INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 2.11-2.12]

POLYNOMIALS

Compiled by Christos Nikolaidis

O. Practice questions

1. [Maximum mark: 8] ***[without GDC]***

Consider the cubic function $f(x) = ax^3 + 2x^2 + 3x + 4$. Find the value of a in each of the following cases

(a) the graph of the function passes through the point (1,10). [2]

(b) $f(x)$ is divisible by $(x-1)$. [2]

(c) when $f(x)$ is divided by $(x-1)$, the remainder is 10. [2]

(d) Confirm the result in (c) by using long division. [2]

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2*. [Maximum mark: 13] ***[without GDC]***

Consider the cubic function $f(x) = ax^3 + bx^2 + 3x + 4$. Find the values of a and b in each of the following cases

(a) $f(x)$ is divisible by $(x-1)$ and leaves a remainder 6 when divided by $(x+1)$. [4]

(b) $f(x)$ is divisible by (x^2-1). [4]

(c) $f(x)$ leaves a remainder $-3x+3$ when divided by (x^2-1). [5]

3. [Maximum mark: 10] ***[with / without GDC]***

Consider the cubic function $f(x) = 2x^3 + ax^2 + bx + c$. Find the values of a, b, c

(a) if the graph of the function passes through the points (1,0), (–1 ,2), and (0,3). [5]

(b) if the graph of the function passes through the points (1,0), (–1,0), and (3,0). [5]

4. [Maximum mark: 8] ***[without GDC]***

Consider the polynomial $f(x) = 4x^3 - 4x^2 - 5x + 3$.

(a) Show that $x = -1$ is a root of $f(x)$. [1]

(b) Find the other two roots of $f(x)$ by using long division. [5]

(c) Express the polynomial in the form $f(x) = (x - a)(bx - c)(dx - e)$, where $a, b, c, d, e \in Z$. [2]

5. [Maximum mark: 12] ***[without GDC]***

Complete the following table: for each polynomial find the sum and the product of the roots (allowing non-real roots and repetition of roots) as well as the remainder when the polynomial is divided by $(x-1)$.

Polynomial	Sum of roots	Product of roots	Remainder by $(x-1)$
$f(x)=2x^4+6x^3+5x^2-7x+8$			
$f(x)=2x^5+6x^3+5x^2-7x+8$			
$f(x)=x^{10}-x^9-1$			

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6. [Maximum mark: 4] ***[without GDC]***

Consider the cubic function $f(x)=ax^3+2x^2+3x+4$.

(a) Find the value of a if the sum of the roots of the cubic polynomial is 10. [2]

(b) Find the value of a if the product of the roots of the cubic polynomial is 10. [2]

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7. [Maximum mark: 4] ***[without GDC]***

Consider the cubic function $f(x)=ax^3+bx^2+3x+4$. Find the values of a and *b* given that the sum of the roots is 10 and the product of the roots is 12

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8*. [Maximum mark: 14] ***[without GDC]***

The polynomial $f(x) = x^3 - 7x^2 + ax - 9$ is divisible by $(x-1)$.

(a) Find the value of a. [3]

(b) Give full factorization of $f(x)$. [5]

(c) Solve the inequalities

(i) $f(x) > 0$ (ii) $f(x) < 0$ (iii) $f(x) \geq 0$ (iv) $f(x) \leq 0$ [6]

A. Exam style questions (SHORT)

9. [Maximum mark: 4] ***[without GDC]***

Consider $f(x) = x^3 - 2x^2 - 5x + k$. Find the value of k if $(x+2)$ is a factor of $f(x)$.

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10. [Maximum mark: 4] ***[without GDC]***

When the function $f(x) = 6x^4 + 11x^3 - 22x^2 + ax + 6$ is divided by $(x+1)$ the remainder is –20. Find the value of a.

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11. [Maximum mark: 4] ***[without GDC]***

When $x^4 + ax + 3$ is divided by $(x-1)$, the remainder is 8. Find the value of a.

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12. [Maximum mark: 10] ***[without GDC]***

The polynomial $p(x) = x^3 + ax^2 - 3x + b$ is divisible by $(x-2)$ and has a remainder 6 when divided by $(x+1)$.

(a) Find the value of a and of b. [5]

(b) Factorise completely $p(x)$ and state its roots. [5]

13*. [Maximum mark: 6] ***[without GDC]***

The polynomial $p(x) = (ax + b)^3$ leaves a remainder of –1 when divided by $(x + 1)$, and a remainder of 27 when divided by $(x - 2)$. Find the values of a and b.

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14. [Maximum mark: 6] ***[without GDC]***

The polynomial $f(x) = x^3 + 3x^2 + ax + b$ leaves the same remainder when divided by $(x - 2)$ as when divided by $(x + 1)$.

(a) Find the value of a. [5]

(b) State the possible values of b. [1]

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15. [Maximum mark: 7] ***[without GDC]***

Given that $(x-2)$ and $(x+2)$ are factors of $f(x)=x^3+px^2+qx+4$,

(a) find the value of p and of q. [5]

(b) solve the equation $f(x)=0$. [2]

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16. [Maximum mark: 7] ***[without GDC]***

The polynomial $P(x)=2x^3+ax^2-4x+b$ is divisible by $(x-1)$ and by $(x+3)$.

(a) Find the value of a and of *b*. [5]

(b) Factorise $f(x)$. [2]

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17. [Maximum mark: 8] ***[without GDC]***

The polynomial $x^2 - 4x + 3$ is a factor of $x^3 + (a-4)x^2 + (3-4a)x + 3$.

(a) Calculate the value of the constant a. [4]

(b) Factorise completely the cubic polynomial. [2]

(c) Find the remainder when the cubic is divided by $(x-2)$. [2]

18. [Maximum mark: 6] ***[with GDC]***

When $P(x) = 4x^3 + px^2 + qx + 1$ is divided by $(x-1)$ the remainder is -2. When $P(x)$ is divided by $(2x-1)$ the remainder is $\frac{13}{4}$. Find the value of p and of q.

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19*. [Maximum mark: 7] ***[with GDC]***

When $P(x) = 4x^3 + px^2 + qx + 1$ is divided by $(x-1)(2x-1)$ the remainder is $\frac{17-21x}{2}$.

Find the value of p and of q.

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20. [Maximum mark: 8] ***[without GDC]***

The polynomial $f(x) = x^3 - 4x^2 + 3x + a$ is divisible by $(x-1)$.

(a) Find the value of a. [3]

(b) Give full factorization of $f(x)$. [3]

(c) Solve the inequality $f(x) \leq 0$. [2]

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21. [Maximum mark: 8] ***[without GDC]***

The polynomial $f(x) = x^3 - 2x^2 + x + a$ is divisible by $(x-1)$.

(a) Find the value of a. [3]

(b) Give full factorization of $f(x)$. [3]

(c) Solve the inequality $f(x) \leq 0$. [2]

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22. [Maximum mark: 8] ***[without GDC]***

When $f(x) = x^3 + x^2 + x + a$ is divided by $(x-1)$ the remainder is 3.

(a) Find the value of a. [3]

(b) Give full factorization of $f(x)$. [3]

(c) Solve the inequality $f(x) \leq 0$. [2]

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23. [Maximum mark: 6] ***[without GDC]***

The polynomial $f(x) = 3x^3 - a$ is divisible by $(x-1)$.

(a) Find the value of a. [3]

(b) Solve the inequality $f(x) \leq 0$. [3]

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24. [Maximum mark: 6] ***[without GDC]***

Let α, β be the non-real roots of the quadratic $y = 2x^2 + 4x + 6$

(a) Write down the values of (i) $\alpha + \beta$ (ii) $\alpha\beta$. [2]

(b) Find the value of $\alpha^2 + \beta^2$. [2]

(c) Find the value of $(\alpha - \beta)^2$. [2]

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25. [Maximum mark: 6] ***[without GDC]***

Let α, β be the non-real roots of the quadratic $y = 2x^2 + 4x + 6$.

Find a quadratic with roots $\alpha^2\beta$ and $\alpha\beta^2$

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26. [Maximum mark: 7] ***[without GDC]***

Let α, β be the roots of the quadratic $f(x) = x^2 - 2x + 5$. Without finding α and β

(a) Find the value of $\alpha^2 + \beta^2$ [3]

(b) Find a quadratic with integer coefficients and roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$. [4]

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27. [Maximum mark: 6] ***[without GDC]***

The roots α and β of the quadratic equation $x^2 - kx + (k+1) = 0$

are such that $\alpha^2 + \beta^2 = 13$. Find the possible values of the real number k.

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B. Exam style questions (LONG)

28. [Maximum mark: 12] ***[without GDC]***

The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ is divisible by $x^2 - 3x + 2$.

The sum of its roots is 7 and the product of its roots is 0.

(a) Find the values of a, b, c and d. [8]

(b) Factorize $f(x)$. [4]

29. [Maximum mark: 10] ***[without GDC]***

The polynomial $f(x) = x^4 - 2x^3 + ax^2 + bx + 3$ is divisible by $(x-1)$ and the quotient of the division is the polynomial $q(x)$.

(a) Find the sum and the product of the roots of $f(x)$. [2]

(b) State the degree of $q(x)$. [1]

(c) Find the sum of a and b. [3]

(d) The sum and the product of the roots of $q(x)$. [4]

30. [Maximum mark: 12] ***[without GDC]***

Let $f(x) = ax^4 + bx^3 + cx^2 + dx + 16$.

The sum and the product of the roots of $f(x)$ are both 8.

$f(x)$ is divisible by $(x-1)$ and $f(x) = (x-1)q(x)$.

When $f(x)$ is divided by $(x+1)$ the remainder is 120.

(a) Find the values of a and b. [3]

(b) Find the values of c and d. [5]

(c) Find the sum and the product of the roots of $q(x)$. [4]

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31*. [Maximum mark: 20] ***[with GDC]***

Let α, β be the roots of the quadratic $f(x) = 5x^2 - 2x - 4$. Without finding α and β

(a) Write down the values of (i) $\alpha + \beta$ (ii) $\alpha\beta$. [2]

(b) Find the values of (i) $\alpha^2 + \beta^2$ (ii) $\alpha^3 + \beta^3$ [5]

(c) Find a quadratic with integer coefficients which has roots $\frac{1}{\alpha}, \frac{1}{\beta}$. [5]

(d) Find a quadratic with integer coefficients and roots α^2 and β^2 [4]

(e) Find a quadratic with integer coefficients and roots α^3 and β^3 [4]

32*. [Maximum mark: 16] ***[without GDC]***

For a cubic function $ax^3 + bx^2 + cx + d$ with roots r_1, r_2, r_3, it is given that

$$S_1 = r_1 + r_2 + r_3 = -\frac{b}{a}, \quad S_2 = r_1r_2 + r_2r_3 + r_3r_1 = \frac{c}{a}, \quad S_3 = r_1r_2r_3 = -\frac{d}{a}$$

Let α, β, γ be the roots of the cubic function $f(x) = x^3 - 5x^2 - 7x + 3$

Without evaluating the roots α, β, γ, find

(a) $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. [3]

(b) $\alpha^2 + \beta^2 + \gamma^2$. [4]

(c) $(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2$. [5]

(d) a cubic polynomial which has roots $\alpha^2, \beta^2, \gamma^2$. [4]

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33*. [Maximum mark: 15] ***[with GDC]***

Consider the following cubic polynomial

$$f(x) = (x-1)(x^2 + (2-k)x + k^2)$$

(a) Show that $x = 1$ cannot be a root of the quadratic factor $(x^2 + (2-k)x + k^2)$. [4]

(b) Find the values of k in each of the following cases

(i) if the polynomial has exactly one real root;

(ii) if the polynomial has exactly two distinct real roots;

(iii) if the polynomial has three distinct real roots. [7]

(c) Find the roots of $f(x)$ for each value of k in case (b)(ii). [4]