 marks]*

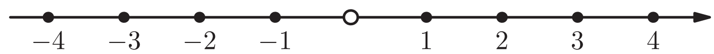
14. (a) Find real numbers a and b such that $(a+bi)^2 = -3-4i$.

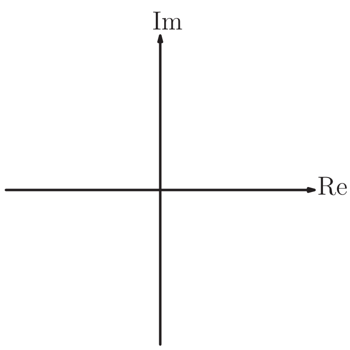
(b) Hence solve the equation: $z^2 + i\sqrt{3}z + i = 0$ *[6 marks]*

15B Geometric interpretation

When you first worked with numbers, you may have used a number line.

When you met negative numbers, this number line was extended to the left.





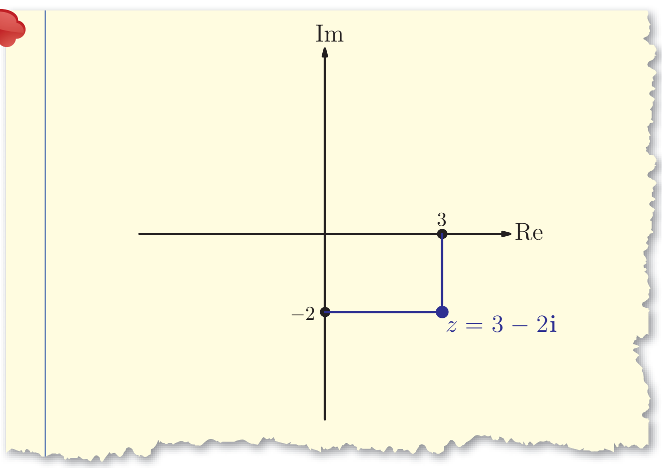
But how can complex numbers be illustrated on this diagram?

The answer was provided by the land surveyor Caspar Wessel who suggested that another axis, perpendicular to the real number line, could represent the imaginary part of a number. However, this idea was popularised by Jean-Robert Argand, and is called an **Argand Diagram**.

Worked example 15.4

Represent $3-2i$ on an Argand Diagram.

3 on the *x*-axis (real part) and –2 on the *y*-axis (imaginary part)



Once you see the power of imaginary numbers, it is tempting to go back and see if adding a third axis produces even more powerful tools. However, there is a result called the Fundamental Theorem of Algebra which proves that complex numbers are sufficient to produce all solutions to all polynomial equations. However, that does not mean that there might not be uses found for a third axis and many mathematicians have worked on this.

Chapter 15

Exercise 15A

1. (a) (i) 5 (ii) -2
   (b) (i) 1 (ii) -1
   (c) (i) 2 (ii) 7
   (d) (i) 0 (ii) -3
   (e) (i) 0 (ii) 0
   (f) (i) $a-1$ (ii) $-4-b$

2. (a) (i) 5i (ii) -8i
   (b) (i) -5 (ii) -1
   (c) (i) -9 (ii) -16
   (d) (i) $5-2\text{i}$ (ii) $-14+10\text{i}$

3. (a) (i) $-1+\text{i}$ (ii) $15+6\text{i}$
   (b) (i) 5i (ii) $14+23\text{i}$
   (c) (i) $8-\text{i}$ (ii) $16+2\text{i}$
   (d) (i) $8+6\text{i}$ (ii) $7-24\text{i}$
   (e) (i) 1 (ii) 13

4. (a) (i) $3+4\text{i}$ (ii) $3-\text{i}$
   (b) (i) $\frac{1}{2}+\frac{1}{5}\text{i}$ (ii) $-\frac{1}{2}+\frac{1}{8}\text{i}$
   (c) (i) $\frac{3}{2}+\frac{3}{2}\text{i}$ (ii) $-3+11\text{i}$

5. (a) (i) 2i (ii) 7i
   (b) (i) $2\sqrt{2}\text{i}$ (ii) $5\sqrt{2}\text{i}$
   (c) (i) $\frac{4}{3}-2\text{i}$ (ii) $\frac{1}{3}+\frac{5}{3}\text{i}$
   (d) (i) $\frac{1}{3}+\frac{1}{2}\text{i}$ (ii) $\frac{5}{4}-\frac{\sqrt{5}}{2}\text{i}$

6. (a) (i) $x=\pm 3\text{i}$
   (ii) $x=\pm 6i$
   (b) (i) $x=\pm\sqrt{10}\text{i}$
   (ii) $x=\pm\sqrt{13}\text{i}$
   (c) (i) $x=1\pm 2\text{i}$
   (ii) $x=\frac{1}{2}\pm\frac{\sqrt{39}}{2}\text{i}$
   (d) (i) $x=1\pm\frac{\sqrt{51}}{3}\text{i}$
   (ii) $x=-\frac{3}{5}\pm\frac{4}{5}\text{i}$

7. (a) (i) $-\text{i}$ (ii) 1
   (b) (i) 16 (ii) 125i
   (c) (i) -8 (ii) 8i
   (d) (i) i (ii) i

8. (a) (i) $a=-\frac{7}{13},\ b=\frac{7}{13}$
   (ii) $a=\frac{12}{37},\ b=\frac{-12}{37}$
   (b) (i) $a=8,\ b=-24$
   (ii) $a=1,\ b=0$
   (c) (i) $a=1,\ b=0$
   (ii) $a=-6,\ b=6$

9. (a) (i) $z=\pm\left(\sqrt{2}-\sqrt{2}i\right)$
   (ii) $z=\pm\left(\frac{3\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}\text{i}\right)$
   (b) (i) $z=\pm\left(\sqrt{3}+\text{i}\right)$
   (ii) $z=\pm\left(\sqrt{\frac{\sqrt{26}+5}{2}}+i\sqrt{\frac{\sqrt{26}-5}{2}}\right)$

10. $a=\pm\sqrt{3},\ b=\mp\frac{\sqrt{3}}{3}$

11. $a=8,\ b=1$
    $a=-1,\ b=10$

12. $z=-\frac{1}{2}+\frac{1}{2}i$

13. (a) $x=-\frac{1}{5},\ y=-\frac{2}{5}$
    (b) $=-\frac{3}{5}-\frac{6}{5}\text{i}$

14. (a) $a=\pm 1,\ b=\mp 2$
    (b) $z=\frac{1}{2}-\frac{\sqrt{3}+2}{2}\text{i}$,
    or $-\frac{1}{2}+\frac{2-\sqrt{3}}{2}\text{i}$