## [MAA 2.13-2.15] RATIONAL AND MODULUS FUNCTIONS – INEQUALITIES

## SOLUTIONS

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### O. Practice questions

**1.** **Note:** the third one is a line.

| | $f(x)=\dfrac{3x+8}{2x+7}$ | $f(x)=\dfrac{8}{2x+7}$ | $f(x)=\dfrac{3x+8}{7}$ |
|---|---|---|---|
| **Roots** | $x=-\dfrac{8}{3}$ | none | $x=-\dfrac{8}{3}$ |
| $y$ **-intercept** | $y=\dfrac{8}{7}$ | $y=\dfrac{8}{7}$ | $y=\dfrac{8}{7}$ |
| **V.A.** | $x=-\dfrac{7}{2}$ | $x=-\dfrac{7}{2}$ | none |
| **H.A.** | $y=\dfrac{3}{2}$ | $y=0$ | none |
| **Domain** | $x\neq-\dfrac{7}{2}$ | $x\neq-\dfrac{7}{2}$ | $x\in\mathbb{R}$ |
| **Range** | $y\neq\dfrac{3}{2}$ | $y\neq 0$ | $y\in\mathbb{R}$ |

**2.**

| | $f(x)=\dfrac{(x-3)(x-4)}{(x+1)(x-2)}$ | $f(x)=\dfrac{(2x-3)(x-4)}{(x+1)(x-2)}$ | $f(x)=\dfrac{2x-3}{(x+1)(x-2)}$ |
|---|---|---|---|
| **Roots** | $x=3$, $x=4$ | $x=\dfrac{3}{2}$, $x=4$ | $x=\dfrac{3}{2}$ |
| $y$ **-intercept** | $y=-6$ | $y=-6$ | $y=\dfrac{3}{2}$ |
| **V.A.** | $x=-1$, $x=2$ | $x=-1$, $x=2$ | $x=-1$, $x=2$ |
| **H.A.** | $y=1$ | $y=2$ | $y=0$ |
| **Domain** | $x\neq-1$, $x\neq 2$ | $x\neq-1$, $x\neq 2$ | $x\neq-1$, $x\neq 2$ |

**3.** (a) Vertical asymptote: $x=-\dfrac{3}{2}$

For the oblique asymptote we divide $4x^2+4x+1$ by $2x+3$

Oblique asymptote: $y=2x-1$

(b) Vertical asymptote: $x=-1$, $x=2$

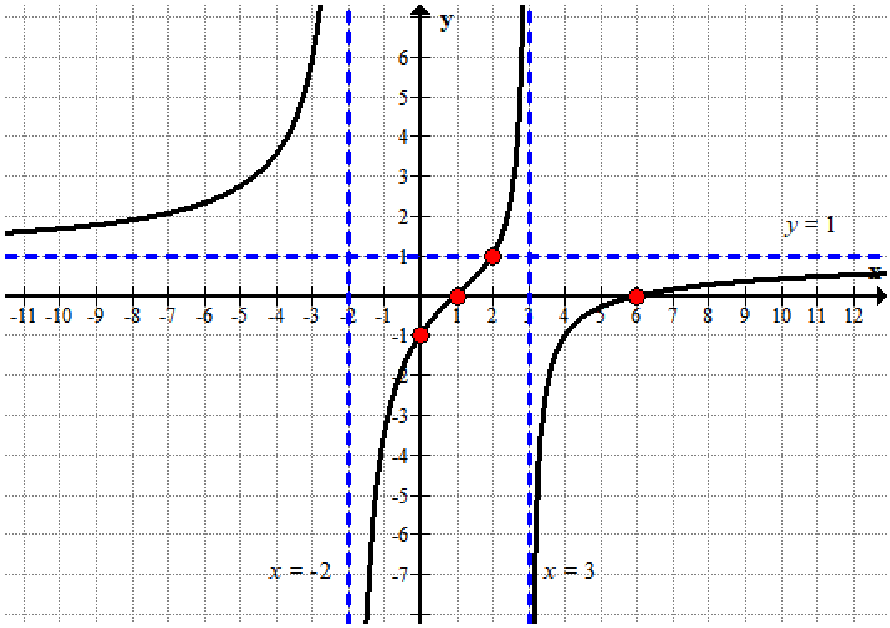
For the oblique asymptote we divide $2x^3+5x^2+4x+1$ by $x^2-x-2$

Oblique asymptote: $y=2x+7$

**4.** (a) $f(x)=1 \Leftrightarrow (x-1)(x-6)=(x+2)(x-3)$

$x^2-7x+6=x^2-x-6 \Leftrightarrow 12=6x \Leftrightarrow x=2$

(b)




(b) Domain: $x \neq -2$, $x \neq 3$  Range: $y \in \mathbb{R}$

**5.** (a) If we divide $(x+2)(x-3)=x^2-x-6$ by $3x-6$ we find the quotient $\frac{1}{3}x+\frac{1}{3}$

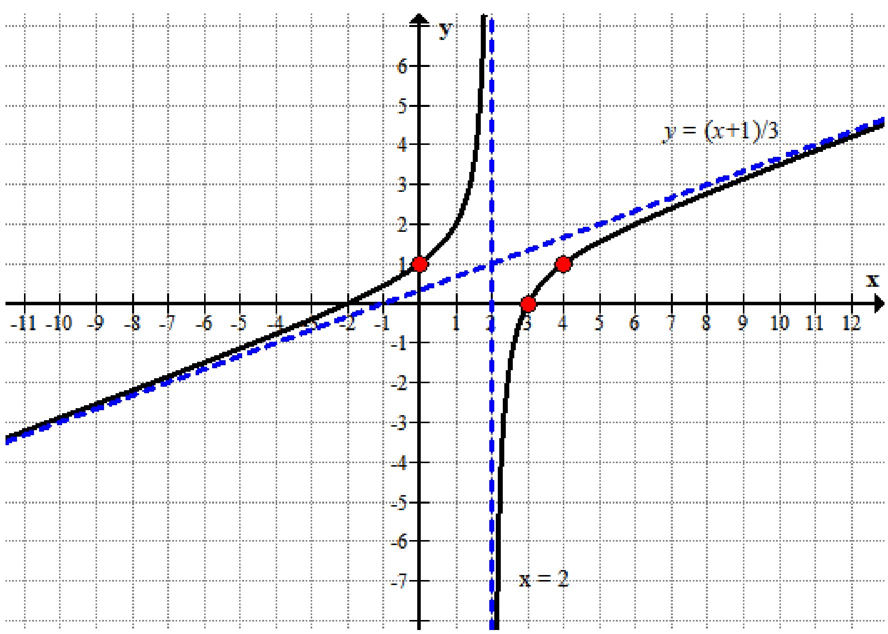
(b) $f(x)=1 \Leftrightarrow (x+2)(x-3)=3x-6$

$\Leftrightarrow x^2-x-6=3x-6$

$\Leftrightarrow x^2-4x=0 \Leftrightarrow x(x-4)=0$

$x=0, x=4$

(c)




(d) Domain: $x \neq -2$, $x \neq 3$  Range: $y \in \mathbb{R}$

**6.** (a) $x^2 - x - 2 = (x-2)(x+1)$

$$f(x) = \frac{6}{x^2 - x - 2} = \frac{A}{x-2} + \frac{B}{x+1}$$

Multiply by the quadratic: $A(x+1) + B(x-2) = 6$

For $x = -1$: $-3B = 6 \Leftrightarrow B = -2$

For $x = 2$: $3A = 6 \Leftrightarrow A = 2$

Hence: $f(x) = \dfrac{2}{x-2} - \dfrac{2}{x+1}$

(b) As above but multiplication by the quadratic gives: $A(x+1) + B(x-2) = 3x + 6$

For $x = -1$: $-3B = 3 \Leftrightarrow B = -1$

For $x = 2$: $3A = 12 \Leftrightarrow A = 4$

Hence: $f(x) = \dfrac{4}{x-2} - \dfrac{1}{x+1}$

(c) $2x^2 - 2x - 4 = 2(x-2)(x+1)$

$$f(x) = \frac{x}{2x^2 - 2x - 4} = \frac{A}{x-2} + \frac{B}{x+1}$$

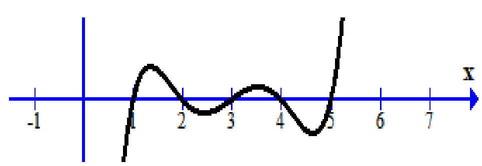
Multiply by the quadratic: $2A(x+1) + 2B(x-2) = x$

For $x = -1$: $-6B = -1 \Leftrightarrow B = \dfrac{1}{6}$

For $x = 2$: $6A = 2 \Leftrightarrow A = \dfrac{1}{3}$

Hence: $f(x) = \dfrac{1/3}{x-2} - \dfrac{1/6}{x+1}$

**7.** Since $a > 0$ (leading coefficient), positive towards $+\infty$




(a)

| $x$ | -∞ | 1 | 2 | 3 | 4 | 5 | +∞ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | – | + | – | + | – | + | |

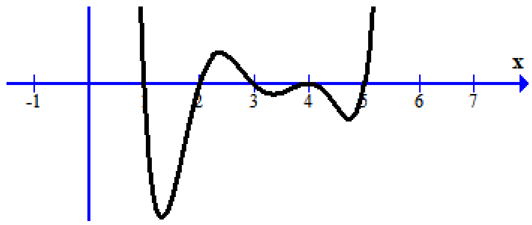
(b) $f(x) > 0 \Leftrightarrow x \in ]1,2[ \cup ]3,4[ \cup ]5,+\infty[$

(c) $f(x) > 0 \Leftrightarrow x \in [1,2] \cup [3,4] \cup [5,+\infty[$

. (d) $x \in ]1,2[ \cup ]3,4[ \cup ]5,+\infty[$ [i.e. as in (b)]

(e) $x \in [1,2[ \cup [3,4[ \cup [5,+\infty[$ [i.e. as in (c) but we exclude $x = 2$ and $x = 4$]

**8.** Since $a>0$ (leading coefficient), positive towards $+\infty$




(a)

| $x$ | $-\infty$ | | 1 | | 2 | | 3 | | 4 | | 5 | | $+\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $g(x)$ | | + | | – | | + | | – | | – | | + | |

(b) $g(x)>0 \Leftrightarrow x\in]-\infty,1[\ \cup\ ]2,3[\ \cup\ ]5,+\infty[$

(c) $g(x)\geq 0 \Leftrightarrow x\in]-\infty,1]\ \cup\ [2,3]\ \cup\ \{4\}\ \cup[5,+\infty[$

(d) $g(x)<0 \Leftrightarrow x\in]1,2[\ \cup\ ]3,4[\ \cup\ ]4,5[$

(e) $g(x)<0 \Leftrightarrow x\in[1,2]\cup[3,5]$

(f) $x\in]-\infty,1]\ \cup\ ]2,3]\ \cup[5,+\infty[$

**9.** For all of them we consider the polynomial $5(x-1)(x-2)^2(x-3)^3$

| $f(x)$ | | 1 | | 2 | | 3 | |
|---|---|---|---|---|---|---|---|
| sign | + | | – | | – | | + |

(the solution for $f(x)\geq 0$ is $x\in(-\infty,1]\cup\{2\}\cup[3,+\infty)$

For each inequality we exclude the roots of the denominator. The corresponding solutions are

(i) $x\in(-\infty,1]\cup\{2\}\cup(3,+\infty)$ (ii) $x\in(-\infty,1]\cup[3,+\infty)$ (iii) $x\in(-\infty,1)\cup\{2\}\cup[3,+\infty)$

**10.** **Without GDC:** $x+\dfrac{2}{x}\geq 3 \Leftrightarrow x+\dfrac{2}{x}-3\geq 0 \Leftrightarrow \dfrac{x^2+2-3x}{x}\geq 0 \Leftrightarrow \dfrac{(x-1)(x-2)}{x}\geq 0$

| $f(x)$ | | 0 | | 1 | | 2 | |
|---|---|---|---|---|---|---|---|
| sign | – | $\Vert$ | + | | – | | + |

**With GDC:** The solutions of the equation are 1,2, the denominator is 0 at $x=0$ so the graph gives a table as above.

Therefore, $x\in(0,1]\cup[2,+\infty)$

**NOTICE: Questions 11-15 by using GDC are easy. We just observe the graph.**

**11.** (a) **METHOD A:** Taking squares

$$|x-5|^2=3^2 \Leftrightarrow (x-5)^2=3^2 \Leftrightarrow x^2-10x+25-9=0 \Leftrightarrow x^2+10x-16=0$$

$x=2$, $x=8$

**METHOD B:** Definition of modulus

$x-5=\pm 3 \Leftrightarrow x=5\pm 3$ . therefore, $x=2$, $x=8$

(b) **METHOD A:** Taking squares

$$|x-5|^2<3^2 \Leftrightarrow (x-5)^2<3^2 \Leftrightarrow x^2-10x+25-9<0 \Leftrightarrow x^2+10x-16<0$$

$x\in]2,8[$

**METHOD B:** Definition of modulus

$-3<x-5<3 \Leftrightarrow 2<x<8$, that is $x\in]2,8[$

**12.** Taking squares (since both sides are positive)

(a) $(x-5)^2=(x-3)^2 \Leftrightarrow x^2-10x+25=x^2-6x+9 \Leftrightarrow 16=4x \Leftrightarrow x=4$

(b) $(x-5)^2<(x-3)^2 \Leftrightarrow x^2-10x+25<x^2-6x+9 \Leftrightarrow 16<4x \Leftrightarrow x>4$

**13.** (we CANNOT take squares). The zero of the modulus is $x=5$, so we consider two cases

(a) **Case 1:** If $x \geq 5$,

The equation takes the form $x-5=x-3 \Leftrightarrow 5=3$ which is impossible. No solutions!

**Case 2:** If $x<5$,

The equation takes the form $5-x=x-3 \Leftrightarrow 8=2x \Leftrightarrow x=4$ which is accepted.

Therefore, there is only one solution: $x=4$

(b) **Case 1:** If $x \geq 5$,

The inequality takes the form $x-5<x-3 \Leftrightarrow -5<-3$ which is always true.

In this region, the whole region is accepted $x \geq 5$

**Case 2:** If $x<5$,

The inequality takes the form $5-x<x-3 \Leftrightarrow 8<2x \Leftrightarrow x>4$.

In this region, the solution is $4<x<5$

Therefore, the final solution is: $x \in ]4,+\infty[$

**14.** Although we can consider cases ( $x \geq 0$ and $x<0$ ), we observe that it is a hidden quadratic,

Let $y=|x|$. The inequality takes the form:

$y^2-3y+2>0 \Leftrightarrow y<1$ or $y>2$

$|x|<1$ gives $-1<x<1$, $|x|>2$ gives $x<-2$ or $x>2$

Final solution: $x \in ]-\infty,-2[ \cup ]-1,1[ \cup ]2,+\infty[$

**15.** The zeroes of the moduli are $x=0$ and $x=5$

Hence, we consider three regions:

| | **0** | | **5** | |
|---|---|---|---|---|
| √ | | √ | | √ |

(a) **Case 1:** If $x \geq 5$,

The equation takes the form $x-5=x-3 \Leftrightarrow 5=3$ which is impossible. No solutions!

**Case 2:** If $0 \leq x<5$,

The equation takes the form $-x+5=x-3 \Leftrightarrow 8=2x \Leftrightarrow x=4$ which is accepted.

**Case 3:** If $x<0$,

The equation takes the form $-x+5=-x-3 \Leftrightarrow 5=-3$ which is impossible.

Therefore, there is only one solution: $x=4$

(b) **Case 1:** If $x \geq 5$,

The inequality takes the form $x-5<x-3 \Leftrightarrow -5<-3$ which is always true.

The whole region is accepted $x \geq 5$

**Case 2:** If $0 \leq x<5$,

The inequality takes the form $-x+5<x-3 \Leftrightarrow 8<2x \Leftrightarrow x>4$.

In this region, the solution is $4<x<5$

**Case 3:** If $x<0$,

The inequality takes the form $-x+5<-x-3 \Leftrightarrow 5<-3$ which is impossible.

The whole region is rejected.

Therefore, the final solution is: $x \in ]4,+\infty[$

**16.** (a) The zeroes of the moduli are $x=0$ and $x=5$.

We find

$f(0)=5-0+3=8$,

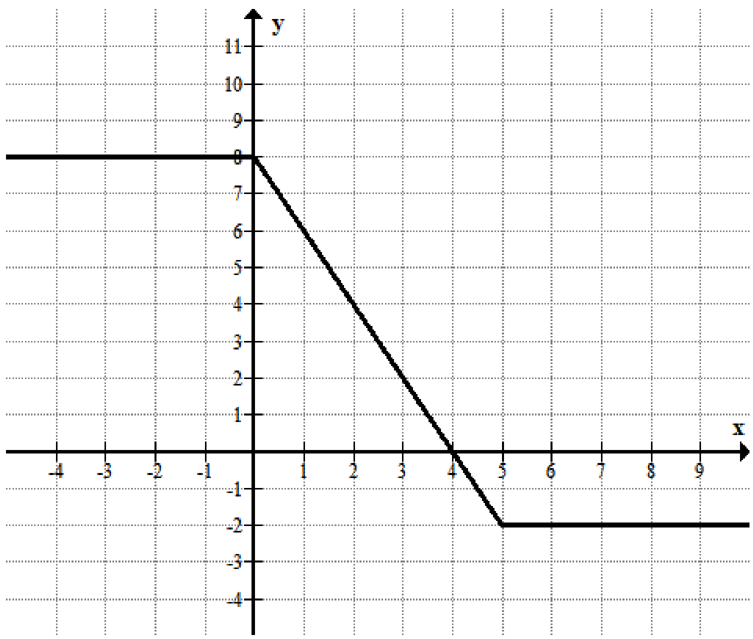
$f(5)=0-5+3=-2$

We also find one value before 0 and one value after 5.

$f(-1)=6-1+3=8$,

$f(6)=1-6+3=-2$

We plot those points on the diagram and draw the graph.




(b) According to the graph

$f(x)>0 \Leftrightarrow x<4$

$f(x)>4 \Leftrightarrow x<2$

$f(x)>-4 \Leftrightarrow x\in\mathbb{R}$

$f(x)>10$ no solution

## A. Exam style questions (SHORT)

**17.** $2x^2-14x+24=0 \Leftrightarrow x^2-7x+12=0 \Leftrightarrow x=3$ or $x=4$

Vertical asymptotes: $x=3$, $x=4$

Horizontal asymptote: $y=\dfrac{3}{2}$

**18.** $x^2-x+1=0$ has no real solutions

No vertical asymptotes

Horizontal asymptote: $y=3$

**19.** Long division of $6x^2+5x+1$ by $3x+7$ gives the quotient $2x-3$

Vertical asymptote: $x=-\dfrac{7}{3}$

Oblique asymptote: $y=2x-3$

**20.** $2x^2-14x+24=0 \Leftrightarrow x^2-7x+12=0 \Leftrightarrow x=3$ or $x=4$

Long division of $6x^3+1$ by $2x^2-14x+24$ gives the quotient $3x+21$

Vertical asymptotes: $x=3$, $x=4$

Oblique asymptote: $y=3x+21$

**21.** $2x^2-14x+24=0 \Leftrightarrow x^2-7x+12=0 \Leftrightarrow x=3$ or $x=4$

$$f(x)=\frac{5}{2x^2-14x+24}=\frac{A}{x-3}+\frac{B}{x-4}$$

Multiply by the quadratic:

$A(x-4)+B(x-3)=5$

For $x=3$: $-A=5 \Leftrightarrow A=-5$

For $x=4$: $B=5$

Therefore, $f(x)=\dfrac{5}{2x^2-14x+24}=\dfrac{-5}{x-3}+\dfrac{5}{x-4}$

**22.** $2x^2-14x+24=0 \Leftrightarrow x^2-7x+12=0 \Leftrightarrow x=3$ or $x=4$

$$f(x)=\frac{5x+1}{2x^2-14x+24}=\frac{A}{x-3}+\frac{B}{x-4}$$

Multiply by the quadratic:

$A(x-4)+B(x-3)=5x+1$

For $x=3$: $-A=16 \Leftrightarrow A=-16$

For $x=4$: $B=21$

Therefore, $f(x)=\dfrac{5x+1}{2x^2-14x+24}=\dfrac{-16}{x-3}+\dfrac{21}{x-4}$

**23.** Number line

| $f(x)$ | $-1$ | $1$ | $2$ | |
|---|---|---|---|---|
| sign | + | – | – | + |

(i) $x\in(-\infty,-1]\cup\{1\}\cup[2,+\infty)$

(ii) $x\in(-\infty,-1)\cup(2,+\infty)$

(iii) we exclude the roots of the denominator $x\in(-\infty,-1]\cup\{1\}\cup(2,+\infty)$

**24.** Number line

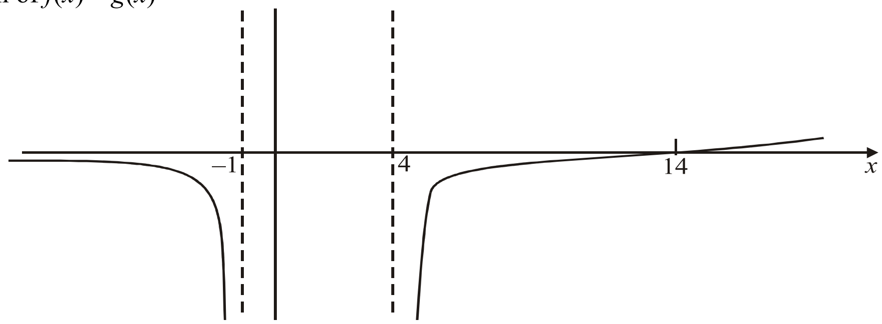
| $f(x)$ | $-1$ | $1/2$ | $2$ | |
|---|---|---|---|---|
| sign | – | + | + ‖ | – |

(i) $x\in[-1,2)$

(ii) $x\in(-\infty,-1]\cup\{1/2\}\cup(2,+\infty)$

**25.** **METHOD 1** (with GDC)

Graph of $f(x) - g(x)$

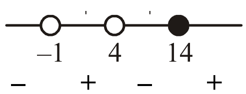



Critical values: $x = 14$ (root), $x = –1$ and $x = 4$

$x < –1$ or $4 < x \le 14$

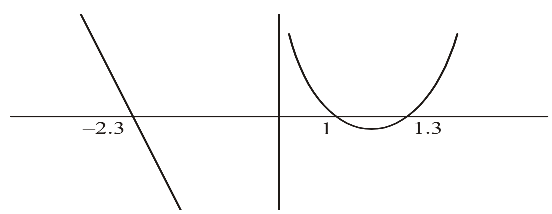
**METHOD 2** (without GDC)

$$\frac{x+4}{x+1} - \frac{x-2}{x-4} \le 0 \Rightarrow \frac{x^2 - 16 - x^2 + x + 2}{(x+1)(x-4)} \le 0 \Rightarrow \frac{x-14}{(x+1)(x-4)} \le 0$$




$x < –1$ or $4 < x \le 14$

**26.**




Roots: $x = –2.30, 1, 1.30$

V.A. $x = 0$

Therefore, critical values: -2.30, 1, 0, 1.30

$x^2 - 4 + \dfrac{3}{x} < 0 \Rightarrow –2.30 < x < 0$ or $1 < x < 1.30$

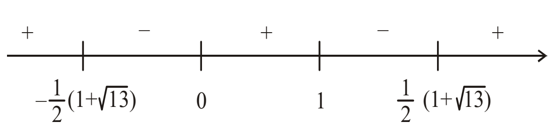
**27.** $x^2 - 4 + \dfrac{3}{x} < 0 \Leftrightarrow \dfrac{x^3 - 4x + 3}{x} < 0$

Division of $x^3 - 4x + 3$ by $x - 1$ gives $x^2 + x - 3$

$$\frac{(x-1)(x^2 + x - 3)}{x} < 0$$

Roots of the quadratic: $\dfrac{1}{2}(–1 \pm \sqrt{13})$

Critical values: 1, $\dfrac{1}{2}(–1 \pm \sqrt{13})$, 0




Therefore, $-\dfrac{1}{2}(\sqrt{13} + 1) < x < 0$ or $1 < x < \dfrac{1}{2}(\sqrt{13} - 1)$

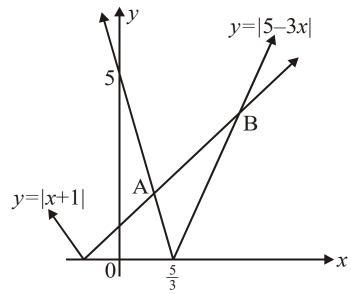
**28.** **METHOD 1** (without GDC, taking squares)

$|5-3x| \le |x+1| \Rightarrow 25 - 30x + 9x^2 \le x^2 + 2x + 1$

$\Rightarrow 8x^2 - 32x + 24 \le 0 \Rightarrow 8(x-1)(x-3) \le 0$

$\Rightarrow 1 \le x \le 3$

**METHOD 2**



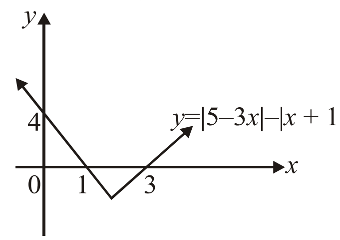

(G1)

We obtain A = (1, 2) and B = (3, 4). (G1)

Therefore, $1 \le x \le 3$.  (A1)(C3)

**METHOD 3**

Sketch the graph of $y = |5 - 3x| - |x + 1|$.




(G2)

From this graph we see that

$y \le 0$ for $1 \le x \le 3$.  (A1)(C3)

**29.** **METHOD 1** (without GDC, taking squares)

$(x-2)^2 \ge (2x+1)^2 \Leftrightarrow x^2 - 4x + 4 \ge 4x^2 + 4x + 1$

$3x^2 + 8x - 3 \le 0 \Leftrightarrow (3x-1)(x+3) \le 0$ (or find roots of equation)

So $x \in \left[-3, \frac{1}{3}\right]$.

**METHOD 2** (without GDC)

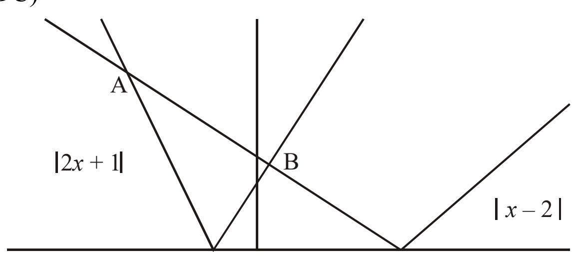
The graphs of $y = |x-2|$ and $y = |2x+1|$ meet where

$(x-2) = (2x+1) \Leftrightarrow x = -3$

$(x-2) = -(2x+1) \Leftrightarrow x = \frac{1}{3}$

Test any value, *e.g.* $x = 0$ satisfies inequality, so $x \in \left[-3, \frac{1}{3}\right]$.

**METHOD 3** (with GDC)




We obtain for A, $x = -3$ and for B, $x = \frac{1}{3}$  From the graph, $x \in \left[-3, \frac{1}{3}\right]$.

**METHOD 4** (with GDC)

Observe the graph of $|x-2| - |2x+1|$  and check when $|x-2| - |2x+1| \ge 0$

**30.** **METHOD 1** (without GDC, taking squares)

$|x+9| \le 2|x-9|$ and then taking squares:

$$x^2+18x+81 \le 4(x^2-18x+81)$$

$$3x^2-5\times 18x+3\times 81 \ge 0 \Leftrightarrow x^2-30x+81 \ge 0 \Leftrightarrow (x-3)(x-27) \ge 0$$

$x \in ]-\infty, 3] \cup [27, \infty[$

**METHOD 2** (without GDC)

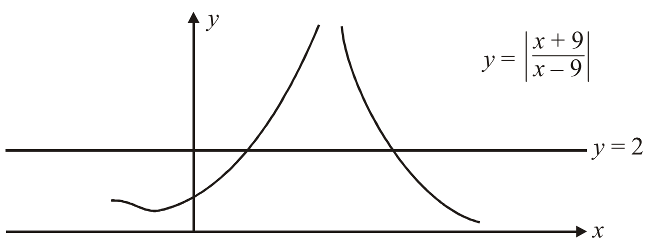
Solve $-2 \le \dfrac{x+9}{x-9} \le 2$

$$\frac{x+9}{x-9} \le 2 \Leftrightarrow \frac{x+9}{x-9}-2 \le 0 \Leftrightarrow \frac{-x+27}{x-9} \le 0 \Leftrightarrow \frac{x-27}{x-9} \ge 0 \quad \text{so } x<9 \text{ or } x \ge 27$$

$$\frac{x+9}{x-9} \ge -2 \Leftrightarrow \frac{x+9}{x-9}+2 \ge 0 \Leftrightarrow \frac{3x-9}{x-9} \ge 0 \Leftrightarrow \frac{x-3}{x-9} \ge 0 \quad \text{so } x \le 3 \text{ or } x>9$$

Therefore, $x \in ]-\infty, 3] \cup [27, \infty[$

**METHOD 3** (with GDC)




$]-\infty, 3] \cup [27, \infty[$

**31.** either algebraically or graphically: $x \le 0.5$ or $x > 3$

**32.**

$(f \circ g)(x) = 2\left(\dfrac{x}{x+1}\right)-1$

$(g \circ f)(x) = \dfrac{2x-1}{2x}$

**EITHER**

$\dfrac{2x}{x+1}-1 \le \dfrac{2x-1}{2x}$

Getting 0 on one side

$\dfrac{2x}{x+1}-1-\dfrac{1-3x}{2x} \le 0 \quad \left(\dfrac{1-3x}{2x(x+1)} \le 0\right)$

$-1 < x < 0$ or $x \ge \dfrac{1}{3}$.

**OR** by observing the graph of $(f \circ g)(x)-(g \circ f)(x)$
and the critical values $x = 1/3$ (root) , $x = 0$, $x = -1$ (asymptotes)

## B. Exam style questions (LONG)

**33.** (a) (i) V.A.: $x=1$, $x=4$, H.A.: $y=0$

(ii) V.A.: $x=1$, $x=4$, H.A.: $y=1$

(iii) V.A.: $x=1$, $x=4$, O.A.: $y=x$

(b) $f(x)=\dfrac{6}{x^2-5x+4}=\dfrac{2}{x-4}-\dfrac{2}{x-1}$

(c) $g(x)=1+\dfrac{6}{x^2-5x+4}$

(d) (i) $g(x)=1+\dfrac{2}{x-4}-\dfrac{2}{x-1}$

(ii) $h(x)=x+\dfrac{4}{x-4}-\dfrac{4}{x-1}$

(e) $5f(x)\le 3g(x) \Leftrightarrow \dfrac{30}{x^2-5x+4}\le\dfrac{3x^2-15x+30}{x^2-5x+4}\Leftrightarrow\dfrac{3x^2-15x}{x^2-5x+4}\ge 0$

$\Leftrightarrow \dfrac{3x(x-5)}{(x-1(x-4)}\ge 0$

| | 0 | | 1 | | 4 | | 5 | |
|---|---|---|---|---|---|---|---|---|
| + | | – | ‖ | + | ‖ | – | | + |

$x\in[-\infty,0]\cup\,]1,4[\,\cup\,[5,+\infty[$

**34.** (a) quotient $q(x)=x+6$, remainder $r(x)=28x+2$

(b) $f(x)=x+6+\dfrac{28x+2}{x^2-6x+8}$

(c) $x^2-6x+8\Leftrightarrow x=2$ or $x=4$

Vertical asymptotes: $x=2$, $x=4$

Oblique asymptote: $y=28x+2$

(d) we express in partial fraction the function $\dfrac{28x+2}{x^2-6x+8}$

$\dfrac{28x+2}{x^2-6x+8}=\dfrac{A}{x-2}+\dfrac{B}{x-4}$

Multiply by the quadratic:

$A(x-2)+B(x-4)=28x+2$

For $x=2$: $-2B=58\Leftrightarrow B=-29$

For $x=4$: $2A=114\Leftrightarrow A=57$

Therefore, $f(x)=x+6+\dfrac{57}{x-2}-\dfrac{29}{x-4}$