Calculus [443 marks]

1. [Maximum mark: 5] SPM.1.AHL.TZ0.7

A particle, A, moves so that its velocity (v ms^{-1}) at time t is given by $v = 2\sin t, t \geq 0$.

The kinetic energy (E) of the particle A is measured in joules (J) and is given by $E = 5v^2$.

(a) Write down an expression for E as a function of time. [1]

Markscheme

$E = 5(2\sin t)^2 \left(= 20\sin^2 t\right)$ ***A1***

[1 mark]

(b) Hence find $\frac{\mathrm{d}E}{\mathrm{d}t}$. [2]

Markscheme

$\frac{\mathrm{d}E}{\mathrm{d}t} = 40\sin t\cos t$ ***(M1)A1***

[2 marks]

(c) Hence or otherwise find the first time at which the kinetic energy is changing at a rate of 5 J s^{-1}. [2]

Markscheme

$t = 0.126$ ***(M1)A1***

[2 marks]

2. [Maximum mark: 8] SPM.1.AHL.TZ0.14

The graph of $y = -x^3$ is transformed onto the graph of $y = 33 - 0.08x^3$ by a translation of a units vertically and a stretch parallel to the x-axis of scale factor b.

(a.i) Write down the value of a. [1]

Markscheme

$a = 33$ ***A1***

[1 mark]

(a.ii) Find the value of b. [3]

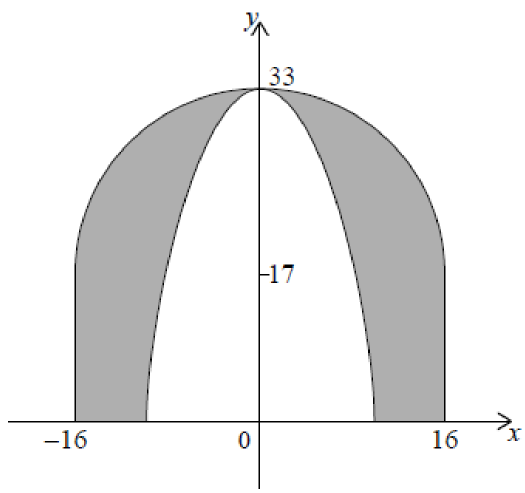
(a.ii) Find the value of b. [2]

Markscheme

$\frac{1}{\sqrt[3]{0.08}} = 2.32$ ***M1A1***

[2 marks]

(b) The outer dome of a large cathedral has the shape of a hemisphere of diameter 32 m, supported by vertical walls of height 17 m. It is also supported by an inner dome which can be modelled by rotating the curve $y = 33 - 0.08x^3$ through 360° about the y-axis between y = 0 and y = 33, as indicated in the diagram.



Find the volume of the space between the two domes. [5]

Markscheme

volume within outer dome

$\frac{2}{3}\pi + 16^3 + \pi \times 16^2 \times 17 = 22\,250.85$ ***M1A1***

volume within inner dome

$\pi \int_0^{33} \left(\frac{33-y}{0.08}\right)^{\frac{2}{3}} \mathrm{d}y = 3446.92$ ***M1A1***

volume between = 22 250.85 – 3446.92 = 18 803.93 m^3 ***A1***

[5 marks]

3. [Maximum mark: 8] EXN.1.AHL.TZ0.15

Consider the function $f(x) = \sqrt{-ax^2 + x + a},\ a \in \mathbb{R}^+$.

(a) Find $f\prime(x)$. [2]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$f\prime(x) = (-2ax + 1) \times \frac{1}{2} \times \left(-ax^2 + x + a\right)^{-\frac{1}{2}}$

Note: M1 is for use of the chain rule.

$= \frac{-2ax+1}{2\sqrt{-ax^2+x+a}}$ **M1A1**

[2 marks]

For $a > 0$ the curve $y = f(x)$ has a single local maximum.

(b) Find in terms of a the value of x at which the maximum occurs. [2]

Markscheme

$-2ax + 1 = 0$ **(M1)**

$x = \frac{1}{2a}$ **A1**

[2 marks]

(c) Hence find the value of a for which y has the smallest possible maximum value. [4]

Markscheme

Value of local maximum $= \sqrt{-a \times \frac{1}{4a^2} + \frac{1}{2a} + a}$ **M1A1**

$= \sqrt{\frac{1}{4a} + a}$

This has a minimum value when $a = 0.5$ **(M1)A1**

[4 marks]

4. [Maximum mark: 17] EXN.2.AHL.TZ0.7

A ball is attached to the end of a string and spun horizontally. Its position relative to a given point, O, at time t seconds, $t \geq 0$, is given by the equation

$\boldsymbol{r} = \begin{pmatrix} 1.5 \cos\left(0.1t^2\right) \\ 1.5 \sin\left(0.1t^2\right) \end{pmatrix}$ where all displacements are in metres.

(a) Show that the ball is moving in a circle with its centre at O and state the radius of the circle. [4]

Markscheme

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$|\boldsymbol{r}| = \sqrt{1.5^2 \cos^2\left(0.1t^2\right) + 1.5^2 \sin^2\left(0.1t^2\right)}$ **M1**

$= 1.5$ as $\sin^2\theta + \cos^2\theta = 1$ **R1**

Note: use of the identity needs to be explicitly stated.

Hence moves in a circle as displacement from a fixed point is constant. **R1**

Radius $= 1.5\,\mathrm{m}$ **A1**

[4 marks]

(b.i) Find an expression for the velocity of the ball at time t. [2]

Markscheme

$\boldsymbol{v} = \begin{pmatrix} -0.3t \sin\left(0.1t^2\right) \\ 0.3t \cos\left(0.1t^2\right) \end{pmatrix}$ **M1A1**

Note: M1 is for an attempt to differentiate each term

[2 marks]

(b.ii) Hence show that the velocity of the ball is always perpendicular to the position vector of the ball. [2]

Markscheme

$$\boldsymbol{v} \bullet \boldsymbol{r} = \begin{pmatrix} 1.5 \cos\left(0.1t^2\right) \\ 1.5 \sin\left(0.1t^2\right) \end{pmatrix} \bullet \begin{pmatrix} -0.3t \sin\left(0.1t^2\right) \\ 0.3t \cos\left(0.1t^2\right) \end{pmatrix}$$ **M1**

Note: M1 is for an attempt to find $\boldsymbol{v} \bullet \boldsymbol{r}$

$$= 1.5 \cos\left(0.1t^2\right) \times \left(-0.3t \sin\left(0.1t^2\right)\right) + 1.5 \sin\left(0.1t^2\right) \times 0.3t \sin\left(0.1t^2\right) = 0$$ **A1**

Hence velocity and position vector are perpendicular. **AG**

[2 marks]

(c.i) Find an expression for the acceleration of the ball at time t. [3]

Markscheme

$$\boldsymbol{a} = \begin{pmatrix} -0.3 \sin\left(0.1t^2\right) - 0.06t^2 \cos\left(0.1t^2\right) \\ 0.3 \cos\left(0.1t^2\right) - 0.06t^2 \sin\left(0.1t^2\right) \end{pmatrix}$$ **M1A1A1**

[3 marks]

The string breaks when the magnitude of the ball's acceleration exceeds $20\,\text{ms}^{-2}$.

(c.ii) Find the value of t at the instant the string breaks. [3]

Markscheme

$$\left(-0.3 \sin\left(0.1t^2\right) - 0.06t^2 \cos\left(0.1t^2\right)\right)^2 + \left(0.3 \cos\left(0.1t^2\right) - 0.06t^2 \sin\left(0.1t^2\right)\right)^2 = 400$$ **(M1)(A1)**

Note: M1 is for an attempt to equate the magnitude of the acceleration to 20.

$t = 18.3\ \ (18.256\ldots)\ (\mathrm{s})$ **A1**

[3 marks]

(c.iii) How many complete revolutions has the ball completed from $t = 0$ to the instant at which the string breaks? [3]

Markscheme

Angle turned through is $0.1 \times 18.256^2 =$ **M1**

$= 33.329\ldots$ **A1**

$\frac{33.329}{2\pi}$ **M1**

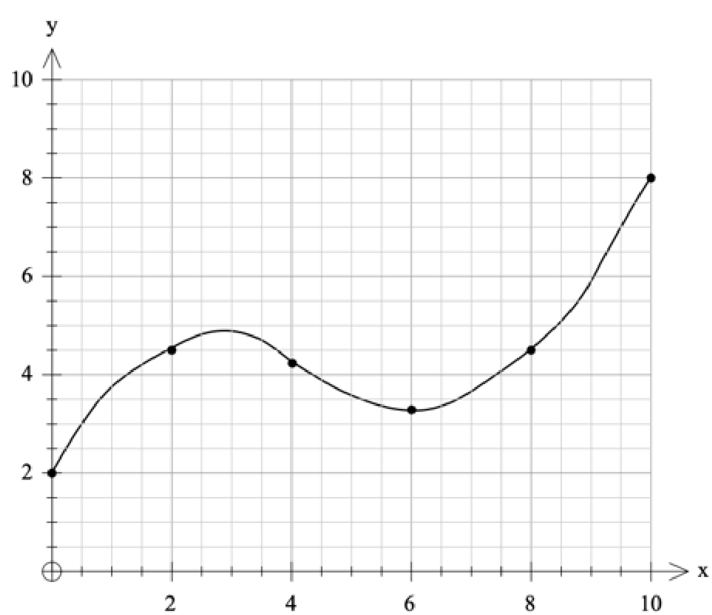
$\frac{33.329}{2\pi} = 5.30\ldots$

5 complete revolutions **A1**

[4 marks]

5. [Maximum mark: 10] EXM.2.AHL.TZ0.12

The curve $y = f(x)$ is shown in the graph, for $0 \leqslant x \leqslant 10$.



The curve $y = f(x)$ passes through the following points.

x	0	2	4	6	8	10
y	2	4.5	4.2	3.3	4.5	8

It is required to find the area bounded by the curve, the x-axis, the y-axis and the line $x = 10$.

(a) Use the trapezoidal rule to find an estimate for the area. [3]

Markscheme

Area = $\frac{2}{2}(2 + 2(4.5 + 4.2 + 3.3 + 4.5) + 8)$ ***M1A1***

Area = 43 ***A1***

[3 marks]

One possible model for the curve $y = f(x)$ is a cubic function.

(b.i) Use all the coordinates in the table to find the equation of the least squares cubic regression curve. [3]

Markscheme

$y = 0.0389x^3 - 0.534x^2 + 2.06x + 2.06$ ***M1A2***

[3 marks]

(b.ii) Write down the coefficient of determination. [1]

Markscheme

$R^2 = 0.991$ ***A1***

[1 mark]

(c.i) Write down an expression for the area enclosed by the cubic regression curve, the x-axis, the y-axis and the line $x = 10$. [1]

Markscheme

Area = $\int_0^{10} y\,dx$ ***A1***

[1 mark]

(c.ii) Find the value of this area. [2]

Markscheme

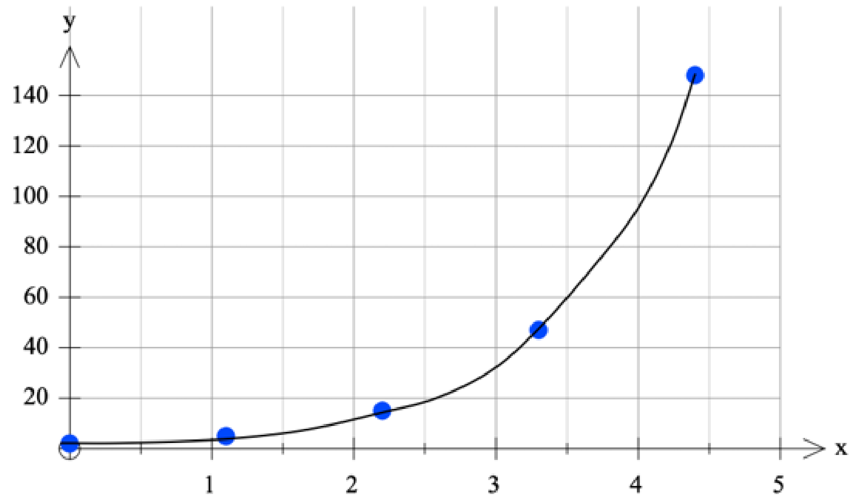
42.5 ***A2***

[2 marks]

6. [Maximum mark: 23] EXM.3.AHL.TZ0.7

This question explores methods to determine the area bounded by an unknown curve.

The curve $y = f(x)$ is shown in the graph, for $0 \leqslant x \leqslant 4.4$.



The curve $y = f(x)$ passes through the following points.

x	0	1.1	2.2	3.3	4.4
y	2	5	15	47	148

It is required to find the area bounded by the curve, the x-axis, the y-axis and the line $x = 4.4$.

(a.i) Use the trapezoidal rule to find an estimate for the area. [3]

Markscheme

Area $= \frac{1.1}{2}(2 + 2(5 + 15 + 47) + 148)$ ***M1A1***

Area = 156 units^2 ***A1***

[3 marks]

(a.ii) With reference to the shape of the graph, explain whether your answer to part (a)(i) will be an over-estimate or an underestimate of the area. [2]

Markscheme

The graph is concave up, ***R1***

so the trapezoidal rule will give an overestimate. ***A1***

[2 marks]

One possible model for the curve $y = f(x)$ is a cubic function.

(b.i) Use all the coordinates in the table to find the equation of the least squares cubic regression curve. [3]

Markscheme

$f(x) = 3.88x^3 - 12.8x^2 + 14.1x + 1.54$ ***M1A2***

[3 marks]

(b.ii) Write down the coefficient of determination. [1]

Markscheme

$R^2 = 0.999$ ***A1***

[1 mark]

(c.i) Write down an expression for the area enclosed by the cubic function, the x-axis, the y-axis and the line $x = 4.4$. [2]

Markscheme

Area $= \int_0^{4.4} \left(3.88x^3 - 12.8x^2 + 14.1x + 1.54\right) dx$ ***A1A1***

[2 marks]

(c.ii) Find the value of this area. [2]

Markscheme

Area = 145 units^2 (Condone 143–145 units^2, using rounded values.) ***A2***

[2 marks]

A second possible model for the curve $y = f(x)$ is an exponential function, $y = pe^{qx}$, where $p,\ q \in \mathbb{R}$.

(d.i) Show that $\ln y = qx + \ln p$. [2]

Markscheme

$\ln y = \ln(pe^{qx})$ ***M1***

$\ln y = \ln p + \ln(e^{qx})$ ***A1***

$\ln y = qx + \ln p$ ***AG***

[2 marks]

(d.ii) Hence explain how a straight line graph could be drawn using the coordinates in the table. [1]

Markscheme

Plot $\ln y$ against p. ***R1***

[1 mark]

(d.iii) By finding the equation of a suitable regression line, show that $p = 1.83$ and $q = 0.986$. [5]

Markscheme

Regression line is $\ln y = 0.986x + 0.602$ ***M1A1***

So q = gradient = 0.986 ***R1***

$p = e^{0.602} = 1.83$ ***M1A1***

[5 marks]

(d.iv) Hence find the area enclosed by the exponential function, the x-axis, the y-axis and the line $x = 4.4$. [2]

Markscheme

Area $= \int_0^{4.4} 1.83e^{0.986x}\,dx = 140$ units^2 ***M1A1***

[2 marks]

7. [Maximum mark: 7] 24M.1.AHL.TZ1.9

(a) Find $\int \frac{8}{2x+3}\mathrm{dx}$. [3]

Markscheme

attempt to integrate by substitution or inspection ***(M1)***

$4\ \ln\ |2x+3|+c$ **OR** $4\ \ln\ |x+1.5|+c$ ***A1A1***

Note: Award ***M1*** for $\ln\ (2x+3)$ or $\ln\ (x+1.5)$, ***A1*** for the 4 and ***A1*** for c. The ***A*** marks can only be awarded if the ***M*** mark is awarded. Condone absence of modulus signs.

[3 marks]

(b) Hence find the exact area between the curve $y=\frac{8}{2x+3}$, the x-axis and the lines $x=0$ and $x=6$. Give your answer in the form $a\ \ln\ b$, where $a,\ b\in\mathbb{N}$. [4]

Markscheme

recognizing that area is $[4\ \ln\ (2x+3)]_0^6$ ***(M1)***

$=4\ \ln\ (15)-4\ \ln\ (3)$ ***(A1)***

use of log laws for their expression ***(M1)***

$=4\ \ln\ (5)\ (=2\ \ln\ (25)=1\ \ln\ (625))$ ***A1***

Note: Award ***(M1)A0M0A0*** for an unsupported final answer of $6.43775\ldots$

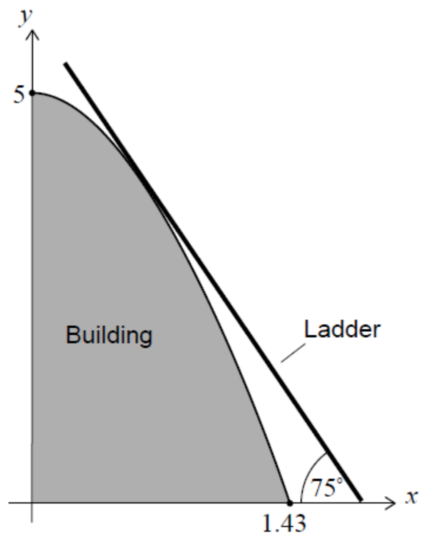
Award at most ***(M1)A1FTM0A0*** if their answer from part (a) does not include $\ln$.

[4 marks]

8. [Maximum mark: 8] 24M.1.AHL.TZ1.16

The cross section of the side of a building can be modelled by a curve with equation $y=5\ \cos\ (1.1x),\ 0\le x\le 1.43$, as shown in the following diagram. Distances are measured in metres.

diagram not to scale



[8]

A builder leans a straight ladder against the building to do repairs. For safety reasons, the angle between the ladder and the horizontal ground must be 75°.

Find the height above the ground at which the ladder touches the building.

Markscheme

attempt to find gradient ***(M1)***

EITHER

gradient of tangent $= -\tan 75^\circ \left(= -3.73205\ldots,\ -2-\sqrt{3}\right)$ ***(A1)(A1)***

Note: Award ***A1*** for negative and ***A1*** for $\tan 75^\circ$ (or equivalent).

OR

gradient of tangent $= \tan 105^\circ\ (= -3.73205\ldots)$ ***(A2)***

THEN

$\frac{dy}{dx} = -5.5 \sin(1.1x)$ ***(A1)***

Note: Award ***(A1)*** for a labelled sketch of the derivative function.

equating derivative to their gradient ***(M1)***

$-5.5\sin(1.1x) = -3.73205\ldots$ **OR** line on graph

$x = 0.677993\ldots$ ***(A1)***

Note: Award ***(A1)(M1)A0*** for an answer of $x = 38.8$, from calculator being in degrees.

Award ***A0M1A0*** if "$\frac{\mathrm{d}}{dx}(5\cos(1.1x)) = -3.73205\ldots$" seen, but leading to an incorrect x-value.

height$= 5\cos(1.1 \times 0.677993\ldots)$ ***(M1)***

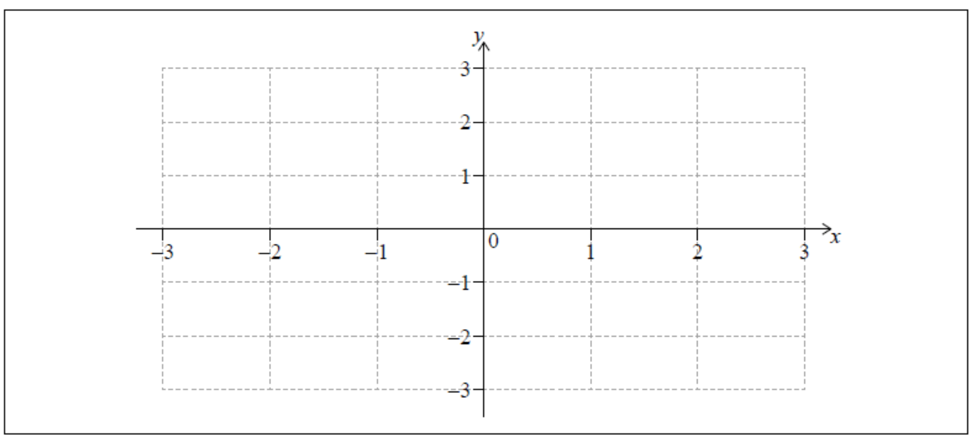
$= 3.67(\mathrm{m}) \quad (3.67274\ldots)$ ***A1***

[8 marks]

9. [Maximum mark: 9] 24M.1.AHL.TZ2.10

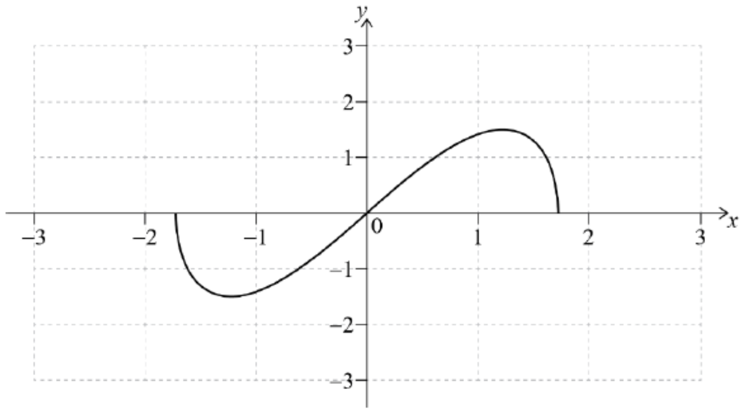
Consider the function $f(x) = x\sqrt{3 - x^2}, -\sqrt{3} \le x \le \sqrt{3}$.

(a) Sketch the graph of $y = f(x)$ on the following pair of axes.



[2]

Markscheme



A1A1

Note: Award ***A1*** smooth curve, through origin, with maximum in 1^{st} quadrant and minimum in 3^{rd} quadrant (approximately correct position).

Award ***A1*** correct (approximate) position of endpoints and rotational symmetry. Condone a (small) gap between end of curve and the x-axis.

[2 marks]

The area between the graph of $y = f(x)$ and the x-axis is rotated through 360° about the x-axis.

(b.i) Write down an integral that represents this volume. [2]

Markscheme

$2\pi\int_0^{\sqrt{3}} \left(\mathrm{x}\sqrt{3-\mathrm{x}^2}\right)^2 \mathrm{dx}$ **OR** $\pi\int_{-\sqrt{3}}^{\sqrt{3}} \left(x\sqrt{3-x^2}\right)^2 \mathrm{d}\ x$ (or equivalent) ***A1A1***

Note: Award ***A1*** for integral with correct limits in the correct places, ***A1*** for completely correct expression;

$\mathrm{d}\ x$ must be included. Condone a correct answer presented as two integrals, despite the demand of the question: $\pi\left(\int_{-\sqrt{3}}^{0} \left(x\sqrt{3-x^2}\right)^2 \mathrm{d}x + \int_0^{\sqrt{3}} \left(x\sqrt{3-x^2}\right)^2 \mathrm{d}x\right)$.

[2 marks]

(b.ii) Calculate the value of this integral. [2]

Markscheme

13.1 $\left(13.0593\ldots,\ \frac{12\pi\sqrt{3}}{5}\right)$ ***A2***

[2 marks]

The graph of the function f is transformed, to give the graph of the function g, in the following way:

- It is first stretched by scale factor 2, parallel to the x-axis with the y-axis invariant.
- It is then stretched by scale factor 0.5, parallel to the y-axis with the x-axis invariant.

(c) Find the volume obtained when the area between the graph of $y = g(x)$ and the x-axis is rotated through 360° about the x-axis. [3]

Markscheme

attempt to express $g(x)$ in terms of $f(x)$ ***(M1)***

$g(x) = \frac{1}{2}\left(\frac{x}{2}\sqrt{3-\left(\frac{x}{2}\right)^2}\right),\ g(x) = \frac{1}{2}f\left(\frac{x}{2}\right)$

correct limits for g seen: $-2\sqrt{3}$ and $2\sqrt{3}$ ***(A1)***

$\pi\int_{-2\sqrt{3}}^{2\sqrt{3}}\left(\frac{1}{2}\left(\frac{x}{2}\sqrt{3-\left(\frac{x}{2}\right)^2}\right)\right)^2 \mathrm{d}x$ **OR** $\pi\int_{-2\sqrt{3}}^{2\sqrt{3}}\left(\frac{1}{2}f\left(\frac{x}{2}\right)\right)^2\mathrm{d}x$ **OR** $2\times\frac{1}{2}^2\times\left(\frac{12\pi\sqrt{3}}{5}\right)$

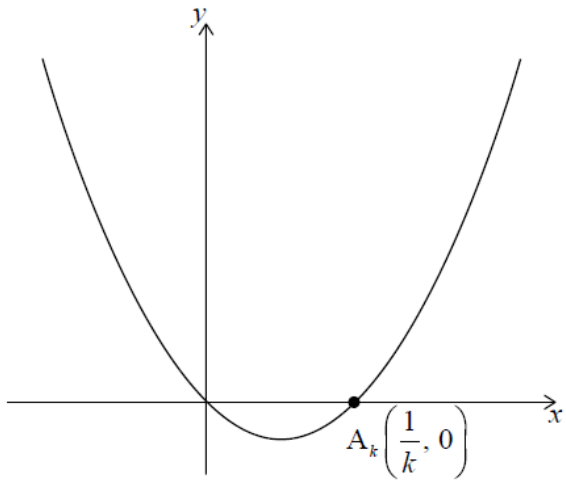
6.53 $\left(6.52967\ldots,\ \frac{6\pi\sqrt{3}}{5}\right)$ ***A1***

Note: Some candidates may answer question by transforming their part (b)(ii): $2\times\frac{1^2}{2}\times\left(\frac{12\pi\sqrt{3}}{5}\right)$. This is a valid method and leads to the correct answer.

[3 marks]

10. [Maximum mark: 9] 24M.1.AHL.TZ2.11

The diagram shows the curve with equation $y_k = kx^2 - x,\ k > 0$, which intersects the x-axis at the origin and at the point $A_k\left(\frac{1}{k},\ 0\right)$.



The normal to the curve at A_k intersects the curve again at point B_k.

(a) Show that the x-coordinate of B_k is $-\frac{1}{k}$. [6]

Markscheme

$y = kx^2 - x$

$\frac{dy}{dx} = 2kx - 1$ ***(A1)***

$m_{\tan} = 1$ ***A1***

finding negative reciprocal of their gradient ***(M1)***

$m_{\text{normal}} = -1$

normal: $y = -\left(x - \frac{1}{k}\right)$ ***A1***

equating normal line to quadratic curve ***M1***

$kx^2 - x = -\left(x - \frac{1}{k}\right)$

correct simplification leading to given result ***A1***

e.g. quadratic formula or difference of two squares

$kx^2 - \frac{1}{k} = 0$

$k^2x^2 - 1 = 0$

$(kx - 1)(kx + 1) = 0$

$x = -\frac{1}{k}$ ***AG***

[6 marks]

Consider the case where $k = 2$.

(b) Calculate the finite area of the region between the curve with equation $y_2 = 2x^2 - x$ and the normal at A_2. [3]

Markscheme

area$= \int_{-\frac{1}{2}}^{\frac{1}{2}} -\left(x - \frac{1}{2}\right) - \left(2x^2 - x\right) \ \mathrm{d}\,x$ ***(M1)(A1)***

Note: Award ***M1*** for evidence of subtracting the functions and ***A1*** for a completely correct integral. (Condone a missing $\mathrm{d}\,x$.)

$= 0.333 \quad \left(0.333333\ldots,\ \frac{1}{3}\right)$ ***A1***

[3 marks]

11. [Maximum mark: 8] 24M.1.AHL.TZ2.13

A particle starts from rest at point O and moves in a straight line with velocity, v, given by

$v = 3\sin(t)\,(1 + \cos(t)),\ t \geq 0$

where v is measured in metres per second and time, t (radians), is measured in seconds.

The particle next comes to instantaneous rest when $t = a$.

(a) Determine the value of a. [2]

Markscheme

$3\sin(a)(1 + \cos(a)) = 0$ ***(M1)***

$a = 3.14 \quad (\pi,\ 3.14159\ldots)$ ***A1***

[2 marks]

(b) Find the maximum velocity of the particle during the interval $0 \leq t \leq a$. [2]

Markscheme

attempt to find the maximum velocity ***(M1)***

e.g. $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$ **OR** sketch of v with maximum marked

$v_{\max} = 3.90\left(\mathrm{m\,s^{-1}}\right) \quad (3.89711\ldots)$ ***A1***

Note: Accept 3.9 rounded to 2 sf.

[2 marks]

(c) By finding the total distance travelled between $t = 0$ and $t = a$, find the average speed of the particle during the interval $0 \leq t \leq a$. [4]

Markscheme

attempt to find integral of speed (condone omission of modulus signs) ***(M1)***

$\int_0^{\pi} |3 \sin(t)(1 + \cos(t))| \, dt$ **OR** 6

attempt to substitute into speed $= \frac{\text{distance}}{\text{time}}$ ***(M1)***

$= \frac{\int_0^{\pi} |3 \sin(t)(1+\cos(t))| \, dt}{\pi}$ ***(A1)***

$= 1.91 \, (\text{ms}^{-1}) \quad \left(1.90985\ldots, \frac{6}{\pi}\right)$ ***A1***

[4 marks]

12. [Maximum mark: 13] 24M.2.AHL.TZ1.3

A shop uses the following model to estimate n, the number of smoothies sold per day, in terms of x, the price of a single smoothie in pesos.

$n = \frac{40\,000}{x^2}$

The maximum number of smoothies the shop can make in a day is 400.

(a) Find the maximum price they could charge per smoothie for the shop to sell 400 in one day. [2]

Markscheme

$\frac{40\,000}{x^2} = 400$ ***(M1)***

$x = 10$ (pesos) (since x is positive) ***A1***

[2 marks]

(b) On a day when the shop sells smoothies at 50 pesos each, use the model to find

(b.i) the number of smoothies sold. [1]

Markscheme

$\left(\frac{40\,000}{50^2}=\right)\ 16$ ***A1***

[1 mark]

(b.ii) the total income from the smoothies sold. [1]

Markscheme

$(50\times 16=)\ 800$ (pesos) ***A1***

[1 mark]

The cost of making each smoothie is 20 pesos. The profit per day (P) is the total income from the sale of smoothies that day minus the cost of making them.

(c.i) Show that, according to the model, $P=\frac{40\,000}{x}-\frac{800\,000}{x^2}$. [2]

Markscheme

EITHER

profit for each smoothie $=x-20$ ***(M1)***

$P=\frac{40\,000}{x^2}\times(x-20)$ ***A1***

OR

profit $=$ revenue $-$ costs $=nx-20n$ ***(M1)***

$P=x\times\frac{40\,000}{x^2}-20\times\frac{40\,000}{x^2}$ ***A1***

Note: Do not award ***A1*** if $\frac{40\,000}{x}$ seen as first term unless explained (in part (a) or (b)), as it is given in question.

THEN

$P=\frac{40\,000}{x}-\frac{800\,000}{x^2}$ ***AG***

[2 marks]

(c.ii) Find $\frac{\mathrm{d}P}{\mathrm{d}x}=0$. [3]

Markscheme

attempt to express P ready for power rule ***(M1)***

$P = 40\,000x^{-1} - 800\,000x^{-2}$

$\frac{\mathrm{d}P}{\mathrm{d}x} = -\frac{40\,000}{x^2} + \frac{1\,600\,000}{x^3}$ **OR** $\frac{\mathrm{d}P}{\mathrm{d}x} = -40\,000x^{-2} + 1\,600\,000\,x^{-3}$ ***A1A1***

Note: The ***(M1)*** can be awarded for either of the correct terms seen.

A1 for each correct term.

At most ***M1A1A0*** if additional terms seen.

[3 marks]

(c.iii) Find the value of x for which $\frac{\mathrm{d}P}{\mathrm{d}x} = 0$. [2]

Markscheme

attempt to find x-value ***(M1)***

e.g. sketch of $\frac{\mathrm{d}P}{\mathrm{d}x}$ with x-intercept indicated **OR** recognition that it occurs at the maximum of P **OR** algebraic approach (requires multiplication by x^3)

$x = 40$ ***A1***

Note: $\frac{-40\,000}{x^2} + \frac{1\,600\,000}{x^3} = 0$ is insufficient to award ***M1***, this is given in the question. There must be an "attempt to find x-value".

Award ***M1A0*** for a coordinate pair $(40,\ 500)$.

[2 marks]

(c.iv) Find the number of smoothies sold when the profit is maximized. [2]

Markscheme

attempt to substitute their x-value into equation for n ***(M1)***

$n = \frac{40\,000}{40^2}$

$= 25$ ***A1***

Note: Given the nature of the function P, the local maximum is also the global maximum. This is often the case in examinations, but should not always be assumed.

[2 marks]

13. [Maximum mark: 14] 24M.2.AHL.TZ1.7

The interior of a vase is modelled by rotating the region bounded by the curve $y = \frac{1}{2}x^2 - 1$, and the lines $x = 0$, $y = 0$ and $y = 15$, through 2π radians about the y-axis. The values of x and y are measured in centimetres.

The vase is filled with water to a height of h cm.

(a) Find an explicit expression for the volume of water in terms of h. [5]

Markscheme

attempt to use $V = \pi \int x^2 \, dy$ ***(M1)***

$x^2 = 2y + 2$ or any reasonable attempt to find x in terms of y ***(M1)***

$V = \pi \int_0^h 2y + 2 \, dy$ ***(A1)***

Note: Correct limits must be seen for the ***A1*** to be awarded however the dy may be omitted (as not a final answer).

If this is given as the final answer to this part the remaining marks can be awarded if seen in part (b).

$\int 2y + 2 \, dy = y^2 + 2y$ ***(A1)***

Note: Accept equivalent with alternate variable

$V = \pi \left[y^2 + 2y\right]_0^h$

$= \pi\left(h^2 + 2h\right)$ ***A1***

Note: The final two ***A1*** marks can be awarded independently of the first ***A1***.

If $h^2 + 2h$ or $y^2 + 2y$ is the final (unsupported) answer award at most ***(M1)(M1)(A0)(A1)A0***.

[5 marks]

The vase is filled at a rate of $20\ \mathrm{cm}^3\ \mathrm{s}^{-1}$.

(b) Find the time taken to completely fill the vase. [2]

Markscheme

volume of vase$= \pi\left(15^2 + 2 \times 15\right) \quad (= 801.106\ldots)$ ***(A1)***

(time to fill vase $= \frac{801.106\ldots}{20} =$) $40.1 \quad (40.0553\ldots)$ (seconds) ***A1***

Note: Accept exact answers in terms of π, e.g. 12.75π or $\frac{51\pi}{4}$

[2 marks]

(c) Find the rate at which the height is changing when $h = 10\ \mathrm{cm}$. [7]

Markscheme

EITHER

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \times \frac{\mathrm{d}h}{\mathrm{d}V}$ ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}h} = \pi(2h + 2)$ ***(A1)***

OR

differentiating $V = \pi\left(h^2 + 2h\right)$ implicitly ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}t} = \pi(2h + 2)\frac{\mathrm{d}h}{\mathrm{d}t}$ ***(A1)***

THEN

$\frac{\mathrm{d}h}{\mathrm{d}t} = 20 \times \frac{1}{\pi(2h+2)}$ ***(M1)(A1)***

Note: Award ***M1*** for attempting to solve for $\frac{\mathrm{d}h}{\mathrm{d}t}$, ***A1*** for a correct expression.

substituting $h = 10$ seen anywhere ***(M1)***

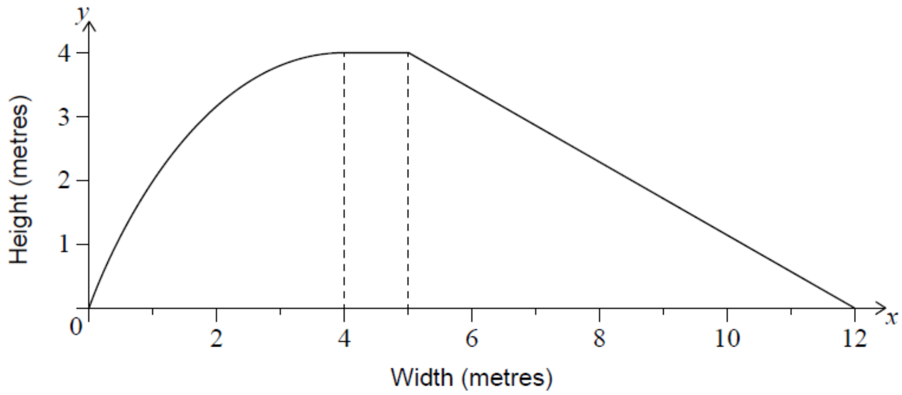
$0.289 \quad (0.289372\ldots)\ \mathrm{cm\ s}^{-1}$ ***A1A1***

Note: Award ***A1*** for the correct value. Award ***A1*** for the correct units, independent of other marks.

[7 marks]

14. [Maximum mark: 15] 24M.2.AHL.TZ2.2

The following diagram shows a model of the side view of a water slide. All lengths are measured in metres.



The curved edge of the slide is modelled by

$f(x) = -\frac{1}{4}x^2 + 2x$ for $0 \leq x \leq 4$.

The remainder of the slide is modelled by

$$g(x) = \begin{cases} 4, & \text{for } 4 \leq \text{x} \leq 5 \\ \frac{48}{7} - \frac{4x}{7}, & \text{for } 5 \leq \text{x} \leq 12 \end{cases}$$

(a) Use the trapezoidal rule with an interval width of 1 to calculate the approximate area under the model of the slide in the interval $0 \leq x \leq 4$. [5]

Markscheme

heights, $0,\ 4,\ 1.75,\ 3$ and 3.75 seen ***(A2)***

Note: Award ***A1A0*** if **two** of $1.75, 3$ or 3.75 are seen.

attempt to use trapezoidal rule formula for their heights ***(M1)***

$\frac{1}{2} \times 1 \times \{0 + 4 + 2(1.75 + 3 + 3.75)\}$ ***(A1)***

Note: Award ***(M1)(A1)*** for correctly expressing this as 3 trapezoids and a triangle. The "$\times 1$" need not be seen.

$= 10.5\,(\mathrm{m}^2)$ ***A1***

[5 marks]

(b) Find $\int\left(-\frac{1}{4}x^2 + 2x\right)\mathrm{d}x$. [3]

Markscheme

$-\frac{1}{12}x^3 + x^2 + c$ ***A1A1A1***

[3 marks]

(c) Calculate the exact area under the entire model of the slide, for $0 \le x \le 12$. [4]

Markscheme

$\int_0^4\left(-\frac{1}{4}x^2 + 2x\right)\mathrm{d}x + 1 \times 4 + \frac{1}{2} \times 7 \times 4$ ***(A1)(M1)(A1)***

Note: Award ***A1*** for correct area of rectangle **OR** triangle, ***M1*** for substituting correct limits into given integral (may be seen in part (b)), and ***A1*** for entire expression correct.

$= 10.6666\ldots\ + 4 + 14$

$= 28\frac{2}{3}\,(\mathrm{m}^2)\ \ \left(\frac{86}{3}\right)$ ***A1***

Note: The answer must be **exact** for the ***A1*** to be awarded. For an answer of 28.7 or 28.66 award ***(A1)(M1)(A1)A0***.

[4 marks]

(d) Find the percentage error in the **total** area under the entire model of the slide when using the approximate value from part (a). [3]

Markscheme

(Total area using part (a) =) 28.5 ***(A1)***

Percentage error $\left|\frac{28.5 - 28.6666\ldots}{28.6666\ldots}\right| \times 100$ ***(M1)***

Note: if their trapezoid value is incorrect but is used correctly in the percentage error formula, award at most ***A0M1A0***. If it is clear from the answer that $\times 100$ has been used, then condone the omission and award the ***M*** mark.

$= 0.581\,(\%) \quad (0.581395\ldots)$ ***A1***

(accept 0.697 from use of 28.7)

[3 marks]

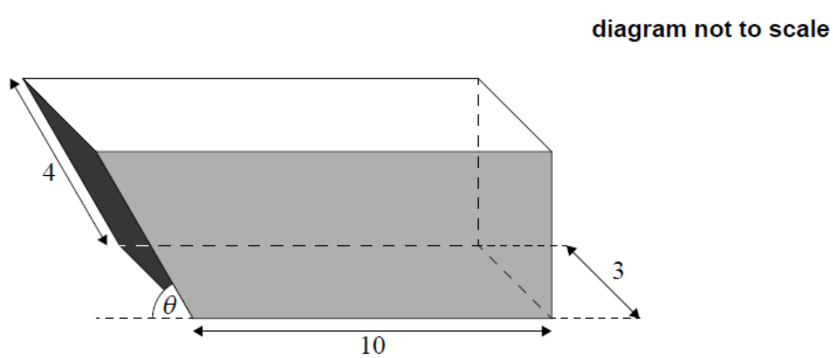
15. [Maximum mark: 22] 24M.2.AHL.TZ2.3

A skip is a container used to carry garbage away from a construction site. For safety reasons the garbage must not extend beyond the top of the skip. The maximum volume of garbage to be removed is therefore equal to the volume of the skip.

[Source: Andyqwe, n.d. *Dumpster truck* [image online] Available at: https://www.gettyimages.co.uk/detail/photo/dumpster-truck-royalty-free-image/157611454 [Accessed 18 April 2023] Source adapted.]

A particular design of skip can be modelled as a prism with a trapezoidal cross section. For the skip to be transported, it must have a rectangular base of length $10\,\text{m}$ and width $3\,\text{m}$. The length of the sloping edge is fixed at $4\,\text{m}$, and makes an angle of θ with the horizontal.

The following diagram shows such a skip.



(a) Find the volume of this skip,

(a.i) if the length of the top edge of the skip is $11\,\mathrm{m}$. [4]

Markscheme

correct approach to find missing length ***(A1)***

$\sqrt{4^2-1^2}\quad\left(=\sqrt{15}\right)$

attempt to find cross-section ***(M1)***

e.g. use of area of trapezoid formula or rectangle $+$ triangle or rectangle $-$ triangle

use of volume of prism formula ***(M1)***

(their cross-section multiplied by 3)

$3\left[\frac{1}{2}(10+11)\left(\sqrt{4^2-1^2}\right)\right]$

$=122\left(\mathrm{m}^3\right)\quad(121.998\ldots)$ ***A1***

[4 marks]

(a.ii) if the height of the skip is $3.2\,\mathrm{m}$. [3]

Markscheme

correct approach to find missing height ***(A1)***

$\sqrt{4^2-3.2^2}\quad(=2.4)$

attempt to find volume ***(M1)***

(multiplication by 3.2 and 3 seen)

$3\left[\frac{1}{2}\left(10+10+\sqrt{4^2-3.2^2}\right)(3.2)\right]$

$=108\left(\mathrm{m}^3\right)\quad(107.52\ldots)$ ***A1***

[3 marks]

(a.iii) if θ is $\frac{\pi}{3}$. [2]

Markscheme

correct approach to find missing lengths ***(A1)***

$\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{3}\right)$ **OR** $\sin\left(\frac{\pi}{3}\right)$ and Pythagoras etc seen in work

$3\left[\frac{1}{2}\left(10+10+4\cos\left(\frac{\pi}{3}\right)\right)4\sin\left(\frac{\pi}{3}\right)\right]$

$= 114\left(\text{m}^3\right) \quad (114.315\ldots)$ ***A1***

[2 marks]

(b) Show that the volume, $V\,\text{m}^3$, of the skip is given by

$24\sin(\theta)(5+\cos(\theta))$. [2]

Markscheme

$V = 3\left[\frac{1}{2}(10+10+4\cos(\theta))4\sin(\theta)\right]$ ***A1***

all correct intermediate working leading to given answer ***A1***

e.g. $V = 6\sin(\theta)(20+4\cos(\theta))$

Note: The ***AG*** line must be seen for the final ***A1*** to be awarded.

[2 marks]

(c) Explain, in context, why $\theta \neq 0$. [1]

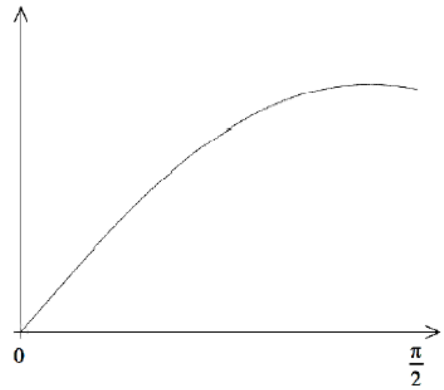
Markscheme

accept any reasoning along the lines: "skip would have zero volume" or "if the angle is zero, then the contents would fall out" ***R1***

[1 mark]

(d.i) Sketch the graph of $V = 24\sin(\theta)(5+\cos(\theta)),\ 0 < \theta < \frac{\pi}{2}$. [2]

Markscheme



A1A1

Note: Award ***A1*** for the correct shape and ***A1*** for the graph on the correct, labelled, domain. Condone omission of θ/V labels (or x/y).

[2 marks]

(d.ii) Find the maximum volume of the skip and the value of θ for which this maximum volume occurs. [2]

Markscheme

$\theta = 1.38 \quad (1.38356\ldots) \quad (79.3^\circ \quad (79.2723\ldots^\circ))$ ***A1***

$V_{\max} = 122 \quad (122.292\ldots)$ ***A1***

Note: Award ***A0A1*** if values are reversed and ***A0A0*** for a coordinate pair.

[2 marks]

(e) Show, by differentiation, that the maximum volume occurs at a value of θ that satisfies the equation $2\cos^2\theta + 5\cos\theta - 1 = 0$. [6]

Markscheme

recognizing that derivative is equal to zero (seen at any stage) ***M1***

$\frac{dV}{d\theta} = 0$ (accept $\frac{dy}{dx} = 0$)

(from graph, turning point is a global maximum)

use of product rule ***M1***

$\left(\frac{dV}{d\theta} =\right) 24\cos(\theta)(5 + \cos(\theta)) + 24\sin(\theta)(-\sin(\theta))$ ***A1***

$= 120\cos(\theta) + 24\cos^2(\theta) - 24\sin^2(\theta)\ \ (=0)$ (or equivalent) ***A1***

substituting $1 - \cos^2(\theta)$ for $\sin^2(\theta)$ ***M1***

e.g $120\cos^2(\theta) + 24\cos^2(\theta) - 24\left(1 - \cos^2(\theta)\right)\ \ (=0)$

correct intermediate steps leading to given answer ***A1***

$2\cos^2(\theta) + 5\cos(\theta) - 1 = 0$ ***AG***

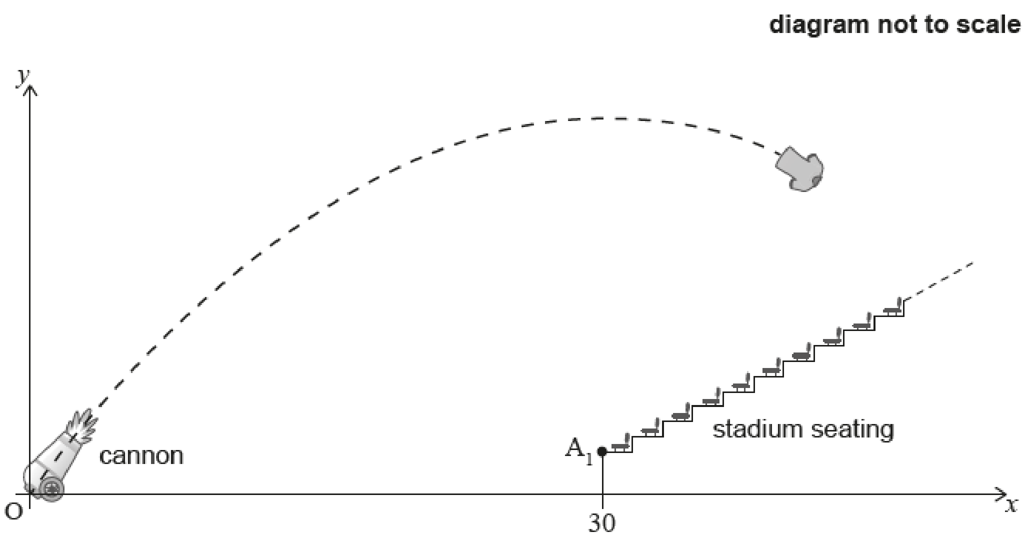
[6 marks]

16. [Maximum mark: 29] 24M.3.AHL.TZ1.2

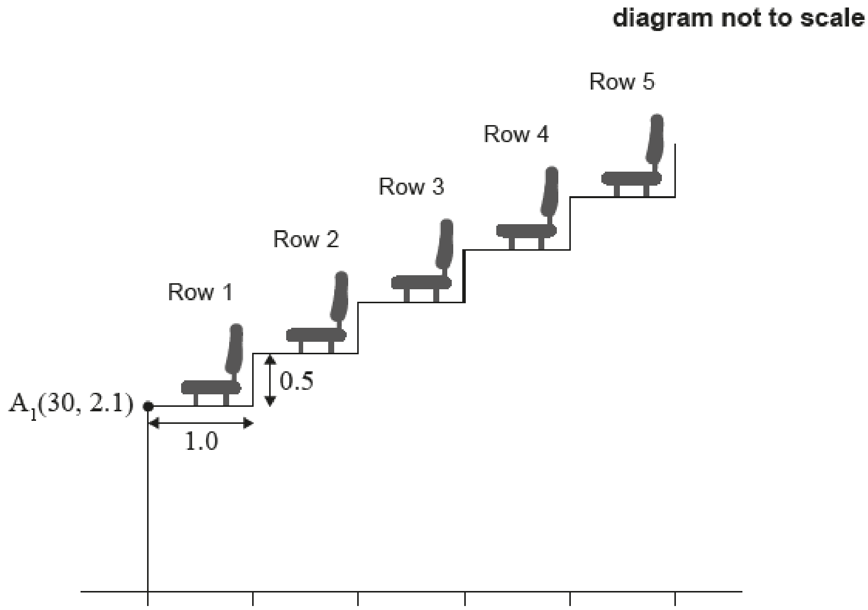
A sports stadium has a T-shirt cannon which is used to launch T-shirts into the crowd. The purpose of this question is to determine whether a person sitting in a particular seat will ever receive a T-shirt.

A T-shirt cannon is placed on the horizontal ground of a stadium playing area. A coordinate system is created such that the origin, O, is the point on the ground from where the T-shirts are launched. In this coordinate system, x and y represent the horizontal and vertical displacement from O, and are measured in metres.

Seat A_1 is the nearest seat to the T-shirt cannon. The coordinates of the front of the foot space for seat A_1 are $(30,\ 2.1)$.



Each seat behind seat A_1 is $1.0\,\mathrm{m}$ further from O horizontally and $0.5\,\mathrm{m}$ higher than the seat in the row below it, as shown on the diagram.



Seat A_1 is in row 1. Let seat A_n be the seat directly behind A_1 in row n.

(a.i) Write down the coordinates of the front of the foot space of seat A_5. [2]

Markscheme

$(34,\ 4.1)$ ***A1A1***

[2 marks]

(a.ii) Find, in terms of n, the coordinates for the front of the foot space of seat A_n. [3]

Markscheme

recognizing the sequence is arithmetic, with a common difference of 0.5 and 1.0 ***(M1)***

$(30+(n-1),\ 2.1+(n-1)\,0.5)$ ***A1A1***

$(=(29+n,\ 1.6+0.5n))$

[3 marks]

While in motion, the T-shirt can be treated as a projectile.

Let t be the time, in seconds, after a T-shirt is launched.

At any time $t>0$, the acceleration of the T-shirt, in $\mathrm{m\,s^{-2}}$, is given by the vector

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix}=\begin{pmatrix}0\\ -9.8\end{pmatrix}.$$

The initial velocity, in $\mathrm{m\,s^{-1}}$, of the T-shirt is given as $\begin{pmatrix} 29.4\cos\theta \\ 29.4\sin\theta \end{pmatrix}$, where θ is the angle to the ground at which the T-shirt is launched and $0^\circ < \theta \leq 90^\circ$.

(b.i) Find an expression for the velocity, $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, at time t. [3]

Markscheme

evidence of integration of the acceleration vector **OR** use of $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}t$ ***(M1)***

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} c_1 \\ -9.8t + c_2 \end{pmatrix}$$

$$\left(\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} =\right)\begin{pmatrix} 29.4\cos\theta \\ 29.4\sin\theta - 9.8t \end{pmatrix}$$ ***A1A1***

Note: The first ***A1*** is for $\dot{x}$ and the second is for $\dot{y}$.

[3 marks]

(b.ii) Hence show that when the T-shirt is launched vertically, the time for it to reach its maximum height is 3 seconds. [3]

Markscheme

$\theta = 90^\circ$ ***(A1)***

$29.4 - 9.8t = 0$ ***M1A1***

Note: Award ***M1*** for setting their $\dot{y}$ to zero (may still include θ), ***A1*** for correct equation, leading to given result.
If they substitute $t = 3$ award at most ***(A1)M0A0***.

maximum point when $t = \frac{29.4}{9.8}$

$= 3$ (seconds) ***AG***

[3 marks]

The displacement of the T-shirt, t seconds after it is launched, is given by the vector equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 29.4(\cos\theta)t \\ 29.4(\sin\theta)t - 4.9t^2 \end{pmatrix}$$

(c) Using the given answer to part (b)(ii) or otherwise, find the maximum height reached by a T-shirt when it is launched vertically. [2]

Markscheme

correct substitution OR use of correct graph ***(M1)***

maximum height is $29.4 \times 3 - 4.9 \times 3^2$

$= 44.1\ (\mathrm{m})$ ***A1***

[2 marks]

(d.i) If there was no seating, and the T-shirt was launched at an angle θ, show that the value of x when it would hit the ground is given by the expression

$x = 176.4 \sin\theta \cos\theta$ [3]

Markscheme

$29.4 \sin\theta t - 4.9t^2 = 0$ ***M1***
$t = 6 \sin\theta$ (or $t = 0$) ***A1***

$x = 29.4 \cos\theta \times 6 \sin\theta$ ***A1***
$= 176.4 \cos\theta \sin\theta$ ***AG***

[3 marks]

(d.ii) Hence find the maximum possible value for x if there was no seating to block the path of the T-shirt. [2]

Markscheme

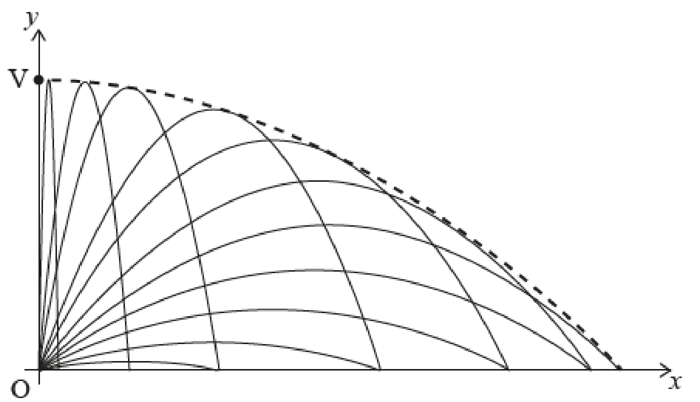
valid method to find maximum (e.g. sketch graph, find derivative) ***(M1)***

maximum value of x is $88.2\ (\mathrm{m})$ ***A1***

Note: Award ***(M1)A0*** for an unsupported answer of "45" or $\frac{\pi}{4}$ $(0.785398\ldots)$.

[2 marks]

In order to calculate the seats in the stadium which can be reached by a T-shirt it is required to find the equation of the curve that forms the boundary of all the points that can be reached. This boundary is represented by the dashed curve in the following diagram, while the solid curves represent some of the possible trajectories for the T-shirts.



It is given that the boundary curve is the parabola $y = ax^2 + bx + c$, with its vertex V on the y-axis.

(e) Using your answers to parts (c) and (d)(ii), or otherwise, find

(e.i) the value of c. [1]

Markscheme

$(c =)\ 44.1$ ***A1***

Note: follow through from 2c

[1 mark]

(e.ii) the value of b. [2]

Markscheme

EITHER

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 2a \times 0 + b = 0$ ***(M1)***

OR

vertex is at $x = -\frac{b}{2a} = 0$ ***(M1)***

THEN

$\Rightarrow b = 0$ ***A1***

[2 marks]

(e.iii) the value of a. [3]

Markscheme

point $(88.2,\ 0)$ used *(A1)*

$0 = a \times 88.2^2 + 44.1$ *(M1)*

$\Rightarrow a = -\frac{44.1}{88.2^2}$

$= -\frac{5}{882} = -0.0056689\ldots$ *A1*

[3 marks]

Note: The method for finding the parameters might appear in a different part, for example use of $(88.2,\ 0)$ might appear in part (ii), or the parts might not be clearly numbered. Accept any correct working wherever it is seen, and ascribe the marks to the correct part.

A spectator is sitting in seat A_{40}.

(f) Show that it is not possible for the spectator to ever get a T-shirt. [5]

Markscheme

use of their arithmetic sequence from (a)(ii) with $n = 40$ *(M1)*

coordinates of seat $\mathrm{A}_{40} = (69,\ 21.6)$ *(A1)*

EITHER

substitution of their 69 into their $y = -\frac{5}{882}x^2 + 44.1$ from part (e) *(M1)*

$y = -\frac{5}{882} \times 69^2 + 44.1$

$= 17.1\ (17.1102\ldots)$ *A1*

$21.6 > 17.1$ *R1*

OR

substitution of their $y = 21.6$ into their $y = -\frac{5}{882}x^2 + 44.1$ from part (e) *(M1)*

$21.6 = -\frac{5}{882}x^2 + 44.1$

$x = 63$ *A1*

$69 > 63$ *R1*

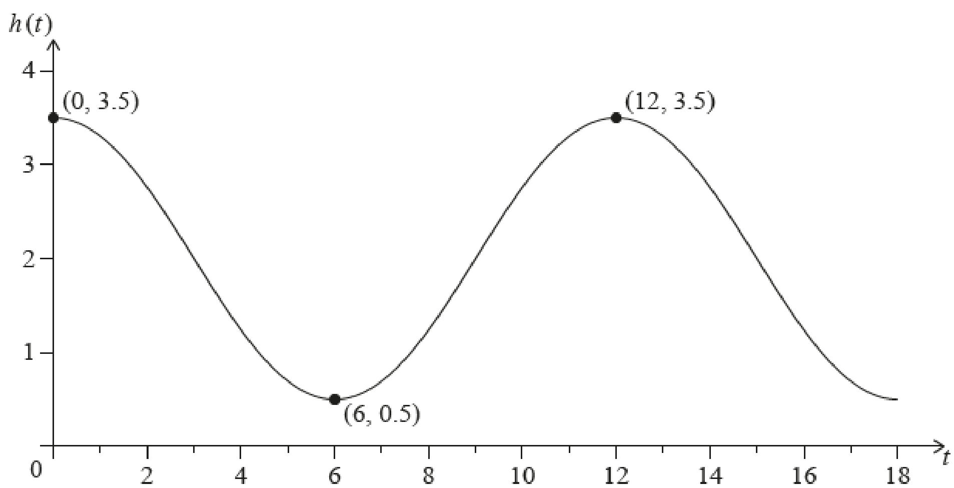
Note: Accept equivalent justification in words, provided both values are seen.
Award ***R1*** for correct reasoning only if ***M1*** has been awarded and their seat coordinates lie outside their equation for the curve.

so the T-shirt cannot reach seat A_{40} ***AG***

[5 marks]

17. [Maximum mark: 8] 23N.1.AHL.TZ0.9

Joon is a keen surfer and wants to model waves passing a particular point P, which is off the shore of his favourite beach. Joon sets up a model of the waves in terms of $h(t)$, the height of the water in metres, and t, the time in seconds from when he begins recording the height of the water at point P.



The function has the form $h(t) = p \cos\left(\frac{\pi}{6}t\right) + q, t \geq 0$.

(a) Find the values of p and q. [2]

Markscheme

$p = 1.5; q = 2$ ***A1A1***

[2 marks]

(b) Find

Markscheme

attempt at using chain rule ***(M1)***

(b.i) $h\prime(t)$. [2]

Markscheme

$h\prime(t) = -\frac{\pi}{4} \sin\left(\frac{\pi}{6}t\right) \ \left(= -0.785 \sin\left(\frac{\pi}{6}t\right)\right)$ ***A1***

[2 marks]

(b.ii) $h\prime\prime(t)$. [1]

Markscheme

$h\prime\prime(t) = -\frac{\pi^2}{24} \cos\left(\frac{\pi}{6}t\right) \ \left(= -0.411233\ldots \cos\left(\frac{\pi}{6}t\right)\right)$ ***A1***

[1 mark]

Joon will begin to surf the wave when the rate of change of h with respect to t, at P, is at its maximum. This will first occur when $t = k$.

(c.i) Find the value of k. [2]

Markscheme

attempt to locate points of inflexion or max value of $h\prime(t)$ ***(M1)***

$h\prime\prime(t) = -\frac{\pi^2}{24} \cos\left(\frac{\pi}{6}t\right) = 0$ **OR** sketch on graph **OR** $t = 3$ **OR** $\frac{\pi}{6}k = \frac{3\pi}{2}$

$(k =)\ 9$ ***A1***

[2 marks]

(c.ii) Find the height of the water at this time. [1]

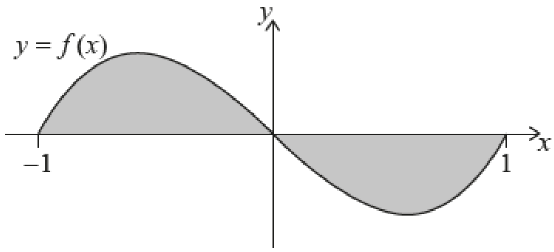
Markscheme

$(h(9) =)\ 2\,(\text{m})$ ***A1***

[1 mark]

18. [Maximum mark: 7] 23N.1.AHL.TZ0.11

Consider the function $f(x) = x^3 - x$, for $-1 \le x \le 1$. The shaded region, R, is bounded by the graph of $y = f(x)$ and the x-axis.



(a.i) Write down an integral that represents the area of R. [1]

Markscheme

EITHER

(area of $R =$) $\int_{-1}^{1} |x^3 - x| \, \mathrm{d}\, x$ ***A1***

OR

(area of $R =$) $2 \times \int_{-1}^{0} x^3 - x \, \mathrm{d}\, x$ **OR** (area of $R =$) $-2 \times \int_{0}^{1} x^3 - x \, \mathrm{d}\, x$ ***A1***

OR

(area of $R =$) $\int_{-1}^{0} x^3 - x \, \mathrm{d}\, x - \int_{0}^{1} x^3 - x \, \mathrm{d}\, x$ ***A1***

[1 mark]

(a.ii) Find the area of R. [1]

Markscheme

(area of $R =$) 0.5 ***A1***

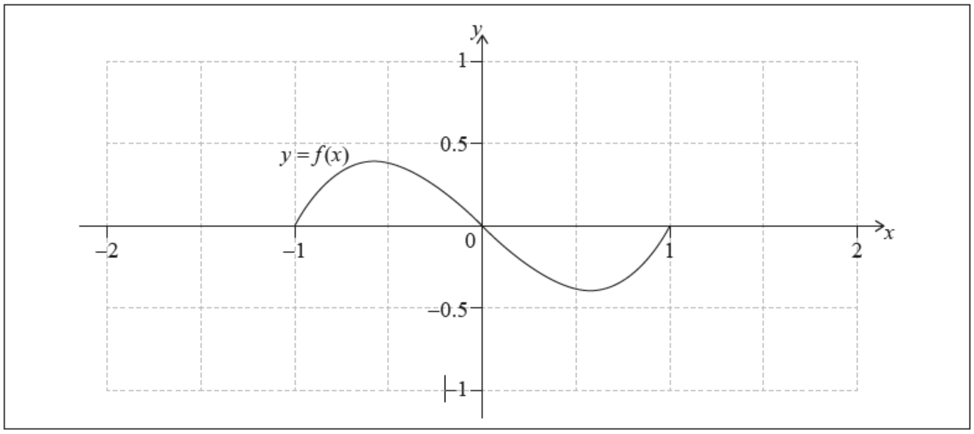
Note: Follow through from part (a)(i) only if answer is greater than zero.

[1 mark]

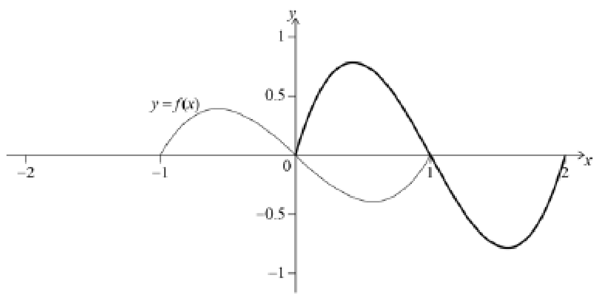
Another function, g, is defined such that $g(x) = 2f(x - 1)$.

(b) On the following set of axes, the graph of $y = f(x)$ has been drawn. On the same set of axes, sketch the graph of $y = g(x)$.

[2]



Markscheme



A1A1

Note: Award ***A1*** for sketch with correct shape on $[0,\ 2]$, ***A1*** for vertical stretch x2. Condone max/min of g extending to $1\,/-1$.

[2 marks]

The region R from the original graph $y=f(x)$ is rotated through 2π radians about the x-axis to form a solid.

(c) Find the volume of the solid. [3]

Markscheme

attempt to use $\pi\int y^2\ \mathrm{d}\ x$ ***(M1)***

volume $=\pi\int_{-1}^{1}\left(\mathrm{x}^3-x\right)^2\ \mathrm{d}\ x$ ***(A1)***

volume $=0.479\ (\text{cubic}\ \ \text{units})\ \left(=0.478718\ldots,\ \frac{16\pi}{105}\right)$ ***A1***

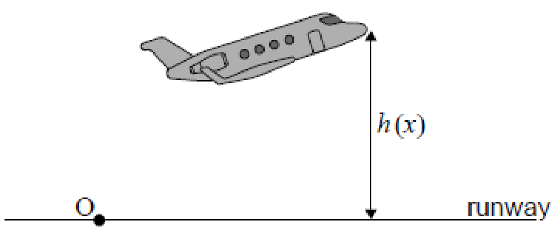
[3 marks]

19. [Maximum mark: 12] 23N.2.AHL.TZ0.4

A plane takes off from a horizontal runway. Let point O be the point where the plane begins to leave the runway and x be the horizontal distance, in km, of the plane from O. The function h models the vertical height, in km, of the nose of the plane from the horizontal runway, and is defined by

$h(x)=\frac{10}{1+150\mathrm{e}^{-0.07x}}-0.06, x\geq 0.$

diagram not to scale



(a.i) Find $h(0)$ [1]

Markscheme

$h(0)=0.00623\,(\mathrm{km})\quad (=0.00622517)$ ***A1***

[1 mark]

(a.ii) Interpret this value in terms of the context. [1]

Markscheme

this is the height of the nose of the plane (above the runway), when the plane is on the runway ***A1***

[1 mark]

(b.i) Find the horizontal asymptote of the graph of $y=h(x)$. [1]

Markscheme

$y=9.94$ ***A1***

Note: Accept $h=9.94$.

[1 mark]

(b.ii) Interpret this value in terms of the context. [1]

Markscheme

EITHER

this is the height that the (nose of the) plane approaches (but does not reach) ***A1***

OR

this is the maximum possible height of the (nose of the) plane ***A1***

OR

the (nose of the) plane does not exceed this height ***A1***

[1 mark]

(c) Find $h\prime(x)$ in terms of x. [4]

Markscheme

METHOD 1 (chain rule)

$h(x)=10\left(1+150\mathrm{e}^{-0.07x}\right)^{-1}-0.06$ ***(M1)***

find $h\prime(x)=-10\left(1+150\mathrm{e}^{-0.07\mathrm{x}}\right)^{-2}\times 150\mathrm{e}^{-0.07x}\times -0.07$ ***A1M1A1***

$\left(=\frac{150\mathrm{e}^{-0.07x}}{\left(1+150\mathrm{e}^{-0.07\mathrm{x}}\right)^{2}}\right)$

Note: Award ***A1*** for correct first term $\left(-10\left(1+150\mathrm{e}^{-0.07x}\right)^{-2}\right)$, ***M1*** for attempt to use the chain rule, ***A1*** for correct use of chain rule $\left(\times 150\mathrm{e}^{-0.07x}\times -0.07\right)$. Award at most ***A1M1A0*** if additional terms are seen. The answer is not required to be simplified beyond what is shown in the markscheme.

METHOD 2 (quotient rule)

$\frac{\left(1+150\mathrm{e}^{-0.007x}\right)(0)-10\left(150\mathrm{e}^{-0.007x}\times -0.007\right)}{\left(1+150\mathrm{e}^{-0.007x}\right)^{2}}$ ***M1A1***

Note: Award ***M1*** for attempt to use quotient rule, ***A1*** for correct use.

$=\frac{-10\left(150\mathrm{e}^{-0.007x}\times -0.007\right)}{\left(1+150\mathrm{e}^{-0.007x}\right)^{2}}$ $\left(=\frac{150\mathrm{e}^{-0.07\mathrm{x}}}{\left(1+150\mathrm{e}^{-0.07\mathrm{x}}\right)^{2}}\right)$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

[4 marks]

A safety regulation recommends that $h\prime(x)$ never exceed 0.2.

(d) Given that this plane flies a distance of at least $200\ \mathrm{km}$ horizontally from point O, determine whether the plane is following this safety regulation. [4]

Markscheme

evidence of a graph of $h\prime(x)$ ***(M1)***

maximum at $x = 71.6 \ \ (= 71.58051\ldots)$ ***(A1)***

$h\prime(71.58051\ldots) = 0.175$ ***A1***

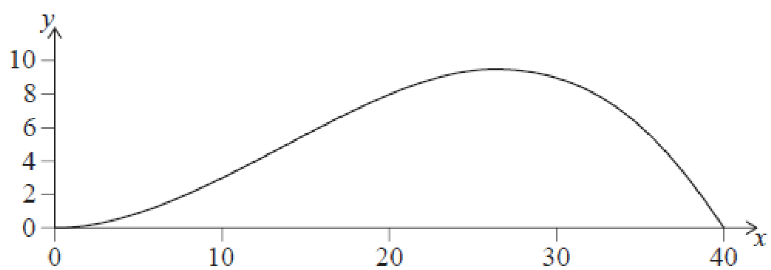
maximum gradient is less than 0.2

and hence the regulation is being followed ***A1***

[4 marks]

20. [Maximum mark: 8] 23M.1.AHL.TZ1.4

The cross section of a scale model of a hill is modelled by the following graph.



The heights of the model are measured at horizontal intervals and are given in the table.

Horizontal distance, x cm	0	10	20	30	40
Vertical distance, y cm	0	3	8	9	0

(a) Use the trapezoidal rule with $h = 10$ to find an approximation for the cross-sectional area of the model. [2]

Markscheme

attempt to substitute $h = 10$ and at least two different values of γ into the trapezoidal rule ***(M1)***

$\frac{10}{2}\left((0+0)+2(3+8+9)\right)$

$= 200\ (\text{cm}^2)$ ***A1***

[2 marks]

It is given that the equation of the curve is $y = 0.04x^2 - 0.001x^3,\ 0 \le x \le 40.$

(b.i) Write down an integral to find the exact cross-sectional area. [2]

Markscheme

$\int_0^{40} 0.04x^2 - 0.001x^3 \mathrm{d}x$ **OR** $\int_0^{40} y \mathrm{d}x$ ***A1A1***

Note: Award **A1** for a correct integral (including $\mathrm{d}x$), ***A1*** for correct limits in the correct location.

[2 marks]

(b.ii) Calculate the value of the cross-sectional area to two decimal places. [2]

Markscheme

$213.33\ (\mathrm{cm}^2)$ ***A2***

Note: Answer must be given to 2 decimal places to award ***A2***. Award ***A1A0*** for a correct answer given to an incorrect accuracy of at least 3 significant figures, e.g. $213\ (\mathrm{cm}^2)$.

[2 marks]

(c) Find the percentage error in the area found using the trapezoidal rule. [2]

Markscheme

attempt to substitute their parts (a) and (b)(ii) into percentage error formula ***(M1)***

$\left|\frac{213.333...-200}{213.333...}\right| \times 100$

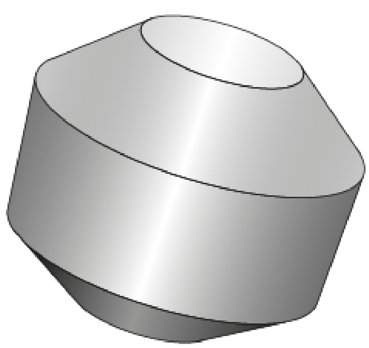
$= 6.25(\%)\ \ (6.23999\ldots(\%))$ ***A1***

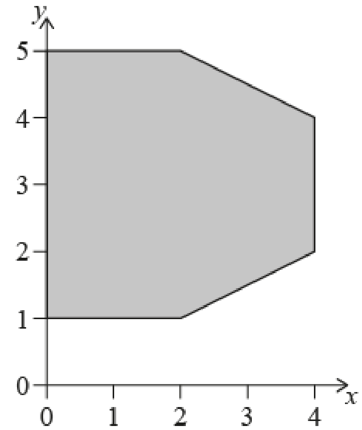
Note: Award ***(M1)A0*** for a final answer of $-6.25(\%)$ or 0.0625.

[2 marks]

21. [Maximum mark: 6] 23M.1.AHL.TZ1.15

The solid shown is formed by rotating the hexagon with vertices $(2,\ 1)$, $(0,\ 1)$, $(0,\ 5)$, $(2,\ 5)$, $(4,\ 4)$ and $(4,\ 2)$ about the y-axis.





Find the volume of this solid. [6]

Markscheme

METHOD 1 Using the volume formula

volume of a "full" or "half" cylinder (seen anywhere) ***(A1)***

$\pi \int_2^4 4^2 \mathrm{d}y, \quad \pi \times 4^2 \times 2, \quad 32\pi \ \left(100.53\ldots\right)$ **OR**

$\pi \int_2^3 4^2 \mathrm{d}y, \quad \pi \times 4^2 \times 1, \quad 16\pi \ \left(50.265\ldots\right)$

one correct equation for the diagonal lines (seen anywhere) ***(A1)***

$y = \frac{1}{2}x$ or $y = 6 - \frac{1}{2}x$

attempt to write one equation x in terms of y ***(M1)***

$x = 2y, \ x = \ 12 - 2y$

EITHER (symmetry plus the volume of the "half" cylinder)

recognition of symmetry between $y = 1$ and $y = 3$ ***(M1)***

$2\pi\left(\int_1^2 (2y)^2 \mathrm{d}y + \int_2^3 4^2 \mathrm{d}y\right)$ ***(A1)***

OR (symmetry plus volume of the "full" cylinder)

recognition of symmetry between $y = 1$ and $y = 2$ ***(M1)***

$2\pi\left(\int_1^2 (2y)^2 \mathrm{d}y\right) + \int_2^4 4^2 \mathrm{d}y$ ***(A1)***

OR (calculation of separate parts) ***(M1)***

$\pi\left(\int_1^2 (2y)^2 \mathrm{d}y + \int_2^4 4^2 \mathrm{d}y + \int_4^5 (-2y+12)^2 \mathrm{d}y\right)$ ***(A1)***

THEN

(volume of the solid $=$) 159 $\left(159.174\ldots,\ \frac{152\pi}{3}\right)$ ***A1***

METHOD 2 Geometric approach using cones and cylinders

volume of a cylinder (seen anywhere) ***(A1)***

$\pi \times 4^2 \times 2,\ 32\pi\ (100.53\ldots)$ (a full cylinder) **OR**

$\pi \times 4^2 \times 1,\ 16\pi\ (50.265\ldots)$ (a half cylinder)

using volume of cone formula to find the volume of the truncated cone ***(M1)***

correct expression to find the volume of the truncated cone (seen anywhere) ***(A1)***

$\frac{1}{3}\left(\pi \times 4^2 \times 2 - \pi \times 2^2 \times 1\right)$

attempt to find an expression for total volume using symmetry or individual parts ***(M1)***

correct expression for total volume ***(A1)***

$2\left(\frac{1}{3}\left(\pi 4^2 \times 2 - \pi 2^2 \times 1\right) + \pi 4^2 \times 1\right)$ **OR**
$\frac{1}{3}\left(\pi 4^2 \times 2 - \pi 2^2 \times 1\right) + \pi 4^2 \times 2 + \frac{1}{3}\left(\pi 4^2 \times 2 - \pi 2^2 \times 1\right)$

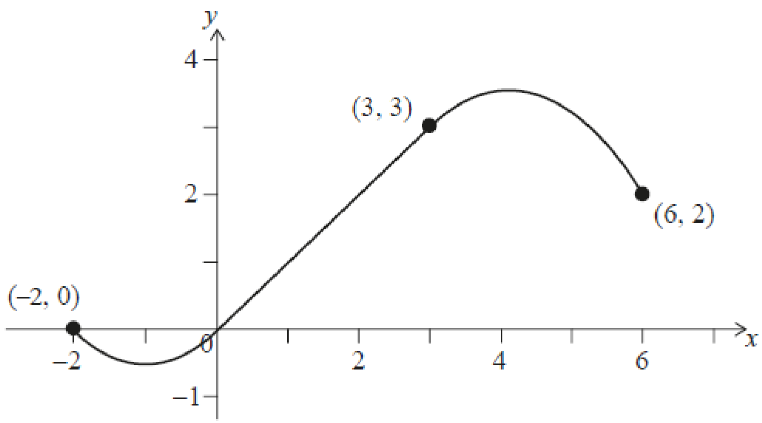
(volume of the solid $=$) 159 $\left(159.174\ldots,\ \frac{152\pi}{3}\right)$ ***A1***

Note: There are other valid approaches possible.

[6 marks]

22. [Maximum mark: 9] 23M.1.AHL.TZ1.10

A decorative hook can be modelled by the curve with equation $y = f(x)$. The graph of $y = f(x)$ is shown and consists of a line segment from $(0,\ 0)$ to $(3,\ 3)$ and two sections formed by quadratic curves.



(a) Write down the equation of the line segment for $0 \le x \le 3$. [1]

Markscheme

$y = x$ ***A1***

[1 mark]

The quadratic curve, with endpoints $(-2,\ 0)$ and $(0,\ 0)$, has the same gradient at $(0,\ 0)$ as the line segment.

(b) Find the equation of the curve between $(-2,\ 0)$ and $(0,\ 0)$. [3]

Markscheme

METHOD 1

equation has the form $y = ax^2 + bx + c$

when $x = 0,\ y = 0$ so $c = 0$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b$

attempt to find the value of b by setting *their* derivative equal to 1 when x is 0 ***(M1)***

$2a(0) + b = 1$

$b = 1$ ***(A1)***

when $x = -2,\ y = 0$

$a = \frac{1}{2}$ (and hence $y = \frac{1}{2}x^2 + x$) ***A1***

METHOD 2

equation has the form $y = ax(x+2)$ OR $y = ax^2 + 2ax$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax + 2a$

attempt to find the value of a by setting *their* derivative equal to 1 when x is 0 ***(M1)***

$a = \frac{1}{2}$ (and hence $y = \frac{1}{2}x^2 + x$) ***A1***

Note: Writing $y = x(x+2)$ is incorrect and gains no marks.

[3 marks]

The second quadratic curve, with endpoints $(3,\ 3)$ and $(6,\ 2)$, has the same gradient at $(3,\ 3)$ as the line segment.

(c) Find the equation of this curve. [4]

Markscheme

equation is $y = ax^2 + bx + c$

finding an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ with unknown coefficients ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b$

setting up two equations using two points AND/OR one equation using the

gradient function ***(M1)***

three correct equations ***(A1)***

$9a + 3b + c = 3$

$36a + 6b + c = 2$

$6a + b = 1$

$a = -\frac{4}{9},\ b = \frac{11}{3},\ c = -4\ \left(a = -0.444444\ldots,\ b = 3.66666\ldots,\ c = -4\right)$ ***A1***

(and hence $y = -\frac{4}{9}x^2 + \frac{11}{3}x - 4$)

[4 marks]

(d) Write down f as a piecewise function. [1]

Markscheme

$$f\left(x\right)=\begin{cases}\frac{1}{2}x^2+x & ,-2\leq x\leq 0\\ x & ,\ 0\leq x\leq 3\\ -\frac{4}{9}x^2+\frac{11}{3}x-4, & 3<x\leq 6\end{cases}$$ ***A1***

Note: Condone open or closed endpoints for all intervals.

Condone y in place of $f(x)$.

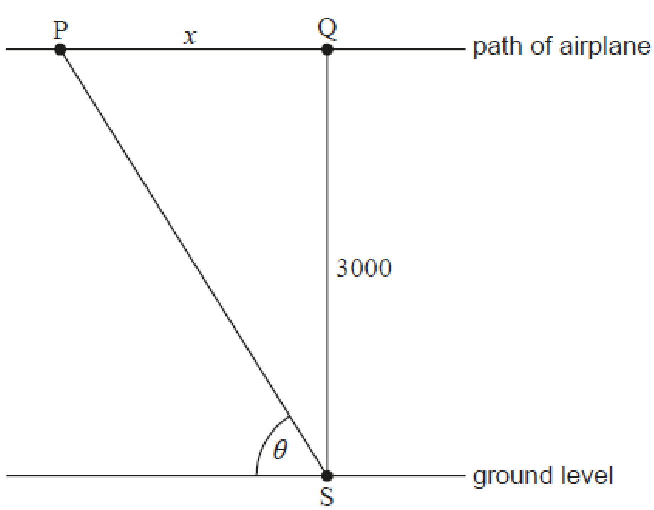
Allow ***FT*** from parts (a), (b) and (c).

[1 mark]

23. [Maximum mark: 9] 23M.1.AHL.TZ1.17

An airplane, P, is flying at a constant altitude of $3000\,\mathrm{m}$ at a speed of $250\,\mathrm{m\,s^{-1}}$. Its path passes over a tracking station, S, at ground level. Let Q be the point $3000\,\mathrm{m}$ directly above the tracking station.

At a particular time, T, as the airplane is flying towards Q, the angle of elevation, θ, of the airplane from S is increasing at a rate of 0.075 radians per second. The distance from Q to P is given by x.



(a) Use related rates to show that, at time T, $\frac{\mathrm{d}x}{\mathrm{d}\theta}=\frac{10000}{3}$. [2]

Markscheme

attempt to use the chain rule to set up a related rate *(M1)*

correct expression ***A1***

$\frac{dx}{d\theta} = \frac{dx}{dt} \div \frac{d\theta}{dt}$ **OR** $\frac{-250}{0.075}$

$= -\frac{10000}{3}$ ***AG***

[2 marks]

(b) Find $x(\theta)$, x as a function of θ. [1]

Markscheme

$x(\theta) = \frac{3000}{\tan\theta}$ ***A1***

[1 mark]

(c) Find an expression for $\frac{dx}{d\theta}$ in terms of $\sin\theta$. [3]

Markscheme

attempt to use chain rule **OR** quotient rule *(M1)*

$\frac{-3000}{\tan^2\theta \times \cos^2\theta}$, $\frac{-3000(\sin\theta(-\sin\theta)-\cos^2\theta)}{\sin^2\theta}$ *(A1)*

$= -\frac{3000}{\sin^2\theta}$ ***A1***

[3 marks]

(d) Hence find the horizontal distance from the station to the plane at time T. [3]

Markscheme

setting their equation in part (c) equal to the given expression in part (a) *(M1)*

$-\frac{3000}{2\sin^2\theta} = -\frac{10000}{3}$

$\theta = 1.24904\ldots$ *(A1)*

$x(1.24904\ldots) = 1000\,\text{m}$ ***A1***

[3 marks]

24. [Maximum mark: 5] 23M.1.AHL.TZ2.12

A spherical balloon is being inflated such that its volume is increasing at a rate of $15\ \text{cm}^3\ \text{s}^{-1}$.

(a) Find the radius of the balloon when its volume is $288\pi\ \text{cm}^3$. [2]

Markscheme

equating volume of sphere formula to 288π ***(M1)***

$\frac{4}{3}\pi r^3 = 288\pi$

$\Rightarrow r = 6\ (\text{cm})$ ***A1***

[2 marks]

(b) Hence or otherwise, find the rate of change of the radius at this instant. [3]

Markscheme

$\frac{\text{d}V}{\text{d}r} = 4\pi r^2$ (seen anywhere) ***(A1)***

$\frac{\text{d}V}{\text{d}t} = \frac{\text{d}V}{\text{d}r}\frac{\text{d}r}{\text{d}t}$ ***(M1)***

$\frac{\text{d}V}{\text{d}t} = 4\pi r^2\frac{\text{d}r}{\text{d}t}$

$15 = 4\pi \times 6^2 \times \frac{\text{d}r}{\text{d}t}$

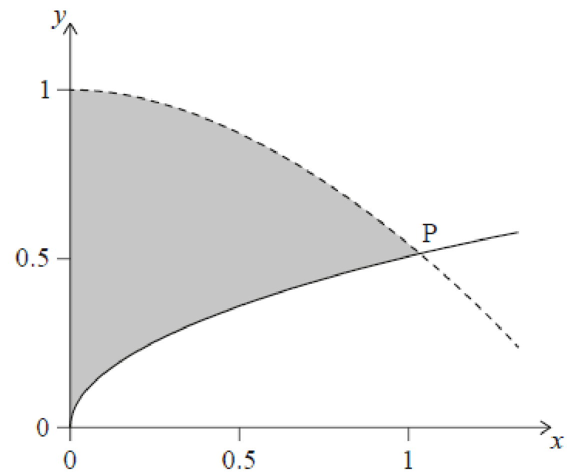
$\frac{\text{d}r}{\text{d}t} = \frac{15}{144\pi}\ (\text{cm s}^{-1})\ (0.0332,\ 0.0331572\ldots)$ ***A1***

[3 marks]

25. [Maximum mark: 9] 23M.1.AHL.TZ2.16

The following diagram shows parts of the curves of $y = \cos x$ and $y = \frac{\sqrt{x}}{2}$.

P is the point of intersection of the two curves.



(a) Use your graphic display calculator to find the coordinates of P. [2]

Markscheme

$(1.04,\ 0.509)\ \ ((1.03667\ldots,\ 0.509085\ldots))$ ***A1A1***

[2 marks]

The shaded region is rotated 360° **about the y-axis** to form a volume of revolution V.

(b) Express V as the sum of two definite integrals. [5]

Markscheme

attempt to make x the subject for either function ***(M1)***

$x = 4y^2,\ x = \cos^{-1} y$ ***A1A1***

attempt to use $V = \pi \int x^2 \mathrm{d}y$ ***(M1)***

$V = \pi \int_0^{0.509085\ldots} \left(4y^2\right)^2 \mathrm{d}y + \pi \int_{0.509085\ldots}^{1} \left(\cos^{-1} y\right)^2 \mathrm{d}y$ ***A1***

[5 marks]

(c) Hence find the value of V. [2]

Markscheme

$= 1.15$ (units^3) ***A2***

Note: Do not ***FT*** from part (b) to part (c).

Award ***A1*** for 1.1. with no previous working.

[2 marks]

26. [Maximum mark: 16] 23M.2.AHL.TZ2.3

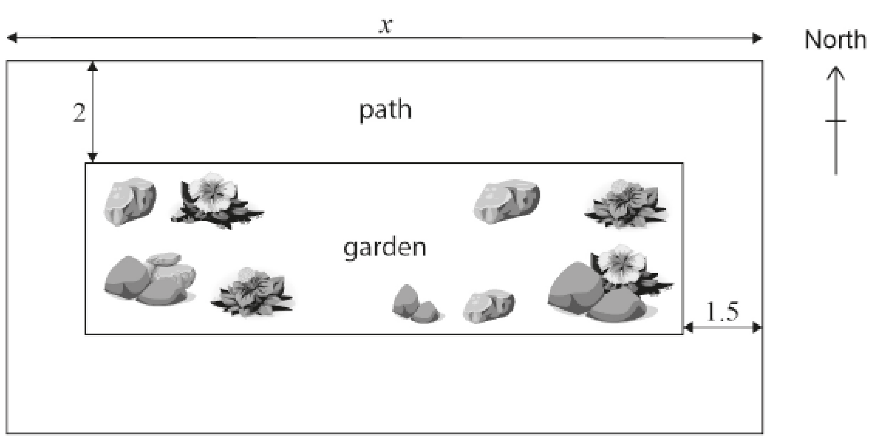
A particular park consists of a rectangular garden, of area $A\,\text{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\text{m}^2$.

The width of the path at the north and south side of the park is $2\,\text{m}$.

The width of the path at the west and east side of the park is $1.5\,\text{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale

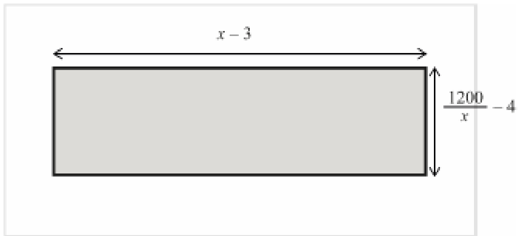


(a) Show that $A = 1212 - 4x - \frac{3600}{x}$. [5]

Markscheme

Note: In methods 1 and 2, full marks are available for candidates who work with a dummy variable, e.g. y, that represents the width of the park and hence is equal to $\frac{1200}{x}$. The substitution to express an answer in only x may come as late as the final line.

METHOD 1 (finding dimensions of garden)



(width of park =) $\frac{1200}{x}$ ***(A1)***

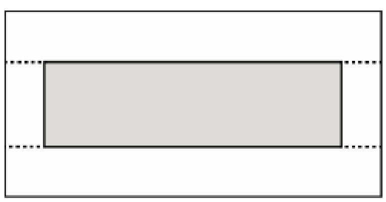
(length of garden =) $x - 3$, (width of garden =) $\frac{1200}{x} - 4$ ***(A1)(A1)***

$A = (x-3) \times \left(\frac{1200}{x} - 4\right)$ ***A1***

$= 1200 - 4x - \frac{3600}{x} + 12$ ***A1***

$= 1212 - 4x - \frac{3600}{x}$ ***AG***

METHOD 2 (subtracting the area of the path)



width of park $= \frac{1200}{x}$ ***(A1)***

attempt to cut path into 4 (or 8) pieces ***(M1)***

four (or eight) areas of the path expressed in terms of x ***(A1)***

$A = 1200 - 2x - 2x - 1.5\left(\frac{1200}{x} - 4\right) - 1.5\left(\frac{1200}{x} - 4\right)$ ***A1***

correct manipulation leading to given result ***A1***

$= 1212 - 4x - \frac{1800}{x} - \frac{1800}{x}$

$= 1212 - 4x - \frac{3600}{x}$ ***AG***

Note: To award ***(M1)(A1)*** without a diagram the division of the park must be clear.

[5 marks]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\mathrm{m}^2$. [4]

Markscheme

setting $1212 - 4x - \frac{3600}{x} = 800$ (accept a sketch) ***(M1)***

$x = 9.64\ (9.64011\ldots)\ (\mathrm{m})$ **OR** $x = 9.34\ (93.3598\ldots)\ (\mathrm{m})$ ***A1***

(width =) $124\ (124.479\ldots)\ (\mathrm{m})$ ***A1***

(width =) $12.9\ (12.8534\ldots)\ (\mathrm{m})$ ***A1***

Note: To award the final ***A1*** both values of x **and** both values of the width must be seen. Accept 12.8 for second value of width from candidate dividing 1200 by 3 sf value of 93.4.

[4 marks]

(c) Find an expression for $\frac{\mathrm{d}A}{\mathrm{d}x}$. [3]

Markscheme

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} =\right) - 4 + \frac{3600}{x^2}$ **OR** $-4 + 3600x^{-2}$ ***A1A1A1***

Note: Award ***A1*** for -4, **A1** for $+3600$, and ***A1*** for x^{-2} or x^2 in denominator.

[3 marks]

(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden. [2]

Markscheme

setting *their* $\frac{\mathrm{d}A}{\mathrm{d}x}$ equal to 0 **OR** sketch of *their* $\frac{\mathrm{d}A}{\mathrm{d}x}$ with x-intercept highlighted ***M1***

$(x =)\ 30\ (\mathrm{m})$ ***A1***

Note: To award ***A1FT*** the candidate's value of x must be within the domain given in the problem $(3 < x < 300)$.

[2 marks]

(e) Find the maximum possible area of the garden. [2]

Markscheme

EITHER

evidence of using GDC to find maximum of graph of $A = 1212 - 4x - \frac{3600}{x}$ ***(M1)***

OR

substitution of *their* x into A ***(M1)***

OR

dividing 1200 by *their* x to find width of park **and** subtracting 3 from *their* x and 4 from the width to find park dimensions ***(M1)***

Note: For the last two methods, only follow through if $3 <$ their $x < 300$.

THEN

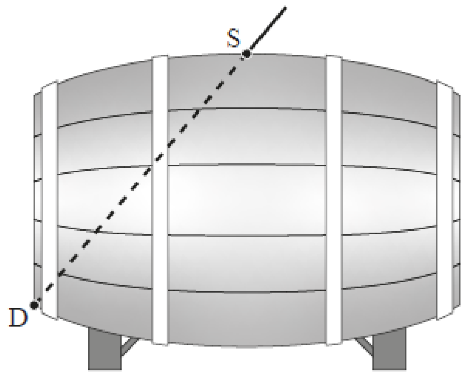
$(A =)\ 972\ (\mathrm{m}^2)$ **A1**

[2 marks]

27. [Maximum mark: 26] 23M.3.AHL.TZ1.1

In this question you will use a historic method of calculating the cost of a barrel of wine to determine which shape of barrel gives the best value for money.

In Austria in the 17th century, one method for measuring the volume of a barrel of wine, and hence determining its cost, was by inserting a straight stick into a hole in the side, as shown in the following diagram, and measuring the length SD. The longer the length, the greater the cost to the customer.



Let SD be d metres and the cost be C gulden (the local currency at the time). When the length of SD was 0.5 metres, the cost was 0.80 gulden.

(a) Given that C was directly proportional to d, find an equation for C in terms of d. [3]

Markscheme

$C = kd$ ***(M1)***

$0.80 = 0.5k$ **OR** $\frac{C}{d} = \frac{0.8}{0.5}$ ***(A1)***

$k = 1.6$

$C = 1.6d$ **OR** $C = \frac{d}{0.625}$ ***A1***

Note: For the final ***A1*** do not accept $C = \frac{0.8}{0.5}d$ or a correct equation which does not have C as the subject.

[3 marks]

A particular barrel of wine cost 0.96 gulden.

(b) Show that $d = 0.6$. [1]

Markscheme

$d = \frac{0.96}{1.6}$ ***M1***

Note: Award ***M1*** for the substitution of $C = 0.96$ into a correct equation, award ***M0*** for substitution of $d = 0.6$.

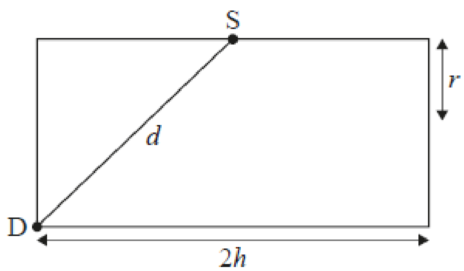
$= 0.6$ ***AG***

[1 mark]

This method of determining the cost was noticed by a mathematician, Kepler, who decided to try to calculate the dimensions of a barrel which would give the maximum volume of wine for a given length SD.

Initially he modelled the barrel as a cylinder, with S at the midpoint of one side. He took the length of the cylinder as $2h$ metres and its radius as r metres, as shown in the following diagram of the cross-section.

diagram not to scale



(c) Find an expression for r^2 in terms of d and h. [3]

Markscheme

attempt at using Pythagoras ***M1***

$d^2 = h^2 + (2r)^2$ ***(A1)***

$r^2 = \frac{1}{4}\left(d^2 - h^2\right)$ ***A1***

[3 marks]

Let the volume of this barrel be $V\,\mathrm{m}^3$.

(d) Show that $V = \frac{\pi}{2}\left(d^2h - h^3\right)$. [2]

Markscheme

$V = \pi r^2 \left(2h\right)$ ***(A1)***

$V = \frac{\pi}{4}\left(d^2 - h^2\right)\left(2h\right)$ ***(M1)***

Note: The ***M1*** is for the substitution of their expression for r^2 into their formula for V.

$V = \frac{\pi}{2}\left(d^2h - h^3\right)$ ***AG***

[2 marks]

The remainder of this question considers the shape of barrel that gives the best value when $d = 0.6$.

(e.i) Use the formula from part (d) to find the volume of this barrel when $h = 0.4$. [2]

Markscheme

$V = \frac{\pi}{2}\left(d^2h - h^3\right)$

$= \frac{\pi}{2}\left(0.6^2 \times 0.4 - 0.4^3\right)$ ***(M1)***

Note: The ***M1*** is for substituting correct values of both d and h in the formula from part (d).

$= 0.126\left(\mathrm{m}^3\right) 0.\left(12566\ldots,\ \frac{\pi}{2},\ 0.04\pi\right)$ ***A1***

[2 marks]

(e.ii) Use differentiation to show that $h = \sqrt{0.12}$ when $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$. [3]

Markscheme

$V = \frac{\pi}{2}\left(0.36h - h^3\right)$ ***M1***

Note: Award ***M1*** for substitution of $d = 0.6$. This may be seen anywhere.

$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi}{2}\left(0.36 - 3h^2\right) = 0$ ***A1***

Note: Award ***A1*** for correct differentiation.

$h^2 = \frac{0.36}{3}$ $(= 0.12)$ ***A1***

Note: Do not award the final ***A1*** if the working is done with approximate values or if $\sqrt{0.12}$ is substituted into $\frac{\mathrm{d}V}{\mathrm{d}h}$.

$h = \sqrt{0.12}$ ***AG***

[3 marks]

(e.iii) Given that this value of h maximizes the volume, find the largest possible volume of this barrel. [2]

Markscheme

substituting $h = \sqrt{0.12}$ into equation for V **OR** use of graph ***(M1)***

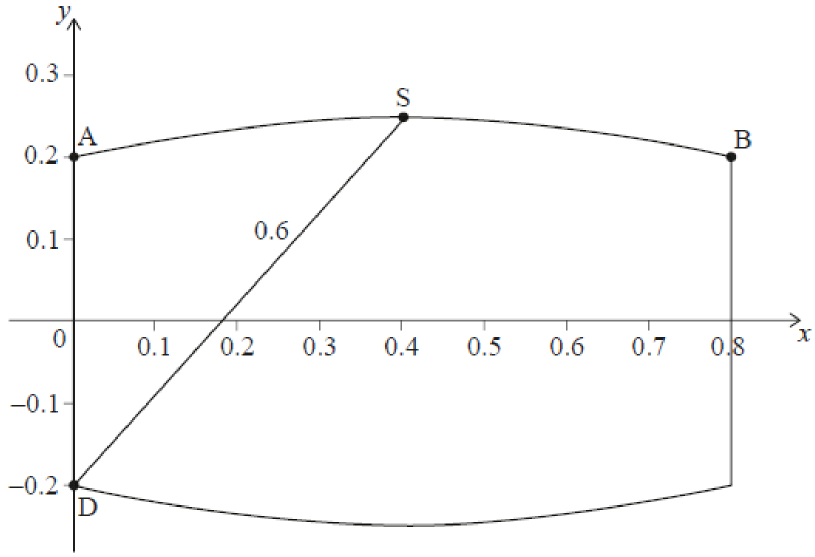
$V = \frac{\pi}{2}\left(0.36 - 0.12\right)\sqrt{0.12}$

$0.131\ (\mathrm{m}^3)\ (0.130593\ldots, 0.0416\pi)$ ***(A1)***

[2 marks]

Kepler then considered a non-cylindrical barrel whose base and lid are circles with radius $0.2\,\mathrm{m}$ and whose length is $0.8\,\mathrm{m}$.

He modelled the curved surface of this barrel by rotating a quadratic curve, ASB, with equation $y = ax^2 + bx + c, 0 \leq x \leq 0.8$, about the x-axis. The origin of the coordinate system is at the centre of one of the circular faces as shown in the following diagram. S is at the vertex of the quadratic curve and $\mathrm{SD} = 0.6$.



Kepler wished to find out if his barrel would give him more wine than any cylindrical barrel with $d = 0.6$.

The coordinates of A and B are $(0,\ 0.2)$ and $(0.8,\ 0.2)$ respectively.

(f) Find the equation of the quadratic curve, ASB. [6]

Markscheme

x-coordinate of S is 0.4 ***(A1)***

let the y-coordinate be y_s

attempt at Pythagoras ***(M1)***

$(y_s + 0.2)^2 = 0.6^2 - 0.4^2$

$y_s + 0.2 = \sqrt{0.2}$

$y_s = 0.247\ (0.247213\ldots)$ ***(A1)***

Note: The ***(M1)*** mark can be implied by a y-coordinate of 0.447 or 0.647 seen.

any valid method to find equation
e.g. quadratic regression, vertex form, simultaneous equation. ***(M1)***

Note: Award only if the student has found three points on the curve.

EITHER

$y = -0.295x^2 + 0.236x + 0.2$ ***A2***

$\left(y = -0.295081\ldots x^2 + 0.236065\ldots x + 0.2\right)$

Note: Award ***A1*** if one coefficient ($-0.295081\ldots$ or $0.236065\ldots$) is correct or if "$y =$" is missing, ***A2*** for completely correct equation.
Award ***A1*** for $y = -0.294x^2 + 0.235x + 0.2$ obtained from using the 3 sf value of y_s.

OR

$y = -0.295(x - 0.4)^2 + 0.247$ ***A2***

$\left(y = -0.295081\ldots(x - 0.4)^2 + 0.247213\ldots\right)$

Note: Award ***A1*** for -0.295, ***A2*** for completely correct equation.

[6 marks]

(g) Show that the volume of this barrel is greater than the maximum volume of any cylindrical barrel with $d = 0.6$. [3]

Markscheme

volume $= \pi \int_0^{0.8} \left(-0.295081\ldots x^2 + 0.236065\ldots x + 0.2\right)^2 \mathrm{d}x$ ***M1***

Note: Award ***M1*** for the minimum of an integral with the correct limits and their function squared.

$= 0.135\ (0.135161\ldots)$ ***A1***

$0.135 > 0.131$ ***R1***

Note: Award ***R1*** independently of the previous marks for a correct comparison of their (clearly stated) volume with their answer to part (e)(iii).

Hence the volume is greater than any cylinder volume ***AG***

[3 marks]

(h) State one assumption, not already given, that has been made in using these models to find the shape of the barrel that gives the best value. [1]

Markscheme

*Award **A1** for at least one reasonable answer, for example:*

the barrel is full of wine when sold
the barrel/stick is constructed with zero thickness
the stick is straight and inflexible
the hole has no diameter
volume of wine is the only important factor in value
aesthetics are not important

Note: Do not accept statements that relate to the barrel having circular cross-section, for example, 'there are no deformities', 'it is perfectly smooth' as these assumptions have already been made with the chosen model.

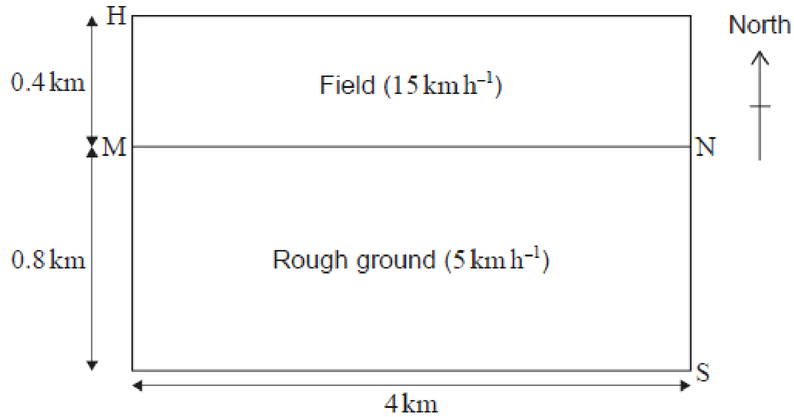
[1 mark]

28. [Maximum mark: 26] 23M.3.AHL.TZ2.1

This question considers the optimal route between two points, separated by several regions where different speeds are possible.

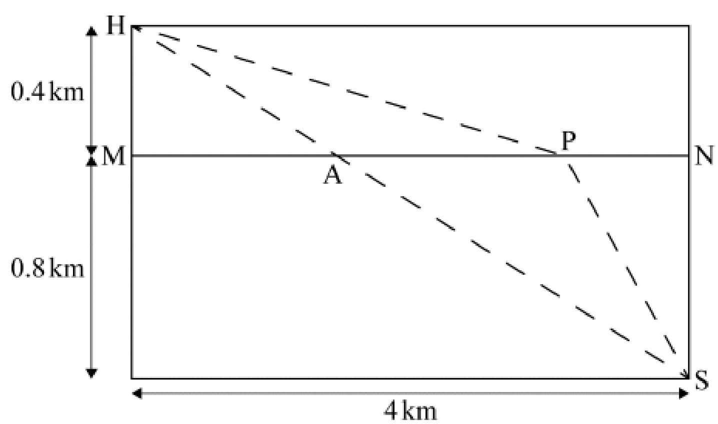
Huw lives in a house, H, and he attends a school, S, where H and S are marked on the following diagram. The school is situated $1.2\ \mathrm{km}$ south and $4\ \mathrm{km}$ east of Huw's house. There is a boundary $[\mathrm{MN}]$, going from west to east, $0.4\ \mathrm{km}$ south of his house. The land north of $[\mathrm{MN}]$ is a field over which Huw runs at 15 kilometres per hour ($\mathrm{km\ h^{-1}}$). The land south of $[\mathrm{MN}]$ is rough ground over which Huw walks at $5\ \mathrm{km\ h^{-1}}$. The two regions are shown in the following diagram.

diagram not to scale



(a) Huw travels in a straight line from H to S. Calculate the time that Huw takes to complete this journey. Give your answer correct to the nearest minute. [6]

Markscheme



METHOD 1

$\mathrm{M\hat{H}S} = \left(\tan^{-1}\frac{4}{1.2}\right) = 73.3007\ldots^\circ$ OR $1.27933\ldots$ ***(A1)***

use of trigonometry to find HA or AS ***(M1)***

$\mathrm{HA} = \frac{0.4}{\cos \mathrm{M\hat{H}S}}$ **AND** $\mathrm{AS} = \frac{0.8}{\cos \mathrm{M\hat{H}S}}$ ***(A1)***

$(\mathrm{HA} = 1.39204\ldots$ and $\mathrm{AS} = 2.78408\ldots)$

use of $\text{time} = \frac{\text{distance}}{\text{speed}}$ for either of their distances ***(M1)***

time taken $= \left(\frac{\mathrm{AH}}{15} + \frac{\mathrm{AS}}{5}\right)$

$0.649618\ldots$ (hours) ***(A1)***

$(38.99712\ldots$ minutes)

therefore 39 (mins) ***A1FT***

Note: Allow ***FT***, within the question part, from their time in hours for the final ***A1***.

METHOD 2

EITHER

use of similar triangles to identify either length MA or AN ***(M1)***

$\left(\frac{4}{3} \text{ or } \frac{8}{3}\right)$

attempt to use Pythagoras for either triangle AMH or ANS ***(M1)***

$\mathrm{AH}^2 = 0.4^2 + \left(\frac{4}{3}\right)^2$ **AND** $\mathrm{AS}^2 = 0.8^2 + \left(\frac{8}{3}\right)^2$ ***(A1)***

OR

attempt to use Pythagoras for larger triangle ***(M1)***

$\mathrm{SH}^2 = 4^2 + 1.2^2$

$\mathrm{AH} = \frac{1}{3}\sqrt{4^2 + 1.2^2}$ **AND** $\mathrm{AS} = \frac{2}{3}\sqrt{4^2 + 1.2^2}$ ***(M1)(A1)***

THEN

$(\mathrm{HA} = 1.39204\ldots$ and $\mathrm{AS} = 2.78408\ldots)$

use of $\text{time} = \frac{\text{distance}}{\text{speed}}$ for either of THEIR distances ***(M1)***

time taken $= \left(\frac{\mathrm{AH}}{15} + \frac{\mathrm{AS}}{5}\right)$

$0.649618\ldots$ (hours) ***(A1)***

$(38.99712\ldots$ minutes)

therefore 39 (mins) ***A1FT***

Note: Allow ***FT***, within the question part, from their time in hours for the final ***A1***.

[6 marks]

(b) Huw realizes that his journey time could be reduced by taking a less direct route. He therefore defines a point P on $[\mathrm{MN}]$ that is $x\ \mathrm{km}$ east of M. Huw decides to run from H to P and then walk from P to S. Let $T(x)$ represent the time, in hours, taken by Huw to complete the journey along this route.

(b.i) Show that $T\left(x\right) = \frac{\sqrt{0.4^2+x^2}+3\sqrt{0.8^2+(4-x)^2}}{15}$. [3]

Markscheme

$\mathrm{PH}^2 = 0.4^2 + x^2$ **AND** $\mathrm{PS}^2 = 0.8^2 + (4-x)^2$ ***A1***

Note: This ***A1*** can be implied by a clear expression for the time in each region coming from distance / speed below.

$T(x) = \frac{\mathrm{PH}}{15} + \frac{\mathrm{PS}}{5}$ ***(M1)***

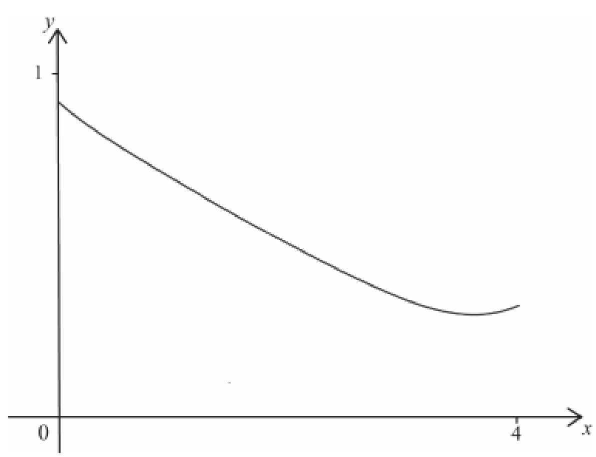
$T\left(x\right) = \frac{\sqrt{0.4^2+x^2}}{15} + \frac{\sqrt{0.8^2+(4-x)^2}}{5}$ ***A1***

$T\left(x\right) = \frac{\sqrt{0.4^2+x^2}+3\sqrt{0.8^2+(4-x)^2}}{15}$ ***AG***

[3 marks]

(b.ii) Sketch the graph of $y = T(x)$. [2]

Markscheme



correct shape with minimum point nearer $x = 4$ than $x = 0$ ***A1***

correct (approximate) y-intercept, $0.843\ldots$ (must be clearly below 1) ***A1***

[2 marks]

(b.iii) Hence determine the value of x that minimizes $T(x)$. [1]

Markscheme

using the GDC, at the minimum $x = 3.72\ (3.71898\ldots)$ ***A1***

Note: Do not accept coordinates of the minimum point.

[1 mark]

(b.iv) Find by how much Huw's journey time is reduced when he takes this optimal route, compared to travelling in a straight line from H to S. Give your answer correct to the nearest minute. [2]

Markscheme

finding their $T(x)$ for their value of x ***M1***

$T(x) = 0.418946\ldots$

so time saved ($= 38.97712\ldots - 25.1367\ldots$ mins) = 14 (mins) ***A1***

[2 marks]

(c.i) Determine an expression for the derivative $T\prime(x)$. [3]

Markscheme

attempt at chain rule ***M1***

$$T\prime\left(x\right) = \frac{1}{15}\left(\frac{x}{\sqrt{0.4^2+x^2}} - \frac{3(4-x)}{\sqrt{0.8^2+(4-x)^2}}\right)$$ ***A1A1***

Note: Award ***A1*** for each correct term. Accept any equivalent form i.e. condone fractions not simplified.

[3 marks]

(c.ii) Hence show that $T(x)$ is minimized when

$$\frac{x}{\sqrt{0.16+x^2}} = \frac{3(4-x)}{\sqrt{0.64+(4-x)^2}}.$$ [1]

Markscheme

setting their $T\prime(x) = 0$ ***M1***

Note: This requires more than just a statement that the derivative equals zero – they must use their attempt at $T\prime(x)$.

$$\frac{1}{15}\left(\frac{x}{\sqrt{0.4^2+x^2}} - \frac{3(4-x)}{\sqrt{0.8^2+(4-x)^2}}\right) = 0$$

$\frac{x}{\sqrt{0.16+x^2}} = \frac{3(4-x)}{\sqrt{0.64+(4-x)^2}}$ ***AG***

[1 mark]

(c.iii) For the optimal route, verify that the equation in part (c)(ii) satisfies the following result:

$$\frac{\cos \mathrm{H\widehat{P}M}}{\cos \mathrm{S\widehat{P}N}} = \frac{\text{speed over field}}{\text{speed over rough ground}}.$$ [2]

Markscheme

METHOD 1

$\cos \mathrm{H\widehat{P}M} = \frac{x}{\sqrt{0.16+x^2}}$ **AND** $\cos \mathrm{S\widehat{P}N} = \frac{4-x}{\sqrt{0.64+(4-x)^2}}$ ***A1***

substituting in the above equation and rearranging ***M1***

$\cos \mathrm{H\widehat{P}M} = 3\cos \mathrm{S\widehat{P}N}$ leading to $\frac{\cos \mathrm{H\widehat{P}M}}{\cos \mathrm{S\widehat{P}N}} = 3 = \left(\frac{15}{5}\right)$

verifying the result ***AG***

METHOD 2

$$\frac{x}{\sqrt{0.16+x^2}} = \frac{3(4-x)}{\sqrt{0.64+(4-x)^2}}$$

attempt to rearrange into a quotient ***M1***

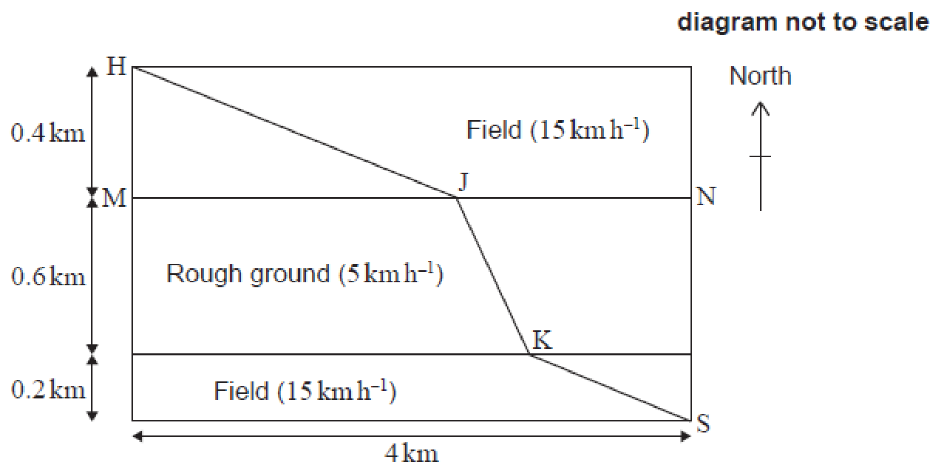
$$\left(\frac{15}{5} = 3 =\right) \frac{\frac{x}{\sqrt{0.16+x^2}}}{\frac{4-x}{\sqrt{0.64+(4-x)^2}}}$$

$$= \frac{\cos \mathrm{H\hat{P}M}}{\cos \mathrm{S\hat{P}N}}$$ ***A1***

verifying the result ***AG***

[2 marks]

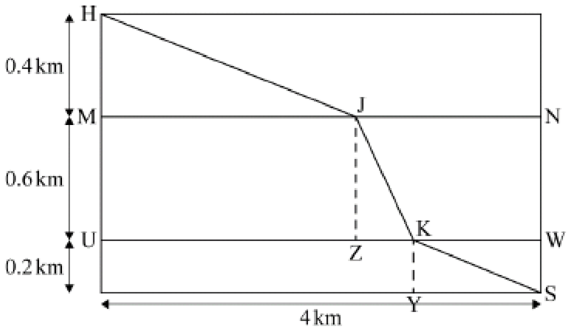
(d) The owner of the rough ground converts the southern quarter into a field over which Huw can run at $15\ \mathrm{km\ h^{-1}}$. The following diagram shows the optimal route, HJKS, in this new situation. You are given that $[\mathrm{HJ}]$ is parallel to $[\mathrm{KS}]$.



Using a similar result to that given in part (c)(iii), at the point J, determine MJ. [6]

Markscheme

METHOD 1



let $\mathrm{MJ} = y\ \mathrm{km}$ and W and Z be the points on the new boundary directly below N and J

attempt to find ZK in terms of MJ ***M1***

$(\mathrm{KW} = 0.5y)$

$\mathrm{ZK} = (4 - 1.5y)\ \mathrm{km}$ ***A1***

attempt to use the result from (c)(iii) at J ***M1***

$\frac{\cos\ \mathrm{H\hat{J}M}}{\cos\ \mathrm{Z\hat{K}J}} = \frac{y}{\sqrt{y^2+0.4^2}} \div \frac{(4-1.5y)}{\sqrt{(4-1.5y)^2+0.6^2}} = \frac{15}{5}$ ***A1***

Note: Accept $\cos\ \mathrm{N\hat{J}K}$ in place of $\cos\ \mathrm{Z\hat{K}J}$.

$\left(\text{leading to } \frac{y}{\sqrt{y^2+0.16}} \div \frac{3(4-1.5y)}{\sqrt{(4-1.5y)^2+0.36}}\right)$

valid method for solving this equation, eg drawing graphs of both sides of the equation, using SOLVER, etc. ***(M1)***

solution is $y = 2.53$ ***A1***

METHOD 2

combining the field into one region with height $0.6\ \mathrm{km}$ ***M1***

$\cos\ \mathrm{H\hat{P}M} = \frac{x}{\sqrt{0.36+x^2}}$

$\cos\ \mathrm{S\hat{P}N} = \frac{4-x}{\sqrt{0.36+(4-x)^2}}$ ***A1***

Note: Both expressions, or their ratio, are required for the ***A1*** to be awarded.

therefore

$\frac{x\sqrt{0.36+(4-x)^2}}{(4-x)\sqrt{0.36+x^2}} = 3$ ***A1***

valid method for solving ***(M1)***

attempting to find MJ in terms of x e.g. $\mathrm{MJ} = \frac{2}{3}x$ ***M1***

so $\mathrm{MJ} = 2.53$ ***A1***

[6 marks]

29. [Maximum mark: 9] 22N.1.AHL.TZ0.14

A particle moves such that its velocity, v metres per second, at time t seconds, is given by $v = t\ \sin(t^2)$.

(a) Find an expression for the acceleration of the particle. [2]

Markscheme

attempt to use product rule ***(M1)***

$a = 2t^2\ \cos(t^2) + \sin(t^2)$ ***A1***

[2 marks]

(b) Hence, or otherwise, find its greatest acceleration for $0 \leq t \leq 8$. [2]

Markscheme

graph of a ***(M1)***

$126\ (\mathrm{ms}^{-2})\ \ (125.699\ldots)$ ***A1***

[2 marks]

The particle starts at the origin.

(c) Find an expression for the displacement of the particle. [3]

Markscheme

attempt at integration by substitution or inspection ***(M1)***

$s = -\frac{1}{2}\cos\left(t^2\right)\ (+c)$ ***A1***

$(s = 0$ when $t = 0) \Rightarrow c = \frac{1}{2}$ ***A1***

$\left(s = -\frac{1}{2}\cos\left(t^2\right) + \frac{1}{2}\right)$

[3 marks]

(d) Hence show that the particle never has a negative displacement. [2]

Markscheme

$\cos\left(t^2\right) \leq 1$ ***A1***

$-\frac{1}{2}\cos\left(t^2\right) \geq -\frac{1}{2}$

so $\frac{1}{2} - \frac{1}{2}\cos\left(t^2\right) \geq 0$ ***R1***

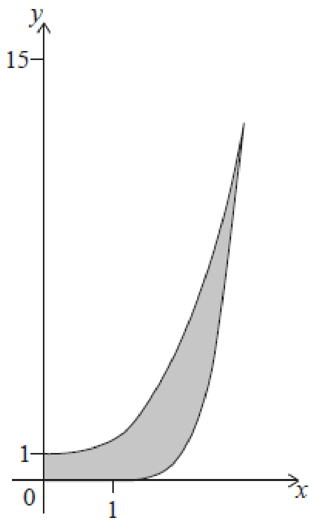
hence the particle never has a negative displacement. ***AG***

Note: Do not accept reasoning based on a sketch of the graph.

[2 marks]

30. [Maximum mark: 13] 22N.2.AHL.TZ0.5

Adesh is designing a glass. The glass has an inner surface and an outer surface. Part of the cross section of his design is shown in the following graph, where the shaded region represents the glass. The two surfaces meet at the top of the glass. 1 unit represents $1\,\text{cm}$.



The inner surface is modelled by $f(x) = \frac{1}{2}x^3 + 1$ for $0 \le x \le p$.

The outer surface is modelled by $g(x)\begin{cases} 0 & \text{for } 0 \le x < 1 \\ (x-1)^4 & \text{for } 1 \le x \le p \end{cases}$.

(a) Find the value of p. [2]

Markscheme

$\frac{1}{2}x^3 + 1 = (x-1)^4$ ***(M1)***

$(p =)\ 2.91\,\text{cm}\ \ (2.91082\ldots)$ ***A1***

[2 marks]

The glass design is finished by rotating the shaded region in the diagram through 360° about the y-axis.

(b) Find the volume of liquid that can be contained inside the finished glass. [5]

Markscheme

attempt to make x (or x^2) the subject of $y = \frac{1}{2}x^3 + 1$ ***(M1)***

$x = \sqrt[3]{2(y-1)}$ (or $x^2 = (2(y-1))^{\frac{2}{3}}$) ***(A1)***

(upper limit =) $13.3(315\ldots)$ ***(A1)***

$V = \int_1^{13.3315\ldots} \pi(2(y-1))^{\frac{2}{3}}\ \mathrm{d}\,y$ ***(M1)***

Note: Award *(M1)* for setting up correct integral squaring their expression for x with both correct lower limit and their upper limit, and π. Condone omission of $\mathrm{d}\,y$.

$= 197\,\text{cm}^3 \quad (196.946\ldots)$ ***A1***

[5 marks]

(c) Find the volume of the region between the two surfaces of the finished glass. [6]

Markscheme

$x = y^{\frac{1}{4}} + 1$ (or $x^2 = \left(y^{\frac{1}{4}} + 1\right)^2$) *(A1)*

$V_2 = \int_0^{13.3315\ldots} \pi\left(y^{\frac{1}{4}} + 1\right)^2 \mathrm{d}\,y$ *(M1)(A1)*

Note: Award *(M1)* for setting up correct integral squaring their expression for x with their upper limit, and π. Award *(A1)* for lower limit of 0, dependent on ***M1***. Condone omission of $\mathrm{d}\,y$. If a candidate found an area in part (b), do not award ***FT*** for another area calculation seen in part (c).

$= 271.87668\ldots$ *(A1)*

Note: Accept $271.038\ldots$ from use of 3sf in the upper limit.

subtracting their volumes *(M1)*

$271.87668\ldots - 196.946\ldots$

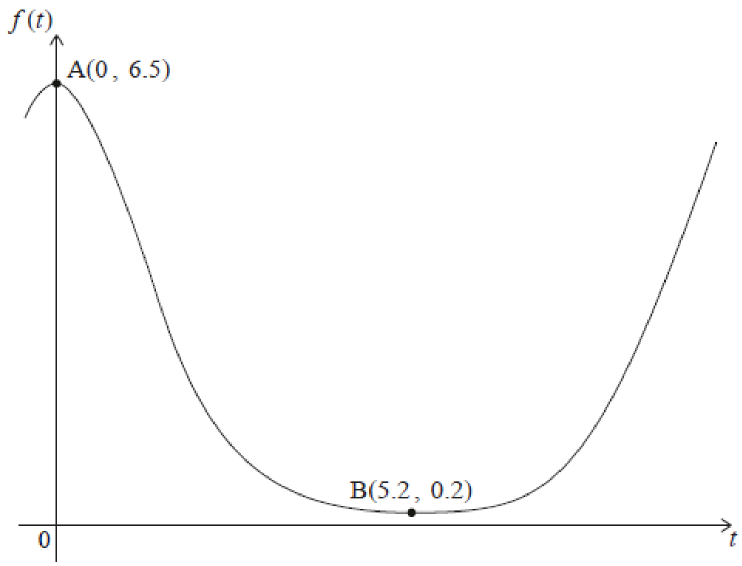
$= 74.9\,\text{cm}^3 \quad (74.93033\ldots)$ ***A1***

Note: Accept any answer that rounds to $75\,(\text{cm}^3)$. If a candidate found an area in part (b), do not award ***FT*** for another area calculation seen in part (c).

[6 marks]

31. [Maximum mark: 8] 22M.1.AHL.TZ1.17

A function f is of the form $f(t) = p\mathrm{e}^{q\cos(rt)},\ p,\ q,\ r \in \mathbb{R}^+$. Part of the graph of f is shown.



The points A and B have coordinates $\mathrm{A}(0,\ 6.5)$ and $\mathrm{B}(5.2,\ 0.2)$, and lie on f.

The point A is a local maximum and the point B is a local minimum.

Find the value of p, of q and of r. [8]

Markscheme

substitute coordinates of A

$f(0) = p\mathrm{e}^{q\cos(0)} = 6.5$

$6.5 = p\mathrm{e}^{q}$ ***(A1)***

substitute coordinates of B

$f(5.2) = p\mathrm{e}^{q\cos(5.2r)} = 0.2$

EITHER

$f'(t) = -pqr\sin(rt)\mathrm{e}^{q\cos(rt)}$ ***(M1)***

minimum occurs when $-pqr\sin(5.2r)\mathrm{e}^{q\cos(5.2r)} = 0$

$\sin(rt) = 0$

$r \times 5.2 = \pi$ ***(A1)***

OR

minimum value occurs when $\cos(rt) = -1$ ***(M1)***

$r \times 5.2 = \pi$ *(A1)*

OR

period $= 2 \times 5.2 = 10.4$ *(A1)*

$r = \frac{2\pi}{10.4}$ *(M1)*

THEN

$r = \frac{\pi}{5.2} = 0.604152\ldots\ (0.604)$ ***A1***

$0.2 = pe^{-q}$ *(A1)*

eliminate p or q *(M1)*

$e^{2q} = \frac{6.5}{0.2}$ **OR** $0.2 = \frac{p^2}{6.5}$

$q = 1.74\ (1.74062\ldots)$ ***A1***

$p = 1.14017\ldots\ (1.14)$ ***A1***

[8 marks]

32. [Maximum mark: 7] 22M.1.AHL.TZ1.16

The wind chill index W is a measure of the temperature, in $^\circ\mathrm{C}$, felt when taking into account the effect of the wind.

When Frieda arrives at the top of a hill, the relationship between the wind chill index and the speed of the wind v in kilometres per hour $(\mathrm{km\,h^{-1}})$ is given by the equation

$W = 19.34 - 7.405v^{0.16}$

(a) Find an expression for $\frac{dW}{dv}$. [2]

Markscheme

use of power rule *(M1)*

$\frac{dW}{dv} = -1.1848v^{-0.84}$ **OR** $-1.18v^{-0.84}$ ***A1***

[2 marks]

(b) When Frieda arrives at the top of a hill, the speed of the wind is 10 kilometres per hour and increasing at a rate of $5\,\mathrm{km\,h^{-1}\,minute^{-1}}$.

Find the rate of change of W at this time. [5]

Markscheme

$\frac{\mathrm{d}v}{\mathrm{d}t} = 5$ ***(A1)***

$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}t} \times \frac{\mathrm{d}W}{\mathrm{d}v}$ ***(M1)***

$\left(\frac{\mathrm{d}W}{\mathrm{d}t} = -5 \times 1.1848v^{-0.84}\right)$

when $v = 10$

$\frac{\mathrm{d}W}{\mathrm{d}t} = -5 \times 1.1848 \times 10^{-0.84}$ ***(M1)***

$-0.856\ \ (-0.856278\ldots)\,^{\circ}\mathrm{C\,min^{-1}}$ ***A2***

Note: Accept a negative answer communicated in words, "decreasing at a rate of...".
Accept a final answer of $-0.852809\ldots\ ^{\circ}\mathrm{C\,min^{-1}}$ from use of -1.18.
Accept 51.4 (or 51.2) $^{\circ}\mathrm{C\,hour^{-1}}$.

[5 marks]

33. [Maximum mark: 6] 22M.1.AHL.TZ1.8

Consider the curve $y = 2x(4 - \mathrm{e}^x)$.

(a.i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

Markscheme

use of product rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2(4 - \mathrm{e}^x) + 2x(-\mathrm{e}^x)$ ***A1***

$= 8 - 2\mathrm{e}^x - 2x\mathrm{e}^x$

[2 marks]

(a.ii) Find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. [2]

Markscheme

use of product rule ***(M1)***

$\frac{d^2y}{dx^2} = -2e^x - 2e^x - 2xe^x$ ***A1***

$= -4e^x - 2xe^x$

$= -2(2+x)e^x$

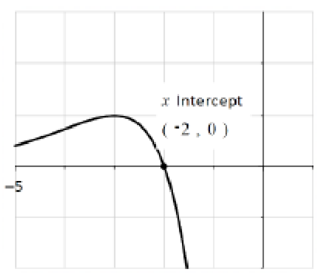
[2 marks]

(b) The curve has a point of inflexion at $(a,\ b)$.

Find the value of a. [2]

Markscheme

$-2(2+a)e^a = 0$ **OR** sketch of $\frac{d^2y}{dx^2}$ with x-intercept indicated **OR** finding the local maximum of $\frac{dy}{dx}$ at $(-2,\ 8.27)$ ***(M1)***



$(a =)\ -2$ ***A1***

[2 marks]

34. [Maximum mark: 8] 22M.1.AHL.TZ1.14

(a.i) Expand $\left(\frac{1}{u}+1\right)^2$. [1]

Markscheme

$\frac{1}{u^2} + \frac{2}{u} + 1$ ***A1***

[1 mark]

(a.ii) Find $\int\left(\frac{1}{(x+2)}+1\right)^2 \mathrm{d}\,x$. [3]

Markscheme

$\int\left(\frac{1}{(x+2)}+1\right)^2 \mathrm{d}\,x$

$=\int\left(\frac{1}{(x+2)^2}+\frac{2}{x+2}+1\right)\mathrm{d}\,x$ **OR** $\int\left(\frac{1}{u^2}+\frac{2}{u}+1\right)\mathrm{d}\,u$ ***(M1)***

$=-\frac{1}{(x+2)}+2\ln|x+2|+x(+c)$ ***A1A1***

Note: Award ***A1*** for first expression, ***A1*** for second two expressions.
Award ***A1A0*** for a final answer of $=-\frac{1}{u}+2\ln(u)+u(+c)$.

[3 marks]

The region bounded by $y=\frac{1}{(x+2)}+1,\ x=0,\ x=2$ and the x-axis is rotated through 2π about the x-axis to form a solid.

(b) Find the volume of the solid formed. Give your answer in the form $\frac{\pi}{4}(a+b\ln(c))$, where $a,\ b,\ c\in\mathbb{Z}$. [4]

Markscheme

volume $=\pi\left[-\frac{1}{(x+2)}+2\ln(x+2)+x\right]_0^2$ ***M1***

$=\pi\left(-\frac{1}{4}+2\ln(4)+2+\frac{1}{2}-2\ln 2\right)$ ***A1***

$=\pi\left(\frac{9}{4}+2\ln(4)-2\ln 2\right)$

use of log laws seen, for example ***M1***

$\pi\left(\frac{9}{4}+4\ln(2)-2\ln 2\right)$ **OR** $\pi\left(\frac{9}{4}+2\ln\left(\frac{4}{2}\right)\right)$

$=\frac{\pi}{4}(9+8\ln(2))$ **OR** $a=9,\ b=8$ and $c=2$ ***A1***

Note: Other correct integer solutions are possible and should be accepted for example $a=9,\ b=c=4$.

[4 marks]

35. [Maximum mark: 4] 22M.1.AHL.TZ2.14

The shape of a vase is formed by rotating a curve about the y-axis.

The vase is $10\,\text{cm}$ high. The internal radius of the vase is measured at $2\,\text{cm}$ intervals along the height:

Height (cm)	Radius (cm)
0	4
2	6
4	8
6	7
8	3
10	5

Use the trapezoidal rule to estimate the volume of water that the vase can hold. [4]

Markscheme

$V = \pi \int_0^{10} y^2 \, \mathrm{d}\, x$ **OR** $\pi \int_0^{10} x^2 \, \mathrm{d}\, y$ ***(M1)***

$h = 2$

$\approx \pi \times \frac{1}{2} \times 2 \times \left(\left(4^2 + 5^2\right) + 2 \times \left(6^2 + 8^2 + 7^2 + 3^2\right)\right)$ ***M1A1***

$= 1120\,\text{cm}^3 \quad (1121.548\ldots)$ ***A1***

Note: Do not award the second ***M1*** If the terms are not squared.

[4 marks]

36. [Maximum mark: 21] 22M.2.AHL.TZ2.6

At an archery tournament, a particular competition sees a ball launched into the air while an archer attempts to hit it with an arrow.

The path of the ball is modelled by the equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} u_x \\ u_y - 5t \end{pmatrix}$$

where x is the horizontal displacement from the archer and y is the vertical displacement from the ground, both measured in metres, and t is the time, in seconds, since the ball was launched.

- u_x is the horizontal component of the initial velocity
- u_y is the vertical component of the initial velocity.

In this question both the ball and the arrow are modelled as single points. The ball is launched with an initial velocity such that $u_x = 8$ and $u_y = 10$.

(a.i) Find the initial speed of the ball. [2]

Markscheme

$\sqrt{10^2 + 8^2}$ ***(M1)***

$= 12.8 \ \left(12.8062\ldots, \ \sqrt{164}\right) \ \left(\mathrm{m\,s^{-1}}\right)$ ***A1***

[2 marks]

(a.ii) Find the angle of elevation of the ball as it is launched. [2]

Markscheme

$\tan^{-1}\left(\frac{10}{8}\right)$ ***(M1)***

$= 0.896$ **OR** 51.3 $(0.896055\ldots$ **OR** $51.3401\ldots^{\circ})$ ***A1***

Note: Accept 0.897 or 51.4 from use of $\arcsin\left(\frac{10}{12.8}\right)$.

[2 marks]

(b) Find the maximum height reached by the ball. [3]

Markscheme

$y = t(10 - 5t)$ ***(M1)***

Note: The ***M1*** might be implied by a correct graph or use of the correct equation.

METHOD 1 – graphical Method

sketch graph ***(M1)***

Note: The ***M1*** might be implied by correct graph or correct maximum (eg $t = 1$).

max occurs when $y = 5\,\mathrm{m}$ ***A1***

METHOD 2 – calculus

differentiating and equating to zero ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = 10 - 10t = 0$

$t = 1$

$y(= 1(10 - 5)) = 5\,\mathrm{m}$ ***A1***

METHOD 3 – symmetry

line of symmetry is $t = 1$ ***(M1)***

$y(= 1(10 - 5)) = 5\,\mathrm{m}$ ***A1***

[3 marks]

(c) Assuming that the ground is horizontal and the ball is not hit by the arrow, find the x coordinate of the point where the ball lands. [3]

Markscheme

attempt to solve $t(10 - 5t) = 0$ ***(M1)***

$t = 2$ (or $t = 0$) ***(A1)***

$x\ (= 5 + 8 \times 2) = 21\,\mathrm{m}$ ***A1***

Note: Do not award the final ***A1*** if $x = 5$ is also seen.

[3 marks]

(d) For the path of the ball, find an expression for y in terms of x. [3]

Markscheme

METHOD 1

$t = \frac{x-5}{8}$ ***M1A1***

$y = \left(\frac{x-5}{8}\right)\left(10 - 5 \times \frac{x-5}{8}\right)$ ***A1***

METHOD 2

$y = k(x-5)(x-21)$ ***A1***

when $x = 13,\ y = 5$ so $k = \frac{5}{(13-5)(13-21)} = -\frac{5}{64}$ ***M1A1***

$\left(y = -\frac{5}{64}(x-5)(x-21)\right)$

METHOD 3

if $y = ax^2 + bx + c$

$0 = 25a + 5b + c$
$5 = 169a + 13b + c$
$0 = 441a + 21b + c$ ***M1A1***

solving simultaneously, $a = -\frac{5}{64},\ b = \frac{130}{64},\ c = -\frac{525}{64}$ ***A1***

$(y = -\frac{5}{64}x^2 + \frac{130}{64}x - \frac{525}{64})$

METHOD 4

use quadratic regression on $(5,\ 0),\ (13,\ 5),\ (21,\ 0)$ ***M1A1***

$y = -\frac{5}{64}x^2 + \frac{130}{64}x - \frac{525}{64}$ ***A1***

Note: Question asks for expression; condone omission of "$y =$".

[3 marks]

An archer releases an arrow from the point $(0,\ 2)$. The arrow is modelled as travelling in a straight line, in the same plane as the ball, with speed $60\,\mathrm{m\,s^{-1}}$ and an angle of elevation of 10°.

(e) Determine the two positions where the path of the arrow intersects the path of the ball. [4]

Markscheme

trajectory of arrow is $y = x\tan 10 + 2$ ***(A1)***

intersecting $y = x\tan 10 + 2$ and their answer to (d) ***(M1)***

$(8.66,\ 3.53)\ \ ((8.65705\ldots,\ 3.52647\ldots))$ ***A1***

$(15.1,\ 4.66)$ $((15.0859\ldots,\ 4.66006\ldots))$ ***A1***

[4 marks]

(f) Determine the time when the arrow should be released to hit the ball before the ball reaches its maximum height. [4]

Markscheme

when $x_{\text{target}} = 8.65705\ldots,\ t_{\text{target}} = \frac{8.65705\ldots-5}{8} = 0.457132\ldots\text{ s}$ ***(A1)***

attempt to find the distance from point of release to intersection ***(M1)***

$\sqrt{8.65705\ldots^2 + (3.52647\ldots - 2)^2}\ (= 8.79060\ldots\text{ m})$

time for arrow to get there is $\frac{8.79060\ldots}{60} = 0.146510\ldots\text{s}$ ***(A1)***

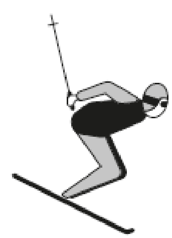
so the arrow should be released when

$t = 0.311\,(\text{s})\ \ (0.310622\ldots\,(\text{s}))$ ***A1***

[4 marks]

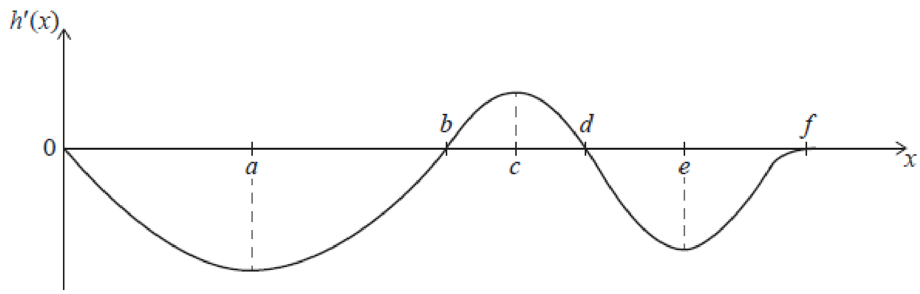
37. [Maximum mark: 5] 21N.1.AHL.TZ0.8

Juri skis from the top of a hill to a finishing point at the bottom of the hill. She takes the shortest route, heading directly to the finishing point (F).



Let $h(x)$ define the height of the hill above F at a horizontal distance x from the starting point at the top of the hill.

The graph of the **derivative** of $h(x)$ is shown below. The graph of $h\prime(x)$ has local minima and maxima when x is equal to $a,\ c$ and e. The graph of $h\prime(x)$ intersects the x-axis when x is equal to $b,\ d$, and f.



(a.i) Identify the x value of the point where $|h\prime(x)|$ has its maximum value. [1]

Markscheme

a ***A1***

[1 mark]

(a.ii) Interpret this point in the given context. [1]

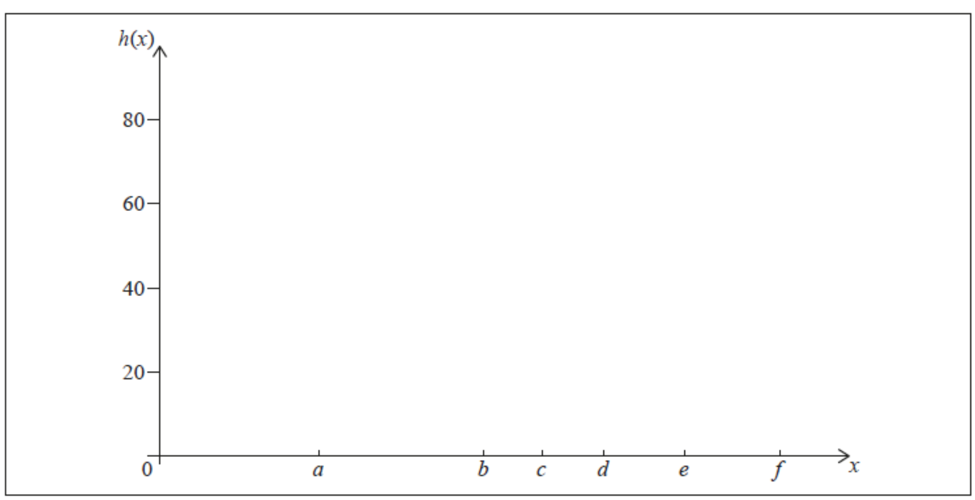
Markscheme

the hill is at its steepest / largest slope of hill ***A1***

[1 mark]

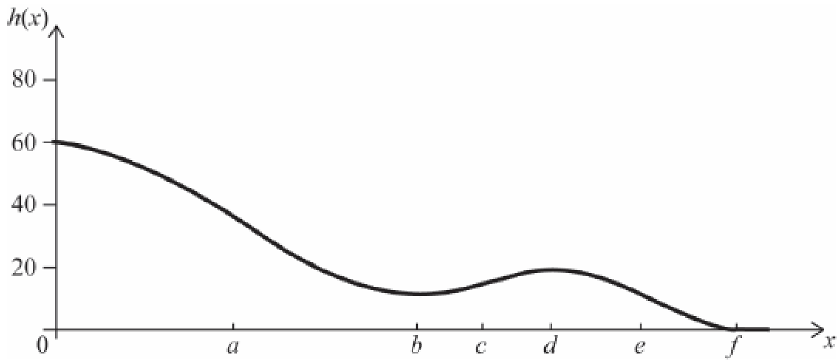
(b) Juri starts at a height of 60 metres and finishes at F, where $x = f$.

Sketch a possible diagram of the hill on the following pair of coordinate axes.



[3]

Markscheme



A1A1A1

Note: Award ***(A1)*** for decreasing function from 0 to b and d to f and increasing from b to d; ***(A1)*** for minimum at b and max at d; ***(A1)*** for starting at height of 60 and finishing at a height of 0 at f. If reasonable curvature not evident on graph (i.e. only straight lines used) award ***A1A0A1***.

[3 marks]

38. [Maximum mark: 7] 21N.1.AHL.TZ0.17

The sides of a bowl are formed by rotating the curve $y = 6 \ln x,\ 0 \le y \le 9$, about the y-axis, where x and y are measured in centimetres. The bowl contains water to a height of $h\,\mathrm{cm}$.

(a) Show that the volume of water, V, in terms of h is $V = 3\pi\left(\mathrm{e}^{\frac{h}{3}} - 1\right)$. [5]

Markscheme

attempt to use $V = \pi \int_a^b x^2 \ \mathrm{d}\ y$ ***(M1)***

$x = \mathrm{e}^{\frac{y}{6}}$ or any reasonable attempt to find x in terms of y ***(M1)***

$V = \pi \int_0^h \mathrm{e}^{\frac{y}{3}} \ \mathrm{d}\ y$ ***A1***

Note: Correct limits must be seen for the ***A1*** to be awarded.

$= \pi\left[3\mathrm{e}^{\frac{y}{3}}\right]_0^h$ ***(A1)***

Note: Condone the absence of limits for this ***A1*** mark.

$= 3\pi\left[\mathrm{e}^{\frac{h}{3}} - \mathrm{e}^0\right]$ ***A1***

$= 3\pi \left[e^{\frac{h}{3}} - 1\right]$ ***AG***

Note: If the variable used in the integral is x instead of y (i.e. $V = \pi \int_0^h e^{\frac{x}{3}}\ \mathrm{d}\ x$) and the candidate has not stated that they are interchanging x and y then award at most ***M1M1A0A1A1AG***.

[5 marks]

(b) Hence find the maximum capacity of the bowl in cm^3. [2]

Markscheme

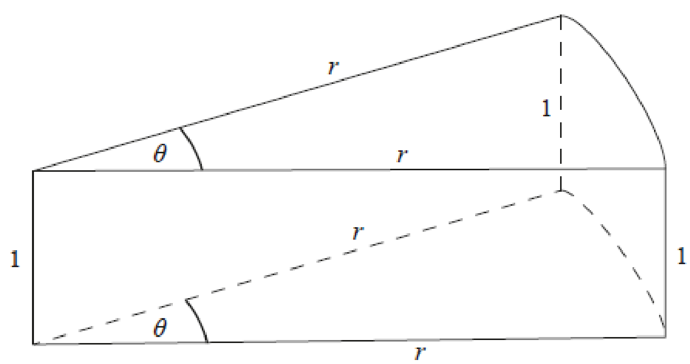
maximum volume when $h = 9\,\mathrm{cm}$ ***(M1)***

max volume $= 180\,\mathrm{cm}^3$ ***A1***

[2 marks]

39. [Maximum mark: 9] 21N.1.AHL.TZ0.15

The following diagram shows a frame that is made from wire. The total length of wire is equal to $15\,\mathrm{cm}$. The frame is made up of two identical sectors of a circle that are parallel to each other. The sectors have angle θ radians and radius $r\,\mathrm{cm}$. They are connected by $1\,\mathrm{cm}$ lengths of wire perpendicular to the sectors. This is shown in the diagram below.



(a) Show that $r = \frac{6}{2+\theta}$. [2]

Markscheme

$15 = 3 + 4r + 2r\theta$ ***M1***

$12 = 2r(2 + \theta)$ ***A1***

Note: Award ***A1*** for any reasonable working leading to expected result e,g, factorizing r.

$r = \frac{6}{2+\theta}$ ***AG***

[2 marks]

The faces of the frame are covered by paper to enclose a volume, V.

(b.i) Find an expression for V in terms of θ. [2]

Markscheme

attempt to use sector area to find volume ***(M1)***

volume $= \frac{1}{2}r^2\theta \times 1$

$= \frac{1}{2} \times \frac{36}{(2+\theta)^2} \times \theta \ \left(= \frac{18\theta}{(2+\theta)^2}\right)$ ***A1***

[2 marks]

(b.ii) Find the expression $\frac{\mathrm{d}V}{\mathrm{d}\theta}$. [3]

Markscheme

$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{(2+\theta)^2 \times 18 - 36\theta(2+\theta)}{(2+\theta)^4}$ ***M1A1A1***

$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{36-18\theta}{(2+\theta)^3}$

[3 marks]

(b.iii) Solve algebraically $\frac{\mathrm{d}V}{\mathrm{d}\theta} = 0$ to find the value of θ that will maximize the volume, V. [2]

Markscheme

$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{36-18\theta}{(2+\theta)^3} = 0$ ***M1***

Note: Award this ***M1*** for simplified version equated to zero. The simplified version may have been seen in part (b)(ii).

$\theta = 2$ ***A1***

[2 marks]