1. [4 *points*]

When the polynomial $P(x) = x^4 - 3x^3 + ax^2 - 4x + 7$ is divided by $(x + 2)$ the remainder is 7. Find the remainder when $P(x)$ is divided by $(x + 1)$.

Using remainder theorem we have $P(-2) = 7$, which gives:

$$16 + 24 + 4a + 8 + 7 = 7$$

which gives $a = -12$. Now we need to calculate $P(-1)$:

$$P(-1) = 1 + 3 - 12 + 4 + 7 = 3$$

so the remainder is 3.

2. [6 *points*]

Given that $2+i$ is one of the roots of the equation:

$$z^4 + 2z^3 - 6z^2 - 22z + 65 = 0$$

find all other roots of this equation.

By conjugate root theorem $2-i$ is also a root. So we can factor out both $(z-2-i)$ and $(z-2+i)$, but we have:

$$(z-2-i)(z-2+i) = (z-2)^2 + 1 = z^2 - 4z + 5$$

Now we divide (using long division):

$$\frac{z^4 + 2z^3 - 6z^2 - 22z + 65}{z^2 - 4z + 5} = z^2 + 6z + 13$$

So $z^4 + 2z^3 - 6z^2 - 22z + 65$ can be expressed as a product of two quadratics:

$$z^4 + 2z^3 - 6z^2 - 22z + 65 = (z^2 - 4z + 5)(z^2 + 6z + 13)$$

The fist quadratic has zeroes for $z = 2 \pm i$ the second quadratic has zeroes for (these can be found using for example the quadratic formula) $z = -3 \pm 2i$.

So the 4 roots of the original equation are: $2 \pm i$ and $-3 \pm 2i$.

3. [5 *points*]

The sum of roots of polynomial equation:

$$x^3 + bx^2 - 5x + d = 0$$

is equal to twice the product of these roots. Given that one of the roots is $x = 1$, find b and d.

Using the formula for the sum ($sum = -b$) and product ($product = -d$) of roots we get that:

$$-b = -2d$$

Using the fact that $x = 1$ is a root ($P(1) = 0$):

$$1 + b - 5 + d = 0$$

We have two equations, solving those gives $b = \frac{8}{3}$ and $d = \frac{4}{3}$.

4. [5 *points*]

Find the remainder when polynomial $P(x)$ is divided by $x^2 + x - 6$, given that $P(2) = 1$ and $P(-3) = -9$.

We can write $P(x)$ as:

$$P(x) = (x^2 + x - 6)Q(x) + ax + b$$

the remainder is of degree at most 1, since we're dividing by a quadratic. Now we substitute $x = 2$ and $x = -3$ to get two equations (note that the quadratic term becomes 0 for these values of x):

$$\begin{cases} 1 = 2a + b \\ -9 = -3a + b \end{cases}$$

Solving gives $a = 2$ and $b = -3$, so the remainder is $2x - 3$.