[MAA 2.11-2.12] POLYNOMIALS

SOLUTIONS

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O. Practice questions

1. (a) $f(1)=10 \Rightarrow a=1$

(b) $f(1)=0 \Rightarrow a=-9$

(c) $f(1)=10 \Rightarrow a=1$

2. (a) $f(1)=0$ and $f(-1)=6$. We finally obtain $a=-6 \quad b=-1$

(b) $f(1)=0$ and $f(-1)=0$. We finally obtain $a=-3$, $b=-4$

(c) By definition $f(x)=(x^2-1)Q(x)-3x+3$

For $x=1$, $f(1)=0$. For $x=-1$, $f(-1)=0$. Hence (as in (a)) $a=-6 \quad b=-1$

3. (a) $f(1)=0$, $f(-1)=2$, $f(0)=3$

We finally obtain $a=-2 \quad b=-3 \quad c=3$

(b) **METHOD A**: Work as in (a), form 3 linear equations and find *a,b,c* by GDC

METHOD B: (This case can be solved without GDC)

Since we know the 3 roots we know factorization of $f(x)$

$$f(x)=2(x-1)(x+1)(x-3)=2(x^2-1)(x-3)$$

$$=2(x^3-3x^2-x+3)=2x^3-6x^2-2x+6$$

Hence, $a=-6$, $b=-2$, $c=6$

4. (a) $f(-1)=-4-4+5+3=0$

(b) $x=\frac{1}{2}$, $x=\frac{3}{2}$

(c) $f(x)=4(x+1)(x-\frac{1}{2})(x-\frac{3}{2})=(x+1)(2x-1)(2x-3)$

5.

Polynomial	Sum of roots	Product of roots	Remainder It is *f*(1)
$f(x)=2x^4+6x^3+5x^2-7x+8$	**-3**	**4**	**14**
$f(x)=2x^5+6x^3+5x^2-7x+8$	**0**	**-4**	**14**
$f(x)=x^{10}-x^9-1$	**1**	**-1**	**-1**

6. (a) $a=-\frac{1}{5}$ (b) $a=-\frac{2}{5}$

7. (a) $a=-\frac{1}{3}$ (b) $b=\frac{10}{3}$

8. (a) $f(1)=0 \Leftrightarrow 1-7+a-9=0 \Leftrightarrow a=15$

(b) $f(x)=x^3-7x^2+15x-9$

The long division by $(x-1)$ gives the quadratic $x^2-6x+9=(x-3)^2$

Hence $f(x)=(x-1)(x-3)^2$

(c) Table of signs:

	1		3	
-		+		+

(i) $1<x<3$ or $x>3$
(ii) $x<1$
(iii) $x \geq 3$
(iv) $x \leq 1$ or $x=3$

A. Exam style questions (SHORT)

9. If $x+2$ is a factor of $f(x)$ then $f(-2)=0 \Rightarrow k=6$

10. By the remainder theorem, $f(-1)=6-11-22-a+6=-20 \Leftrightarrow a=-1$

11. $f(x)=x^4+ax+3$, $f(1)=8 \Rightarrow 1+a+3=8 \Rightarrow a=4$

12. (a) Using the remainder and factor theorems (or long division)

$8+4a-6+b=0$ and $-1+a+3+b=6$

$a=-2, b=6$

(b) $p(x)=(x-2)(x^2-3)=(x-2)(x+\sqrt{3})(x-\sqrt{3})$

roots: $x=2,\ x=-\sqrt{3},\ x=\sqrt{3}$

13. $p(x)=(ax+b)^3$

$p(-1)=-1 \Rightarrow (b-a)^3=-1 \Rightarrow b-a=-1$

$p(2)=27 \Rightarrow (2a+b)^3=27 \Rightarrow 2a+b=3$

Thus, $a=\dfrac{4}{3}, b=\dfrac{1}{3}$.

14. (a) $f(2)=8+12+2a+b=2a+b+20$

$f(-1)=-1+3-a+b=2-a+b$.

These remainders are equal when $2a+20=2-a \Rightarrow a=-6$.

(b) b can be any real number.

15. (a) $f(2)=0$ and $f(-2)=0$:

$f(2)=8+4p+2q+4=0$
$\Rightarrow 4p+2q=-12$
$f(-2)=-8+4p-2q+4=0$
$\Rightarrow 4p-2q=4$
$\Rightarrow 8p=-8$
$\Rightarrow p=-1$
$\Rightarrow -4+2q=-12$
$\Rightarrow q=-4$

OR Using factorisation**:**

$f(x)=x^3+px+qx+4 \equiv (x-2)(x+2)(x+a)$
Equate co-efficients of x^0: $4=-4a \Rightarrow a=-1$
$\Rightarrow f(x)=(x^2-4)(x-1)=x^3-x^2-4x+4$
$\Rightarrow p=-1$ and $q=-4$

(b) $x=2,\ x=-2,\ x=1$

16. (a)

$(x-1)$ is a factor of $P(x) \Rightarrow P(1)=0$

$\Rightarrow a+b=2$

$(x+3)$ is a factor of $P(x) \Rightarrow P(-3)=0$

$\Rightarrow 9a+b=42$

Solving $\Rightarrow a=5$, $b=-3$

(b) $P(x)=(x-1)(x+3)(2x+1)$

17. (a) **METHOD 1**

$x^2 - 4x + 3 = (x-3)(x-1)$

$1 + (a-4) + (3-4a) + 3 = 0 \Rightarrow a = 1$

(**OR** $27 + 9(a-4) + 3(3-4a) + 3 = 0 \Rightarrow a = 1$)

METHOD 2

Using the information given it follows that

$x^3 + (a-4)x^2 + (3-4a)x + 3 \equiv (x^2 - 4x + 3)(x+1)$

Comparing coefficients of x^2 (or x)

$a - 4 = -3$ (or $3 - 4a = -1$) $\Rightarrow a = 1$

(b) Sum of roots = -3 $\Rightarrow$ the third root is -1.

Hence $(x-3)(x-1)(x+1)$

(c) By using the remainder theorem, the remainder is $f(2)=(2-3)(2-1)(2+1)=-3$

18.

$P(x)=4x^3+px^2+qx+1$

$P(1)=4(1)^3+p(1)^2+q(1)+1=-2$

$\Rightarrow p+q=-7$

$P\left(\frac{1}{2}\right)=4\left(\frac{1}{2}\right)^3+p\left(\frac{1}{2}\right)^2+q\left(\frac{1}{2}\right)+1=\frac{13}{4}$

$\Rightarrow p+2q=7$

Solving gives $p=-21, q=14$

19. By the definition of long division: $P(x)=(x-1)(2x-1)Q(x)+\dfrac{17-21x}{2}$

$P(1)=-2$, $P\left(\frac{1}{2}\right)=\frac{13}{4}$

$p=-21$, $q=14$

20. (a) $f(1)=0 \Leftrightarrow 1-4+3+a=0 \Leftrightarrow a=0$

(b) $f(x)=x^3-4x^2+3x=x(x^2-4x+3)=x(x-1)(x-3)$

(c) $x \le 0$ or $1 \le x \le 3$

21. (a) $f(1)=0 \Leftrightarrow 1-2+1+a=0 \Leftrightarrow a=0$

(b) $f(x)=x^3-2x^2+x=x(x^2-2x+1)=x(x-1)^2$

(c) $x \le 0$ or $x=1$

22. (a) $f(1)=3 \Leftrightarrow 3+a=3 \Leftrightarrow a=0$

(b) $f(x)=x^3+x^2+x=x(x^2+x+1)$ (the quadratic is irreducible)

(c) $x \le 1$

23. (a) $f(1)=0 \Leftrightarrow 3-a=0 \Leftrightarrow a=3$

(b) **METHOD A**

$3x^3-3 \le 0 \Leftrightarrow 3x^3 \le 3 \Leftrightarrow x^3 \le 1 \Leftrightarrow x \le 1$

METHOD B

The factorisation is $f(x)=3x^3-3=3(x-1)(x^2+x+1)$ (the quadratic is irreducible)

$x \le 1$

24. (a) (i) $\alpha+\beta=-\frac{4}{2}=-2$, (ii) $\alpha\beta=\frac{6}{2}=3$

(b) $(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta \Rightarrow \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$\Rightarrow \alpha^2+\beta^2=4-6=-2$

(c) $(\alpha-\beta)^2=\alpha^2+\beta^2-2\alpha\beta=-2-6=-8$

25. $S=\alpha+\beta=-\frac{4}{2}=-2$, $P=\alpha\beta=\frac{6}{2}=3$

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=4-6=-2$

For the new roots:

$S'=\alpha^2\beta+\alpha\beta^2=\alpha\beta(\alpha+\beta)=SP=-14$

$P'=\alpha^2\beta\,\alpha\beta^2=(\alpha\beta)^3=P^3=27$

The corresponding quadratic is $x^2-S'x+P'$, that is $x^2+14x+27$ (or any multiple)

26. $S=\alpha+\beta=2$, $P=\alpha\beta=5$

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=4-10=-6$

For the new roots:

$$S'=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{-6}{5}$$

$$P'=\frac{\alpha}{\beta}\frac{\beta}{\alpha}=1$$

The corresponding quadratic is $x^2+\frac{6}{5}x+1$, or $y=5x^2+6x+5$ (integer coefficients)

27. $\alpha+\beta=k$, $\alpha\beta=k+1$

$(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta \Rightarrow k^2=13+2(k+1)$

$k^2-2k-15=0$,

$(k-5)(k+3)=0$

$k=5$ or $k=-3$

B. Exam style questions (LONG)

28. (a) Sum = $-\frac{a}{1}=7 \Rightarrow a=-7$,

Product = $\frac{d}{1}=0 \Rightarrow d=0$

$f(1)=0 \Rightarrow 1+a+b+c+d=0 \Rightarrow b+c=6$

$f(2)=0 \Rightarrow 16+8a+4b+2c+d=0 \Rightarrow 2b+c=20$

Therefore, $b=14 \quad c=-8$

(b) We know three roots, 0,1 and 2. Since the sum is 7 the third root is 4.

Hence $f(x)=x(x-1)(x-2)(x-4)$

29. (a) Sum = 2 , Product = 3

(b) deg = 3 (cubic)

(c) $f(1)=0 \Rightarrow a+b=-2$

(d) Sum = 1, Product = 3

30. (a) Product = $\frac{16}{a}=8 \Rightarrow a=2$

Sum = $-\frac{b}{a}=8 \Rightarrow b=-16$

$f(1)=0 \Rightarrow a+b+c+d+16=0 \Rightarrow c+d=-2$

$f(-1)=120 \Rightarrow a-b+c-d+16=0 \Rightarrow c-d=86$

Therefore, $c=42$, $d=-44$

Therefore $a=2$, $b=-16$, $c=42$, $d=-44$

(b) Sum = 7, Product = 8

31. (a) $S=\alpha+\beta=\frac{2}{5}, \quad P=\alpha\beta=-\frac{4}{5}$

(b) We use $(\alpha+\beta)^2=\alpha^2+2\alpha\beta+\beta^2 \Rightarrow \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

Thus $\alpha^2+\beta^2=\frac{4}{25}+\frac{8}{5}=\frac{44}{25}$

We use $(\alpha+\beta)^3=\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3 \Rightarrow \alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

Thus $\alpha^3+\beta^3=\frac{8}{125}+3\cdot\frac{4}{5}\cdot\frac{2}{5}=\frac{128}{125}$

(c) $S'=\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=-\frac{1}{2}$ and $P'=\frac{1}{\alpha}\frac{1}{\beta}=\frac{1}{\alpha\beta}=-\frac{5}{4}$

Quadratic: $x^2+\frac{1}{2}x-\frac{5}{4}$ or $4x^2+2x-5$

(d) Since $S' = \alpha^2 + \beta^2 = \dfrac{44}{25}$ and $P' = \alpha^2\beta^2 = (\alpha\beta)^2 = \dfrac{16}{25}$

Quadratic: $25x^2 - 44x + 16$

(e) Since $S' = \alpha^3 + \beta^3 = \dfrac{128}{125}$ and $P' = \alpha^3\beta^3 = (\alpha\beta)^3 = \dfrac{64}{125}$

Quadratic: $125x^2 - 128x + 64$

32. (a) $S_1 = \alpha + \beta + \gamma = 5$,

$S_2 = \alpha\beta + \beta\gamma + \gamma\alpha = -7$

$S_3 = \alpha\beta\gamma = -3$

(b) We use $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha$

$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

Thus $\alpha^2 + \beta^2 + \gamma^2 = 25 + 2(-7) = 39$

(c) We use $(\alpha\beta + \beta\gamma + \gamma\alpha)^2 = (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 + 2\alpha\beta^2\gamma + 2\alpha\beta\gamma^2 + 2\alpha^2\beta\gamma$

$\Rightarrow (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

Thus $(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 = (-7)^2 - 2(-3)(5) = 79$

(d) $S_1' = \alpha^2 + \beta^2 + \gamma^2 = 39$, $S_2' = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = 79$, $S_3' = \alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = 9$

The cubic polynomial is $x^3 - 39x^2 + 79x - 9$ (or any multiple).

33. (a) If $x = 1$ was a root then $1^2 + (2 - k) + k^2 = 0 \Leftrightarrow k^2 - k + 3 = 0$
which impossible since $\Delta = -11 < 0$

(b) The discriminant of the quadratic factor is $\Delta = (2-k)^2 - 4k^2 = -3k^2 - 4k + 4$

with roots $k = -2$, $k = \dfrac{2}{3}$

(i) $\Delta<0$: $k < -2$ *or* $k > \dfrac{2}{3}$

(ii) $\Delta=0$: $k = -2$ *or* $k = \dfrac{2}{3}$

(iii) $\Delta>0$: $-2 \le k \le \dfrac{2}{3}$

(c) if $k = -2$, the quadratic factor is $x^2 + 4x + 4 = (x+2)^2$.
Roots of $f(x)$ are 1 and –2

if $k = \dfrac{2}{3}$, the quadratic factor is $x^2 + \dfrac{4}{3}x + \dfrac{4}{9} = (x + \dfrac{2}{3})^2$.

Roots of $f(x)$ are 1 and $-\dfrac{2}{3}$