Solution

Since the polynomial does not have real coefficients, then $1 - 2i$ is not necessarily also a zero. To find the other zeros, we can perform synthetic substitution

$$\begin{array}{c|cccc} 1+2i & 1 & i-2 & 2i+5 & 8+i \\ & & 1+2i & -7+i & -8-i \\ \hline & 1 & -1+3i & -2+3i & 0 \end{array}$$

This shows that $P(x) = (x - 1 - 2i)(x^2 + (-1 + 3i)x - 2 + 3i)$. The second factor can be factored into $(x + 1)(x - 2 + 3i)$ giving us the other two zeros as -1 and $2 - 3i$.

Note: $x^2 + (-1 + 3i)x - 2 + 3i = 0$ can be solved using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 - 3i \pm \sqrt{(-1 + 3i)^2 - 4(-2 + 3i)}}{2}$$

$$= \frac{1 - 3i \pm \sqrt{-8 - 6i + 8 - 12i}}{2} = \frac{1 - 3i \pm \sqrt{-18i}}{2}$$

To find $\sqrt{-18i}$ we let $(a + bi)^2 = -18i \Rightarrow a^2 - b^2 + 2abi = -18i$, then equating the real parts and imaginary parts to each other: $a^2 - b^2 = 0$ and $2ab = -18$ will yield $\sqrt{-18i} = \pm 3 \mp 3i$, and hence

$$x = \frac{1 - 3i \pm \sqrt{-18i}}{2} = \frac{1 - 3i \pm (\pm 3 \mp 3i)}{2}$$

which will yield $x = -1$ or $x = 2 - 3i$.

Exercise 10.1

Express each of the following numbers in the form $a + bi$.

1 $5 + \sqrt{-4}$

2 $7 - \sqrt{-7}$

3 -6

4 $-\sqrt{49}$

5 $\sqrt{-81}$

6 $-\sqrt{\frac{-25}{16}}$

Perform the following operations and express your answer in the form $a + bi$.

7 $(-3 + 4i) + (2 - 5i)$

8 $(-3 + 4i) - (2 - 5i)$

9 $(-3 + 4i)(2 - 5i)$

10 $3i - (2 - 4i)$

11 $(2 - 7i)(3 + 4i)$

12 $(1 + i)(2 - 3i)$

13 $\frac{3 + 2i}{2 + 5i}$

14 $\frac{2 - i}{3 + 2i}$

15 $\left(\frac{2}{3} - \frac{1}{2}i\right) + \left(\frac{1}{3} + \frac{1}{2}i\right)$

16 $\left(\frac{2}{3} - \frac{1}{2}i\right)\left(\frac{2}{3} + \frac{1}{2}i\right)$

17 $\left(\frac{2}{3} - \frac{1}{2}i\right) \div \left(\frac{1}{3} + \frac{1}{2}i\right)$

18 $(2 + i)(3 - 2i)$

19 $\frac{1}{i}(3 - 7i)$

20 $(2 + 5i) - (-2 - 5i)$

21 $\frac{13}{5 - 12i}$

22 $\frac{12i}{3 + 4i}$

23 $3i\left(3 - \frac{2}{3}i\right)$

24 $(3 + 5i)(6 - 10i)$

25 $\frac{39 - 52i}{24 + 10i}$

26 $(7 - 4i)^{-1}$

27 $(5 - 12i)^{-1}$

28 $\frac{3}{3 - 4i} + \frac{2}{6 + 8i}$

29 $\frac{(7 + 8i)(2 - 5i)}{5 - 12i}$

30 $\frac{5 - \sqrt{-144}}{3 + \sqrt{-16}}$

31 Let $z = a + bi$. Find a and b if $(2 + 3i)z = 7 + i$.

32 $(2 + yi)(x + i) = 1 + 3i$, where x and y are real numbers. Solve for x and y.

33 a) Evaluate $(1 + i\sqrt{3})^3$.

b) Prove that $(1 + i\sqrt{3})^{6n} = 8^{2n}$, *where* $n \in Z^+$.

c) Hence, find $(1 + i\sqrt{3})^{48}$.

34 a) Evaluate $(-\sqrt{2} + i\sqrt{2})^2$.

b) Prove that $(-\sqrt{2} + i\sqrt{2})^{4k} = (-16)^k$, *where* $k \in Z^+$.

c) Hence, find $(-\sqrt{2} + i\sqrt{2})^{46}$.

35 If z is a complex number such that $|z + 4i| = 2|z + i|$, find the value of $|z|$. ($|z| = \sqrt{x^2 + y^2}$ where $z = x + iy$.)

36 Find the complex number z and write it in the form $a + bi$ if $z = 3 + \frac{2i}{2 - i\sqrt{2}}$.

37 Find the values of the two real numbers x and y such that $(x + iy)(4 - 7i) = 3 + 2i$.

38 Find the complex number z and write it in the form $a + bi$ if $i(z + 1) = 3z - 2$.

39 Find the complex number z and write it in the form $a + bi$ if $\frac{2 - i}{1 + 2i}\sqrt{z} = 2 - 3i$.

40 Find the values of the two real numbers x and y such that $(x + iy)^2 = 3 - 4i$.

41 a) Find the values of the two real numbers x and y such that $(x + iy)^2 = -8 + 6i$.

b) Hence, solve the following equation

$$z^2 + (1 - i)z + 2 - 2i = 0.$$

42 If $z \in \mathbb{C}$, find all solutions to the equation $z^3 - 27i = 0$.

43 Given that $z = \frac{1}{2} + 2i$ is a zero of the polynomial $f(x) = 4x^3 - 16x^2 + 29x - 51$, find the other zeros.

44 Find a polynomial function with integer coefficients and lowest possible degree that has $\frac{1}{2}$, -1 and $3 + i\sqrt{2}$ as zeros.

45 Find a polynomial function with integer coefficients and lowest possible degree that has -2, -2 and $1 + i\sqrt{3}$ as zeros.

46 Given that $z = 5 + 2i$ is a zero of the polynomial $f(x) = x^3 - 7x^2 - x + 87$, find the other zeros.

47 Given that $z = 1 - i\sqrt{3}$ is a zero of the polynomial $f(x) = 3x^3 - 4x^2 + 8x + 8$, find the other zeros.

48 Let $z \in \mathbb{C}$. If $\frac{z}{z^*} = a + bi$, show that $|a + bi| = 1$.

49 Given that $z = (k + i)^4$ where k is a real number, find all values of k such that

a) z is a real number

b) z is purely imaginary.

50 Solve the system of equations.

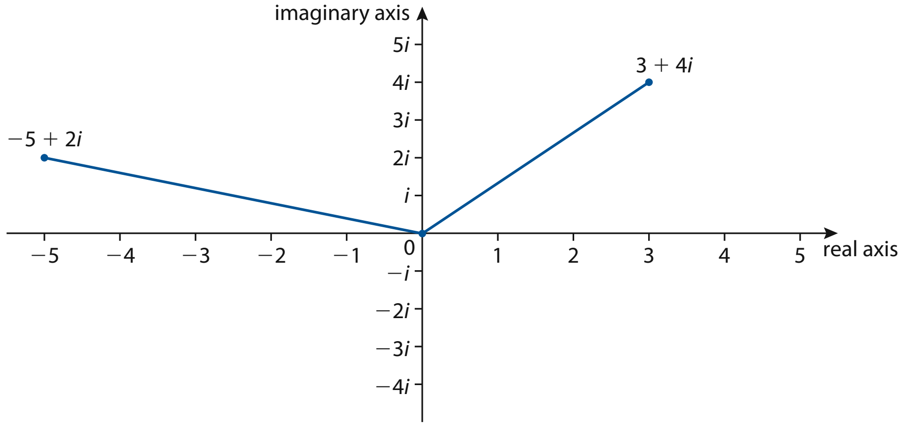
$$iz_1 + 2z_2 = 3 - i$$
$$2z_1 + (2 + i)z_2 = 7 + 2i$$

51 Solve the system of equations.

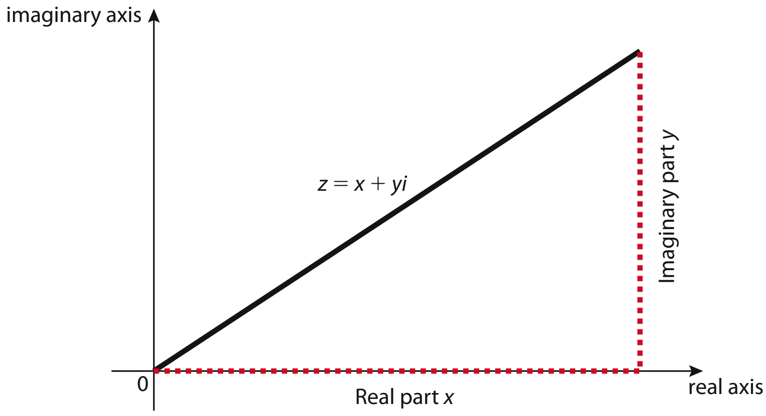
$$iz_1 - (1 + i)z_2 = 3$$
$$(2 + i)z_1 + iz_2 = 4$$

10.2 The complex plane

Our definition of complex numbers as ordered pairs of real numbers enables us to look at them from a different perspective. Every ordered pair (x, y) determines a unique complex number $x + yi$, and vice versa. This correspondence is embodied in the geometric representation of complex numbers. Looking at complex numbers as points in the plane equipped with additional structure changes the plane into what we call **complex plane**, or **Gauss plane**, or **Argand plane** (**diagram**). The complex plane has two axes, the horizontal axis is called the **real axis**, and the vertical axis is the **imaginary axis**. Every complex number $z = x + yi$ is represented by a point (x, y) in the plane. The real part is measured along the real axis and the imaginary part along the imaginary axis.



The diagram above illustrates how the two complex numbers $3 + 4i$ and $-5 + 2i$ are plotted in the complex plane.



b) $\overrightarrow{IR} = \begin{pmatrix} 5 \\ -\frac{25}{6} \end{pmatrix}$

8 a) $\begin{pmatrix} 745 \\ 1000 \end{pmatrix}$ b) 600 km/h c) at 1.5 hrs

d) $\begin{pmatrix} 325 \\ 940 \end{pmatrix}$ e) 451 km

9 $2n^2 - n + 12 = 0$ does not have real solutions, so it is not possible.

10 $\alpha = \frac{\pi}{2} - 2\theta$ **11** 0

Chapter 10

Exercise 10.1

1 $5 + 2i$ **2** $7 - \sqrt{7}i$ **3** $-6 + 0i$

4 $-7 + 0i$ **5** $0 + 9i$ **6** $0 - \frac{5}{4}i$

7 $-1 - i$ **8** $-5 + 9i$ **9** $14 + 23i$

10 $-2 + 7i$ **11** $34 - 13i$ **12** $5 - i$

13 $\frac{16}{29} - \frac{11}{29}i$ **14** $\frac{4}{13} - \frac{7}{13}i$ **15** 1

16 $\frac{25}{36}$ **17** $-\frac{1}{13} - \frac{18}{13}i$ **18** $8 - i$

19 $-7 - 3i$ **20** $4 + 10i$ **21** $\frac{5}{13} + \frac{12}{13}i$

22 $\frac{48}{25} + \frac{36}{25}i$ **23** $2 + 9i$ **24** 68

25 $\frac{8}{13} - \frac{63}{26}i$ **26** $\frac{7}{65} + \frac{4}{65}i$ **27** $\frac{5}{169} + \frac{12}{169}i$

28 $\frac{12}{25} + \frac{8}{25}i$ **29** $\frac{498}{169} + \frac{553}{169}i$ **30** $-\frac{33}{25} - \frac{56}{25}i$

31 $\frac{17}{13} - \frac{19}{13}i$ **32** $x = -\frac{1}{2}, y = -2$; and $x = 1, y = 1$

33 a) -8 c) 2^{48}

34 a) $-4i$ c) 2^{46}

35 $x^2 + y^2 = 4$ **36** $\frac{9 - \sqrt{2}}{3} + \frac{2}{3}i$

37 $x = -\frac{2}{65}, y = \frac{29}{65}$ **38** $\frac{1}{2}(1 + i)$

39 $5 + 12i$ **40** $(x, y) = (2, -1)$ or $(x, y) = (-2, 1)$

41 a) $(x, y) = (1, 3)$ or $(x, y) = (-1, -3)$

b) $2i, -1 - i$

42 $\left\{-3i, \frac{3(\sqrt{3} + i)}{2}, \frac{3(-\sqrt{3} + i)}{2}\right\}$

43 $\frac{1}{2} - 2i, 3$ **44** $f(x) = 2x^4 - 11x^3 + 15x^2 + 17x - 11$

45 $f(x) = x^4 + 2x^3 + 8x + 16$

46 $5 - 2i, -3$ **47** $1 + i\sqrt{3}, -\frac{2}{3}$ **48** Verify

49 a) $k = 0 \pm 1$ b) $k = \pm\sqrt{3 \pm 2\sqrt{2}}$

50 $z_1 = 1 + i, z_2 = 2 - i$ **51** $z_1 = \frac{7 - 4i}{3}, z_2 = \frac{1 + 6i}{3}$

Exercise 10.2

1 $2\sqrt{2} \text{ cis} \left(\frac{\pi}{4}\right)$ **2** $2 \text{ cis} \left(\frac{\pi}{6}\right)$ **3** $2\sqrt{2} \text{ cis} \left(\frac{7\pi}{4}\right)$

4 $2\sqrt{2} \text{ cis} \left(\frac{11\pi}{6}\right)$ **5** $4 \text{ cis} \frac{5\pi}{3}$ **6** $3\sqrt{2} \text{ cis} \left(\frac{3\pi}{4}\right)$

7 $4 \text{ cis}\left(\frac{\pi}{2}\right)$ **8** $6 \text{ cis} \left(\frac{7\pi}{6}\right)$ **9** $\sqrt{2} \text{ cis} \left(\frac{\pi}{4}\right)$

10 $15 \text{ cis } \pi$ **11** $\frac{1}{5} \text{ cis } (5.64)$ **12** $3\sqrt{2} \text{ cis} \left(\frac{3\pi}{4}\right)$

13 $\pi \text{ cis } (0)$ **14** $e \text{ cis} \left(\frac{\pi}{2}\right)$ **15** $\frac{-\sqrt{3}}{2} + \frac{i}{2}, \frac{\sqrt{3}}{2} + \frac{i}{2}$

16 $1, \frac{1}{2} - \frac{\sqrt{3}\,i}{2}$ **17** $\frac{-\sqrt{3}}{2} + \frac{i}{2}, -i$ **18** $-i, \frac{-1}{2} + \frac{\sqrt{3}\,i}{2}$

19 $\frac{\sqrt{6} + \sqrt{2}}{2} + i\frac{\sqrt{6} - \sqrt{2}}{2}, \frac{9(-\sqrt{6} + \sqrt{2})}{8} - i\frac{9(\sqrt{6} + \sqrt{2})}{8}$

20 $-3\sqrt{3} - 3 + i(3\sqrt{3} - 3), \frac{3\sqrt{3} - 3}{4} - \frac{i(3\sqrt{3} + 3)}{4}$

21 $\frac{-\sqrt{2}}{2}(1 + i), \frac{\sqrt{2}}{2}(1 + i)$

22 $6, \frac{-3}{4} - \frac{3\sqrt{3}i}{4}$

23 $\frac{5\sqrt{6} - 15\sqrt{2}}{48} - i\frac{5\sqrt{6} + 15\sqrt{2}}{48}, \frac{-5\sqrt{6} - 15\sqrt{2}}{64} + i\frac{5\sqrt{6} - 15\sqrt{2}}{64}$

24 $-3\sqrt{3} + 3 + i(3\sqrt{3} + 3), \frac{3\sqrt{3} + 3}{4} + \frac{i(3\sqrt{3} - 3)}{4}$

25 $z_1 = 2 \text{ cis} \frac{\pi}{6}, z_2 = 4 \text{ cis} \frac{-\pi}{3}, \frac{1}{z_1} = \frac{1}{2} \text{ cis} -\frac{\pi}{6}, \frac{1}{z_2} = \frac{1}{4} \text{ cis} \frac{\pi}{3},$

$z_1 z_2 = 8 \text{ cis} \frac{-\pi}{6}, \frac{z_1}{z_2} = \frac{1}{2} \text{ cis} \frac{\pi}{2}$

26 $z_1 = 2\sqrt{2} \text{ cis} \frac{\pi}{6}, z_2 = 4\sqrt{3} \text{ cis} \frac{-\pi}{3}, \frac{1}{z_1} = \frac{\sqrt{2}}{4} \text{ cis} \frac{\pi}{6}, \frac{1}{z_2} = \frac{\sqrt{3}}{12} \text{ cis} \frac{\pi}{3}$

,

$z_1 z_2 = 8\sqrt{6} \text{ cis} \frac{-\pi}{6}, \frac{z_1}{z_2} = \frac{\sqrt{6}}{6} \text{ cis} \frac{\pi}{2}$

27 $z_1 = 8 \text{ cis} \frac{\pi}{6}, z_2 = 3\sqrt{2} \text{ cis} \frac{-3\pi}{4}, \frac{1}{z_1} = \frac{1}{8} \text{ cis} \frac{-\pi}{6}, \frac{1}{z_2} = \frac{\sqrt{2}}{6} \text{ cis} \frac{3\pi}{4},$

$z_1 z_2 = 24\sqrt{2} \text{ cis} \frac{-7\pi}{12}, \frac{z_1}{z_2} = \frac{4\sqrt{2}}{3} \text{ cis} \frac{11\pi}{12}$

28 $z_1 = \sqrt{3} \text{ cis} \frac{\pi}{2}, z_2 = 2\sqrt{2} \text{ cis} \frac{-2\pi}{3}, \frac{1}{z_1} = \frac{\sqrt{3}}{3} \text{ cis} \frac{-\pi}{2},$

$\frac{1}{z_2} = \frac{\sqrt{2}}{8} \text{ cis} \frac{2\pi}{3}, z_1 z_2 = 2\sqrt{6} \text{ cis} \frac{-\pi}{6}, \frac{z_1}{z_2} = \frac{\sqrt{6}}{4} \text{ cis} \frac{-5\pi}{6}$

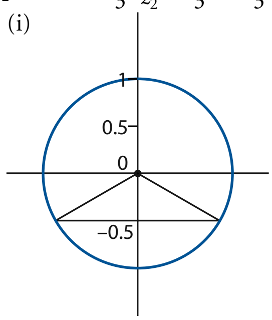
29 $z_1 = \sqrt{10} \text{ cis} \frac{\pi}{4}, z_2 = 2\sqrt{2} \text{ cis} \frac{\pi}{2}, \frac{1}{z_1} = \frac{\sqrt{10}}{10} \text{ cis} \frac{\pi}{4}, \frac{1}{z_2} = \frac{\sqrt{2}}{8} \text{ cis} \frac{\pi}{2}$

$, z_1 z_2 = 4\sqrt{5} \text{ cis} \frac{3\pi}{4}, \frac{z_1}{z_2} = \frac{\sqrt{5}}{2} \text{ cis} \frac{-\pi}{4}$

30 $z_1 = 2 \text{ cis} \frac{\pi}{3}, z_2 = 2\sqrt{3} \text{ cis } 0, \frac{1}{z_1} = \frac{1}{2} \text{ cis} \frac{-\pi}{3}, \frac{1}{z_2} = \frac{\sqrt{3}}{6} \text{ cis } 0,$

$z_1 z_2 = 4\sqrt{3} \text{ cis} \frac{\pi}{3}, \frac{z_1}{z_2} = \frac{\sqrt{3}}{3} \text{ cis} \frac{\pi}{3}$

31 b) (i)



(ii) $\arg(z_1) = \frac{-\pi}{6}, \arg(z_2) = \frac{-5\pi}{6}$

32 Verify

33 a) $\frac{\sqrt{3}}{2} - \frac{3i}{2}$ b) $\frac{-2\sqrt{3}}{3}$ c) $\sqrt{3}i$

34 $|z_1| = 4, \arg(z_1) = \frac{-\pi}{6}, |z_2| = 2\sqrt{2}, \arg(z_2) = \frac{\pi}{4}, |z_3| = 8\sqrt{2},$

$\arg(z_3) = \frac{\pi}{12}$

35 $22 - 2\sqrt{3} \approx 18.5$

36 a) $\{(x, y): x^2 + y^2 = 9\}$, the circle centre (0, 0) radius 3

b) $\{(x, y): x = 0\}$, the y-axis

c) $\{(x, y): x = 4\}$, the line $x = 4$

d) $\{(x, y): (x - 3)^2 + y^2 = 4\}$, the circle centre (3, 0) radius 2

e) $\{(x, y): 1 - x + 3 \text{ and } y = 0\}$, the line segment between (1, 0) and (3, 0)