Calculus [443 marks]

1. [Maximum mark: 5] SPM.1.AHL.TZ0.7

A particle, A, moves so that its velocity (v ms^{-1}) at time t is given by $v = 2 \sin t$, $t \geq 0$.

The kinetic energy (E) of the particle A is measured in joules (J) and is given by $E = 5v^2$.

(a) Write down an expression for E as a function of time. [1]

(b) Hence find $\frac{\mathrm{d}E}{\mathrm{d}t}$. [2]

(c) Hence or otherwise find the first time at which the kinetic energy is changing at a rate of 5 J s^{-1}. [2]

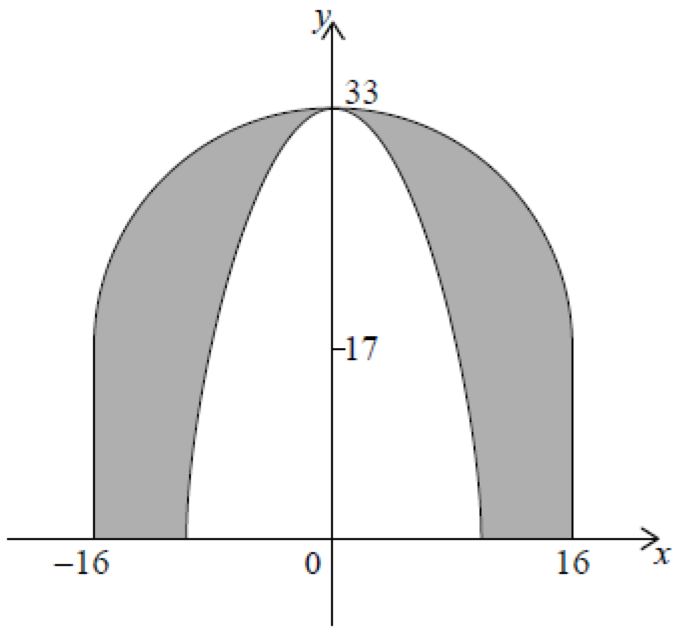
2. [Maximum mark: 8] SPM.1.AHL.TZ0.14

The graph of $y = -x^3$ is transformed onto the graph of $y = 33 - 0.08x^3$ by a translation of a units vertically and a stretch parallel to the x-axis of scale factor b.

(a.i) Write down the value of a. [1]

(a.ii) Find the value of b. [2]

(b) The outer dome of a large cathedral has the shape of a hemisphere of diameter 32 m, supported by vertical walls of height 17 m. It is also supported by an inner dome which can be modelled by rotating the curve $y = 33 - 0.08x^3$ through 360° about the y-axis between $y = 0$ and $y = 33$, as indicated in the diagram.



[5]

Find the volume of the space between the two domes.

3. [Maximum mark: 8] EXN.1.AHL.TZ0.15

Consider the function $f(x) = \sqrt{-ax^2 + x + a},\ a \in \mathbb{R}^+$.

(a) Find $f\prime(x)$. [2]

For $a > 0$ the curve $y = f(x)$ has a single local maximum.

(b) Find in terms of a the value of x at which the maximum occurs. [2]

(c) Hence find the value of a for which y has the smallest possible maximum value. [4]

4. [Maximum mark: 17] EXN.2.AHL.TZ0.7

A ball is attached to the end of a string and spun horizontally. Its position relative to a given point, O, at time t seconds, $t \geq 0$, is given by the equation

$\boldsymbol{r} = \begin{pmatrix} 1.5\cos(0.1t^2) \\ 1.5\sin(0.1t^2) \end{pmatrix}$ where all displacements are in metres.

(a) Show that the ball is moving in a circle with its centre at O and state the radius of the circle. [4]

(b.i) Find an expression for the velocity of the ball at time t. [2]

(b.ii) Hence show that the velocity of the ball is always perpendicular to the position vector of the ball. [2]

(c.i) Find an expression for the acceleration of the ball at time t. [3]

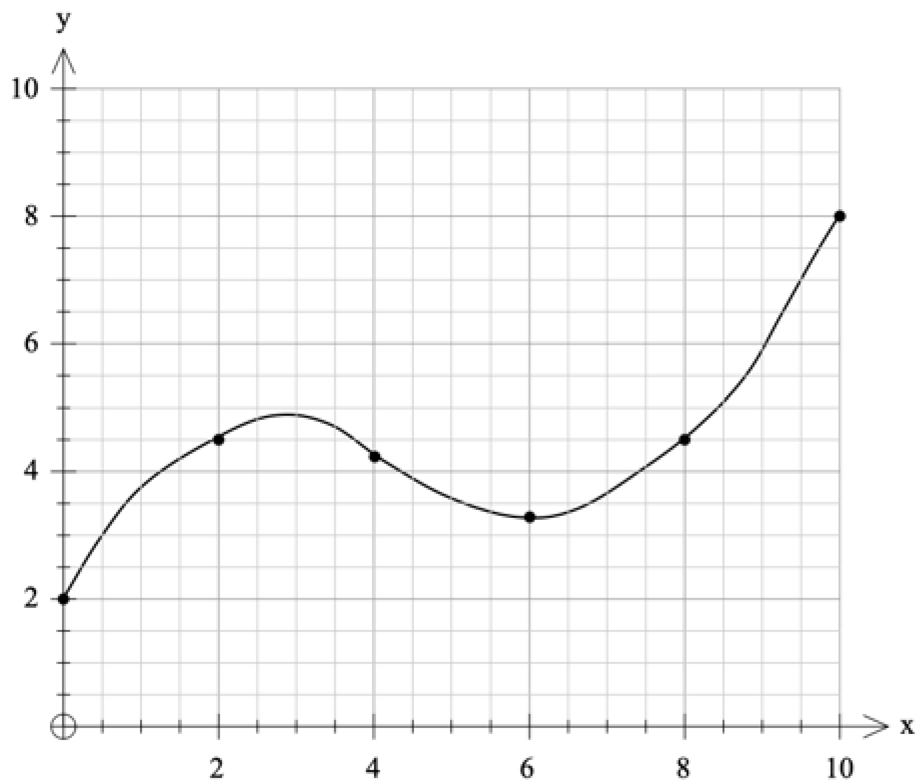
The string breaks when the magnitude of the ball's acceleration exceeds $20\,\mathrm{ms}^{-2}$.

(c.ii) Find the value of t at the instant the string breaks. [3]

(c.iii) How many complete revolutions has the ball completed from $t = 0$ to the instant at which the string breaks? [3]

5. [Maximum mark: 10] EXM.2.AHL.TZ0.12

The curve $y = f(x)$ is shown in the graph, for $0 \leqslant x \leqslant 10$.



The curve $y = f(x)$ passes through the following points.

x	0	2	4	6	8	10
y	2	4.5	4.2	3.3	4.5	8

It is required to find the area bounded by the curve, the x-axis, the y-axis and the line $x = 10$.

(a) Use the trapezoidal rule to find an estimate for the area. [3]

One possible model for the curve $y = f(x)$ is a cubic function.

(b.i) Use all the coordinates in the table to find the equation of the least squares cubic regression curve. [3]

(b.ii) Write down the coefficient of determination. [1]

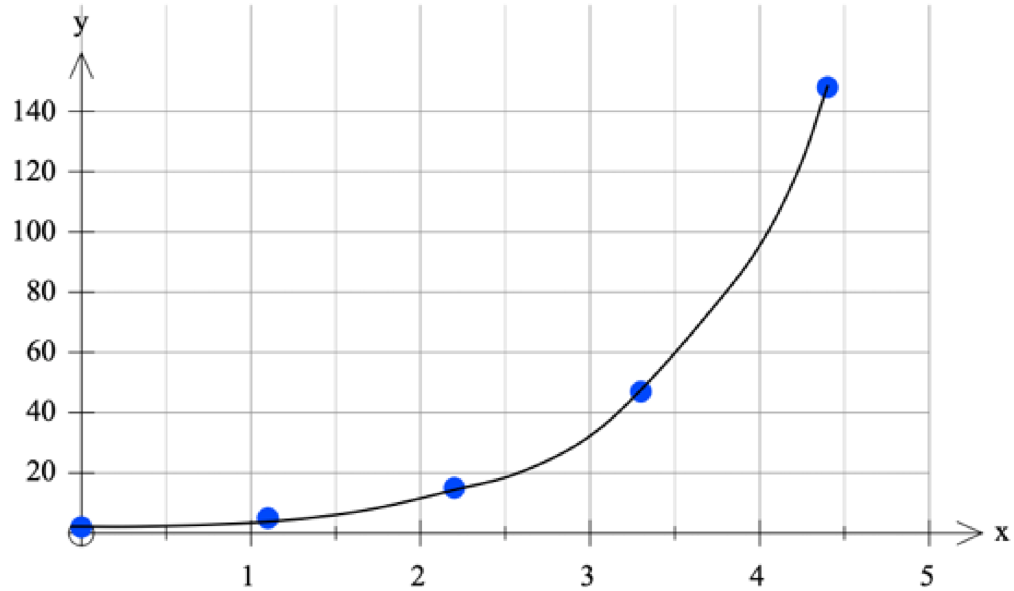
(c.i) Write down an expression for the area enclosed by the cubic regression curve, the x-axis, the y-axis and the line $x = 10$. [1]

(c.ii) Find the value of this area. [2]

6. [Maximum mark: 23] EXM.3.AHL.TZ0.7

This question explores methods to determine the area bounded by an unknown curve.

The curve $y = f(x)$ is shown in the graph, for $0 \leqslant x \leqslant 4.4$.



The curve $y = f(x)$ passes through the following points.

x	0	1.1	2.2	3.3	4.4
y	2	5	15	47	148

It is required to find the area bounded by the curve, the x-axis, the y-axis and the line $x = 4.4$.

(a.i) Use the trapezoidal rule to find an estimate for the area. [3]

(a.ii) With reference to the shape of the graph, explain whether your answer to part (a)(i) will be an over-estimate or an underestimate of the area. [2]

One possible model for the curve $y = f(x)$ is a cubic function.

(b.i) Use all the coordinates in the table to find the equation of the least squares cubic regression curve. [3]

(b.ii) Write down the coefficient of determination. [1]

(c.i) Write down an expression for the area enclosed by the cubic function, the x-axis, the y-axis and the line $x = 4.4$. [2]

(c.ii) Find the value of this area. [2]

A second possible model for the curve $y = f(x)$ is an exponential function, $y = p\mathrm{e}^{qx}$, where $p,\ q \in \mathbb{R}$.

(d.i) Show that $\ln y = qx + \ln p$. [2]

(d.ii) Hence explain how a straight line graph could be drawn using the coordinates in the table. [1]

(d.iii) By finding the equation of a suitable regression line, show that $p = 1.83$ and $q = 0.986$. [5]

(d.iv) Hence find the area enclosed by the exponential function, the x -axis, the y-axis and the line $x = 4.4$. [2]

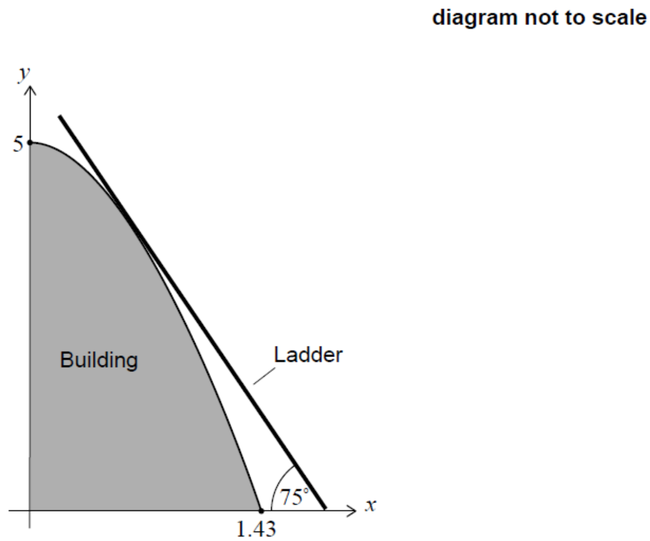
7. [Maximum mark: 7] 24M.1.AHL.TZ1.9

(a) Find $\int \frac{8}{2x+3}\mathrm{dx}$. [3]

(b) Hence find the exact area between the curve $y = \frac{8}{2x+3}$, the x-axis and the lines $x = 0$ and $x = 6$. Give your answer in the form $a\ \ln\ b$, where $a,\ b \in \mathbb{N}$. [4]

8. [Maximum mark: 8] 24M.1.AHL.TZ1.16

The cross section of the side of a building can be modelled by a curve with equation $y = 5\ \cos\ (1.\,1x),\ 0 \leq x \leq 1.\,43$, as shown in the following diagram. Distances are measured in metres.



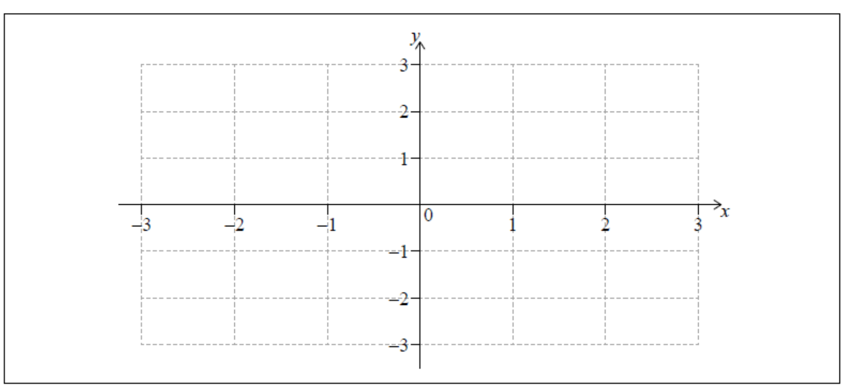
A builder leans a straight ladder against the building to do repairs. For safety reasons, the angle between the ladder and the horizontal ground must be 75°.

Find the height above the ground at which the ladder touches the building. [8]

9. [Maximum mark: 9] 24M.1.AHL.TZ2.10

Consider the function $f(x) = x\sqrt{3-x^2}, -\sqrt{3} \le x \le \sqrt{3}$.

(a) Sketch the graph of $y = f(x)$ on the following pair of axes.



[2]

The area between the graph of $y = f(x)$ and the x-axis is rotated through 360° about the x-axis.

(b.i) Write down an integral that represents this volume. [2]

(b.ii) Calculate the value of this integral. [2]

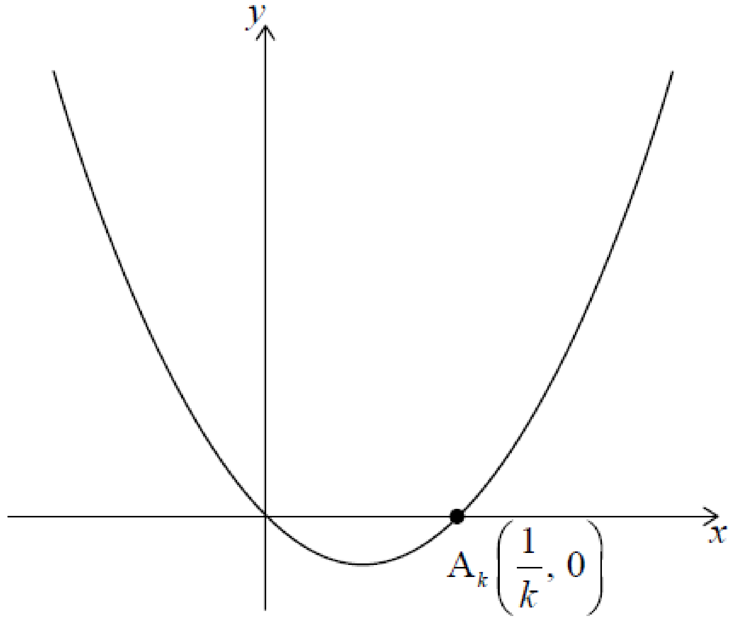
The graph of the function f is transformed, to give the graph of the function g, in the following way:

- It is first stretched by scale factor 2, parallel to the x-axis with the y-axis invariant.
- It is then stretched by scale factor 0.5, parallel to the y-axis with the x-axis invariant.

(c) Find the volume obtained when the area between the graph of $y = g(x)$ and the x-axis is rotated through 360° about the x-axis. [3]

10. [Maximum mark: 9] 24M.1.AHL.TZ2.11

The diagram shows the curve with equation $y_k = kx^2 - x,\ k > 0$, which intersects the x-axis at the origin and at the point $A_k\left(\frac{1}{k},\ 0\right)$.



The normal to the curve at A_k intersects the curve again at point B_k.

(a) Show that the x-coordinate of B_k is $-\frac{1}{k}$. [6]

Consider the case where $k = 2$.

(b) Calculate the finite area of the region between the curve with equation $y_2 = 2x^2 - x$ and the normal at A_2. [3]

11. [Maximum mark: 8] 24M.1.AHL.TZ2.13

A particle starts from rest at point O and moves in a straight line with velocity, v, given by

$$v = 3\sin(t)(1 + \cos(t)),\ t \geq 0$$

where v is measured in metres per second and time, t (radians), is measured in seconds.

The particle next comes to instantaneous rest when $t = a$.

(a) Determine the value of a. [2]

(b) Find the maximum velocity of the particle during the interval $0 \leq t \leq a$. [2]

(c) By finding the total distance travelled between $t = 0$ and $t = a$, find the average speed of the particle during the interval $0 \leq t \leq a$. [4]

12. [Maximum mark: 13] 24M.2.AHL.TZ1.3

A shop uses the following model to estimate n, the number of smoothies sold per day, in terms of x, the price of a single smoothie in pesos.

$n = \frac{40\,000}{x^2}$

The maximum number of smoothies the shop can make in a day is 400.

(a) Find the maximum price they could charge per smoothie for the shop to sell 400 in one day. [2]

(b) On a day when the shop sells smoothies at 50 pesos each, use the model to find

(b.i) the number of smoothies sold. [1]

(b.ii) the total income from the smoothies sold. [1]

The cost of making each smoothie is 20 pesos. The profit per day (P) is the total income from the sale of smoothies that day minus the cost of making them.

(c.i) Show that, according to the model, $P = \frac{40\,000}{x} - \frac{800\,000}{x^2}$. [2]

(c.ii) Find $\frac{\mathrm{d}P}{\mathrm{d}x} = 0$. [3]

(c.iii) Find the value of x for which $\frac{\mathrm{d}P}{\mathrm{d}x} = 0$. [2]

(c.iv) Find the number of smoothies sold when the profit is maximized. [2]

13. [Maximum mark: 14] 24M.2.AHL.TZ1.7

The interior of a vase is modelled by rotating the region bounded by the curve $y = \frac{1}{2}x^2 - 1$, and the lines $x = 0$, $y = 0$ and $y = 15$, through 2π radians about the y-axis. The values of x and y are measured in centimetres.

The vase is filled with water to a height of h cm.

(a) Find an explicit expression for the volume of water in terms of h . [5]

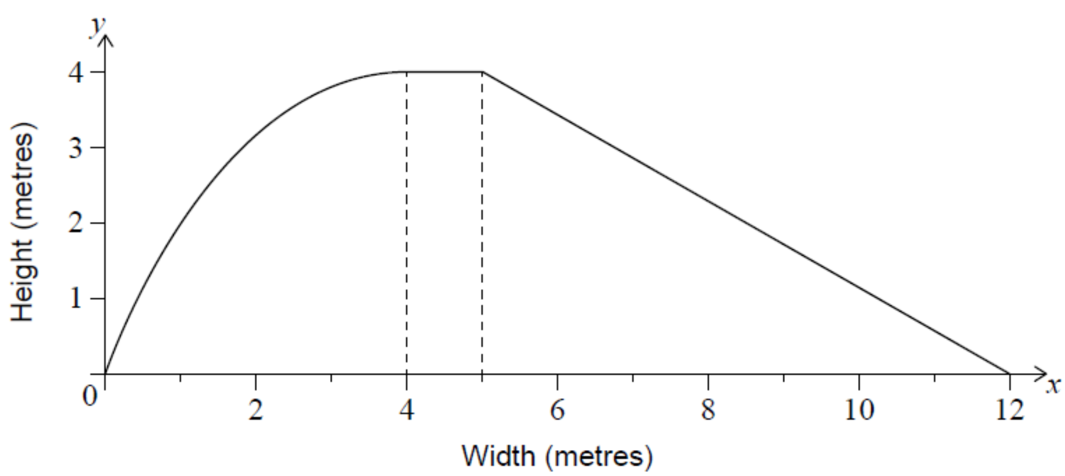
The vase is filled at a rate of $20\ \text{cm}^3\ \text{s}^{-1}$.

(b) Find the time taken to completely fill the vase. [2]

(c) Find the rate at which the height is changing when $h = 10$ cm. [7]

14. [Maximum mark: 15] 24M.2.AHL.TZ2.2

The following diagram shows a model of the side view of a water slide. All lengths are measured in metres.



The curved edge of the slide is modelled by

$f(x) = -\frac{1}{4}x^2 + 2x$ for $0 \leq x \leq 4$.

The remainder of the slide is modelled by

$$g(x) = \begin{cases} 4, \text{ for } 4 \leq \mathrm{x} \leq 5 \\ \frac{48}{7} - \frac{4x}{7}, \text{ for } 5 \leq \mathrm{x} \leq 12 \end{cases}$$

(a) Use the trapezoidal rule with an interval width of 1 to calculate the approximate area under the model of the slide in the interval $0 \leq x \leq 4$. [5]

(b) Find $\int\left(-\frac{1}{4}x^2 + 2x\right)\mathrm{d}x$. [3]

(c) Calculate the exact area under the entire model of the slide, for $0 \leq x \leq 12$. [4]

(d) Find the percentage error in the **total** area under the entire model of the slide when using the approximate value from part (a). [3]

15. [Maximum mark: 22] 24M.2.AHL.TZ2.3

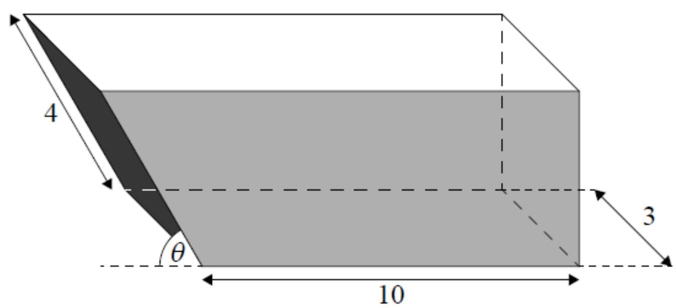
A skip is a container used to carry garbage away from a construction site. For safety reasons the garbage must not extend beyond the top of the skip. The maximum volume of garbage to be removed is therefore equal to the volume of the skip.

[Source: Andyqwe, n.d. *Dumpstertruck* [image online] Available at: https://www.gettyimages.co.uk/detail/photo/dumpster-truck-royalty-free-image/157611454 [Accessed 18 April 2023] Source adapted.]

A particular design of skip can be modelled as a prism with a trapezoidal cross section. For the skip to be transported, it must have a rectangular base of length $10\,\mathrm{m}$ and width $3\,\mathrm{m}$. The length of the sloping edge is fixed at $4\,\mathrm{m}$, and makes an angle of θ with the horizontal.

The following diagram shows such a skip.

diagram not to scale



(a) Find the volume of this skip,

(a.i) if the length of the top edge of the skip is $11\,\mathrm{m}$. [4]

(a.ii) if the height of the skip is $3.2\,\mathrm{m}$. [3]

(a.iii) if θ is $\frac{\pi}{3}$. [2]

(b) Show that the volume, $V\,\mathrm{m}^3$, of the skip is given by

$$24 \sin(\theta)\,(5 + \cos(\theta)).$$ [2]

(c) Explain, in context, why $\theta \neq 0$. [1]

(d.i) Sketch the graph of
$V = 24 \sin(\theta)(5 + \cos(\theta)),\ 0 < \theta < \frac{\pi}{2}$. [2]

(d.ii) Find the maximum volume of the skip and the value of θ for which this maximum volume occurs. [2]

(e) Show, by differentiation, that the maximum volume occurs at a value of θ that satisfies the equation

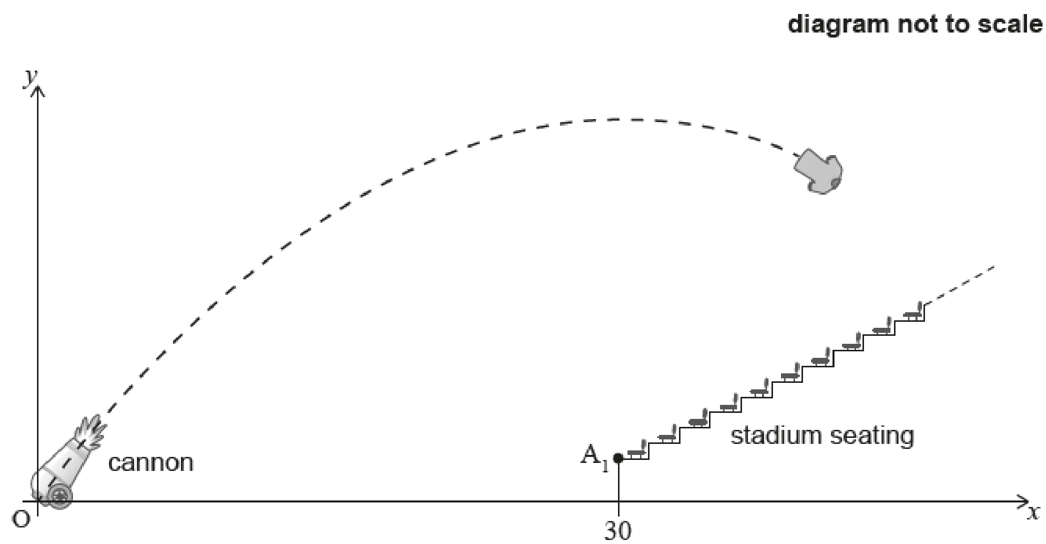
$$2\cos^2\theta + 5\cos\theta - 1 = 0.$$ [6]

16. [Maximum mark: 29] 24M.3.AHL.TZ1.2

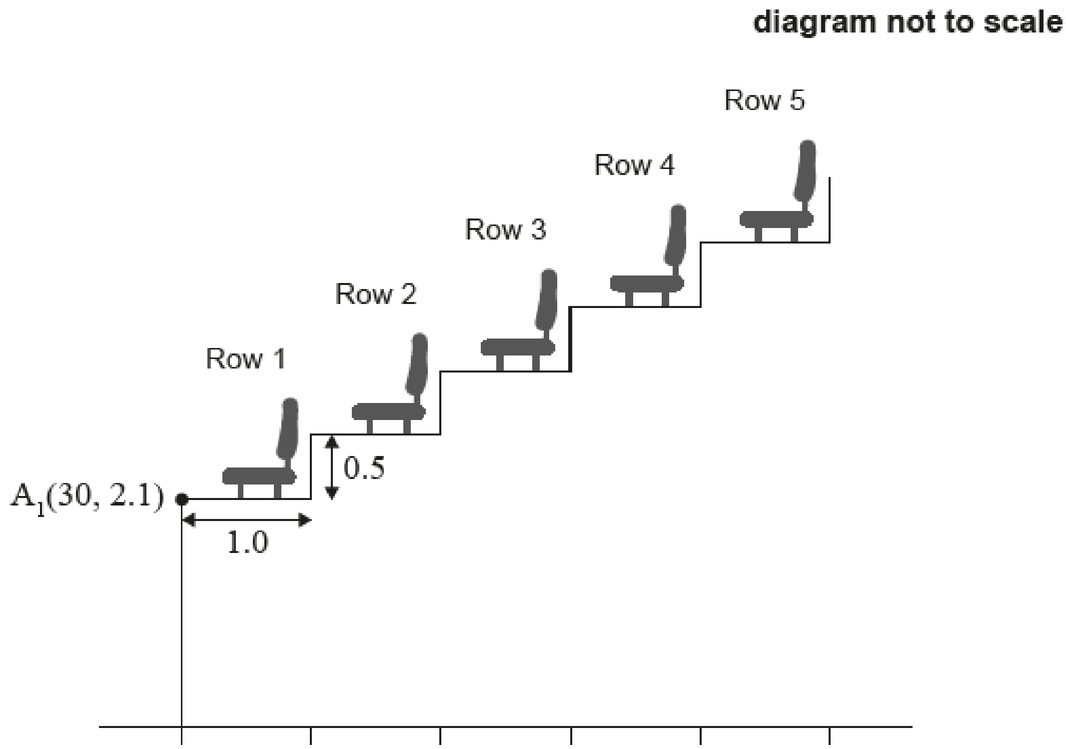
A sports stadium has a T-shirt cannon which is used to launch T-shirts into the crowd. The purpose of this question is to determine whether a person sitting in a particular seat will ever receive a T-shirt.

A T-shirt cannon is placed on the horizontal ground of a stadium playing area. A coordinate system is created such that the origin, O, is the point on the ground from where the T-shirts are launched. In this coordinate system, x and y represent the horizontal and vertical displacement from O, and are measured in metres.

Seat A_1 is the nearest seat to the T-shirt cannon. The coordinates of the front of the foot space for seat A_1 are $(30,\ 2.1)$.



Each seat behind seat A_1 is $1.0\,\mathrm{m}$ further from O horizontally and $0.5\,\mathrm{m}$ higher than the seat in the row below it, as shown on the diagram.



Seat A_1 is in row 1. Let seat A_n be the seat directly behind A_1 in row n.

(a.i) Write down the coordinates of the front of the foot space of seat A_5. [2]

(a.ii) Find, in terms of n, the coordinates for the front of the foot space of seat A_n. [3]

While in motion, the T-shirt can be treated as a projectile.

Let t be the time, in seconds, after a T-shirt is launched.

At any time $t > 0$, the acceleration of the T-shirt, in $\mathrm{m\,s^{-2}}$, is given by the vector

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}.$$

The initial velocity, in $\mathrm{m\,s^{-1}}$, of the T-shirt is given as $\begin{pmatrix} 29.4 \cos \theta \\ 29.4 \sin \theta \end{pmatrix}$, where θ is the angle to the ground at which the T-shirt is launched and $0^{\circ} < \theta \leq 90^{\circ}$.

(b.i) Find an expression for the velocity, $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$, at time t. [3]

(b.ii) Hence show that when the T-shirt is launched vertically, the time for it to reach its maximum height is 3 seconds. [3]

The displacement of the T-shirt, t seconds after it is launched, is given by the vector equation

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}29.4(\cos\theta)t\\ 29.4(\sin\theta)t - 4.9t^2\end{pmatrix}$$

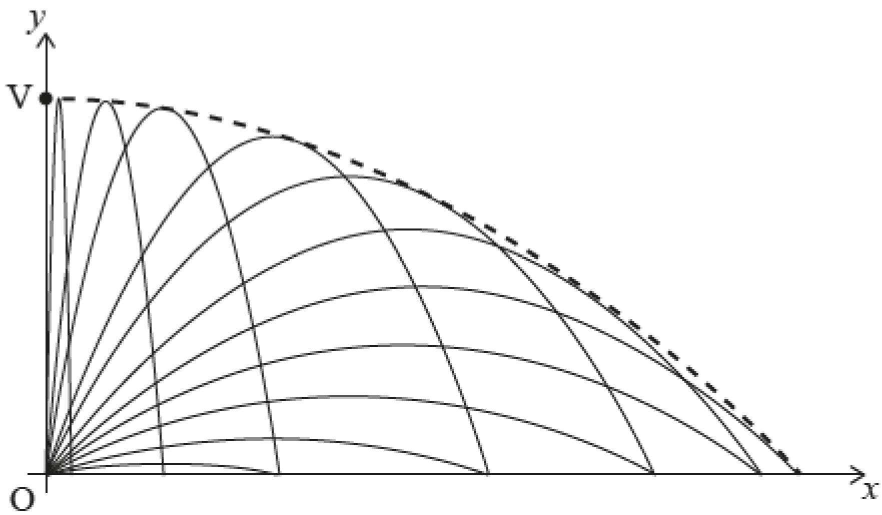
(c) Using the given answer to part (b)(ii) or otherwise, find the maximum height reached by a T-shirt when it is launched vertically. [2]

(d.i) If there was no seating, and the T-shirt was launched at an angle θ, show that the value of x when it would hit the ground is given by the expression

$$x = 176.4\sin\theta\cos\theta$$ [3]

(d.ii) Hence find the maximum possible value for x if there was no seating to block the path of the T-shirt. [2]

In order to calculate the seats in the stadium which can be reached by a T-shirt it is required to find the equation of the curve that forms the boundary of all the points that can be reached. This boundary is represented by the dashed curve in the following diagram, while the solid curves represent some of the possible trajectories for the T-shirts.



It is given that the boundary curve is the parabola $y = ax^2 + bx + c$, with its vertex V on the y-axis.

(e) Using your answers to parts (c) and (d)(ii), or otherwise, find

(e.i) the value of c. [1]

(e.ii) the value of b. [2]

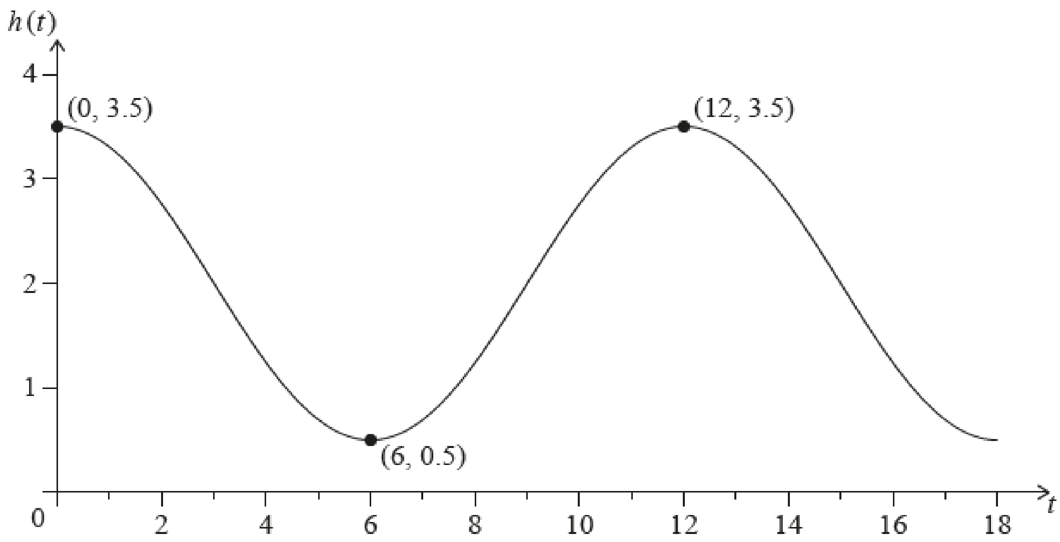
(e.iii) the value of a. [3]

A spectator is sitting in seat A_{40}.

(f) Show that it is not possible for the spectator to ever get a T-shirt. [5]

17. [Maximum mark: 8] 23N.1.AHL.TZ0.9

Joon is a keen surfer and wants to model waves passing a particular point P, which is off the shore of his favourite beach. Joon sets up a model of the waves in terms of $h(t)$, the height of the water in metres, and t, the time in seconds from when he begins recording the height of the water at point P.



The function has the form $h(t) = p\cos\left(\frac{\pi}{6}t\right) + q, t \geq 0$.

(a) Find the values of p and q. [2]

(b) Find

(b.i) $h\prime(t)$. [2]

(b.ii) $h\prime\prime(t)$. [1]

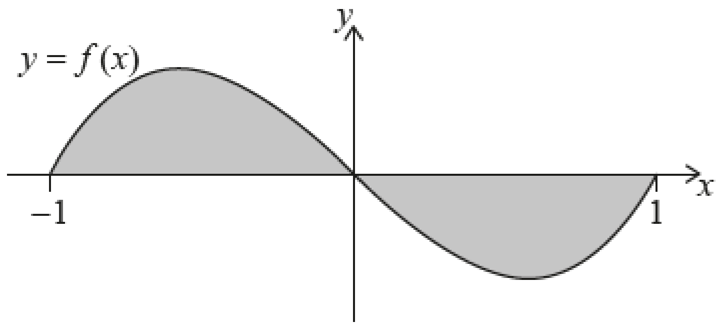
Joon will begin to surf the wave when the rate of change of h with respect to t, at P, is at its maximum. This will first occur when $t = k$.

(c.i) Find the value of k. [2]

(c.ii) Find the height of the water at this time. [1]

18. [Maximum mark: 7] 23N.1.AHL.TZ0.11

Consider the function $f(x) = x^3 - x$, for $-1 \leq x \leq 1$. The shaded region, R, is bounded by the graph of $y = f(x)$ and the x-axis.

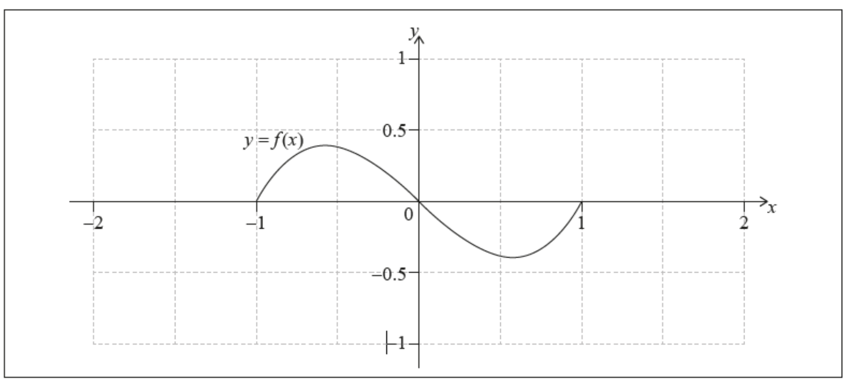


(a.i) Write down an integral that represents the area of R. [1]

(a.ii) Find the area of R. [1]

Another function, g, is defined such that $g(x) = 2f(x - 1)$.

(b) On the following set of axes, the graph of $y = f(x)$ has been drawn. On the same set of axes, sketch the graph of $y = g(x)$.



[2]

The region R from the original graph $y = f(x)$ is rotated through 2π radians about the x-axis to form a solid.

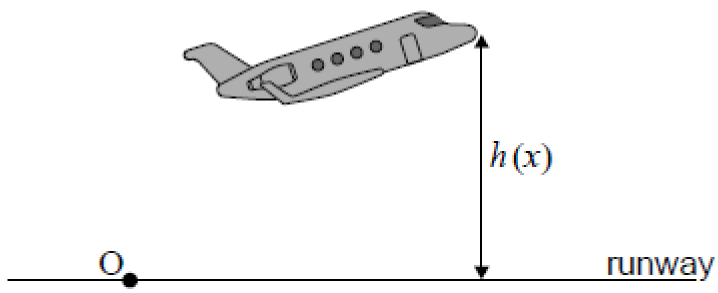
(c) Find the volume of the solid. [3]

19. [Maximum mark: 12] 23N.2.AHL.TZ0.4

A plane takes off from a horizontal runway. Let point O be the point where the plane begins to leave the runway and x be the horizontal distance, in km, of the plane from O. The function h models the vertical height, in km, of the nose of the plane from the horizontal runway, and is defined by

$h(x)=\frac{10}{1+150\mathrm{e}^{-0.07x}}-0.06, x\geq 0.$

diagram not to scale



(a.i) Find $h(0)$ [1]

(a.ii) Interpret this value in terms of the context. [1]

(b.i) Find the horizontal asymptote of the graph of $y=h(x)$. [1]

(b.ii) Interpret this value in terms of the context. [1]

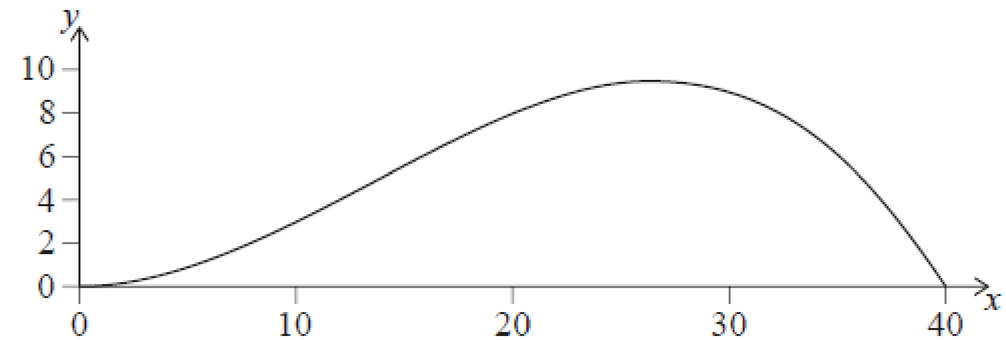
(c) Find $h\prime(x)$ in terms of x. [4]

A safety regulation recommends that $h\prime(x)$ never exceed 0.2.

(d) Given that this plane flies a distance of at least $200\ \mathrm{km}$ horizontally from point O, determine whether the plane is following this safety regulation. [4]

20. [Maximum mark: 8] 23M.1.AHL.TZ1.4

The cross section of a scale model of a hill is modelled by the following graph.



The heights of the model are measured at horizontal intervals and are given in the table.

Horizontal distance, x cm	0	10	20	30	40
Vertical distance, y cm	0	3	8	9	0

(a) Use the trapezoidal rule with $h = 10$ to find an approximation for the cross-sectional area of the model. [2]

It is given that the equation of the curve is
$y = 0.04x^2 - 0.001x^3,\ 0 \leq x \leq 40.$

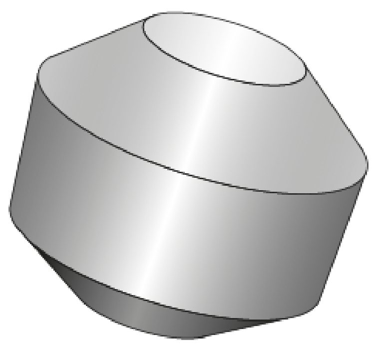
(b.i) Write down an integral to find the exact cross-sectional area. [2]

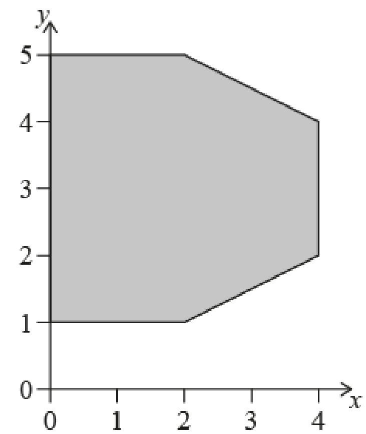
(b.ii) Calculate the value of the cross-sectional area to two decimal places. [2]

(c) Find the percentage error in the area found using the trapezoidal rule. [2]

21. [Maximum mark: 6] 23M.1.AHL.TZ1.15

The solid shown is formed by rotating the hexagon with vertices $(2,\ 1)$, $(0,\ 1)$, $(0,\ 5)$, $(2,\ 5)$, $(4,\ 4)$ and $(4,\ 2)$ about the y-axis.



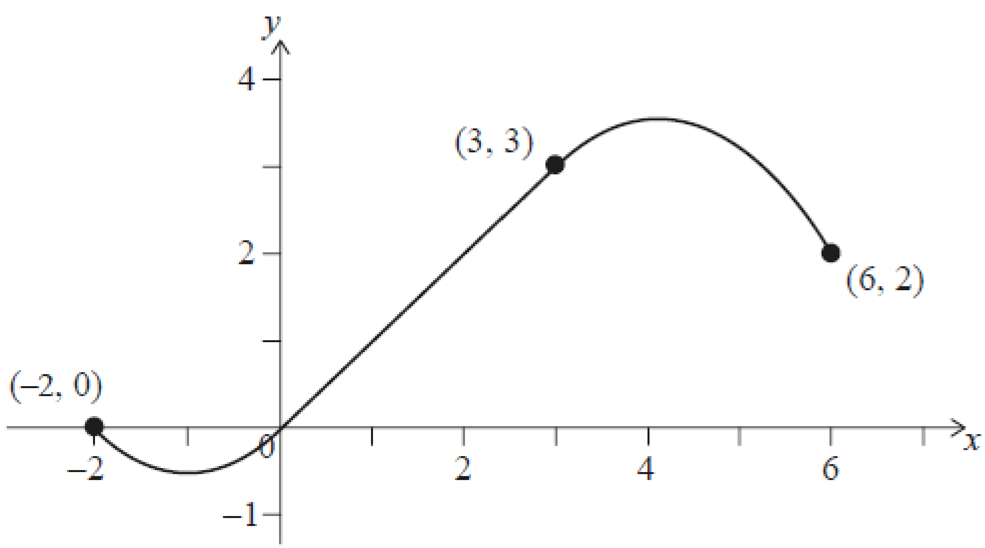


[6]

Find the volume of this solid.

22. [Maximum mark: 9] 23M.1.AHL.TZ1.10

A decorative hook can be modelled by the curve with equation $y = f(x)$. The graph of $y = f(x)$ is shown and consists of a line segment from $(0,\ 0)$ to $(3,\ 3)$ and two sections formed by quadratic curves.



(a) Write down the equation of the line segment for $0 \leq x \leq 3$. [1]

The quadratic curve, with endpoints $(-2,\ 0)$ and $(0,\ 0)$, has the same gradient at $(0,\ 0)$ as the line segment.

(b) Find the equation of the curve between $(-2,\ 0)$ and $(0,\ 0)$. [3]

The second quadratic curve, with endpoints $(3,\ 3)$ and $(6,\ 2)$, has the same gradient at $(3,\ 3)$ as the line segment.

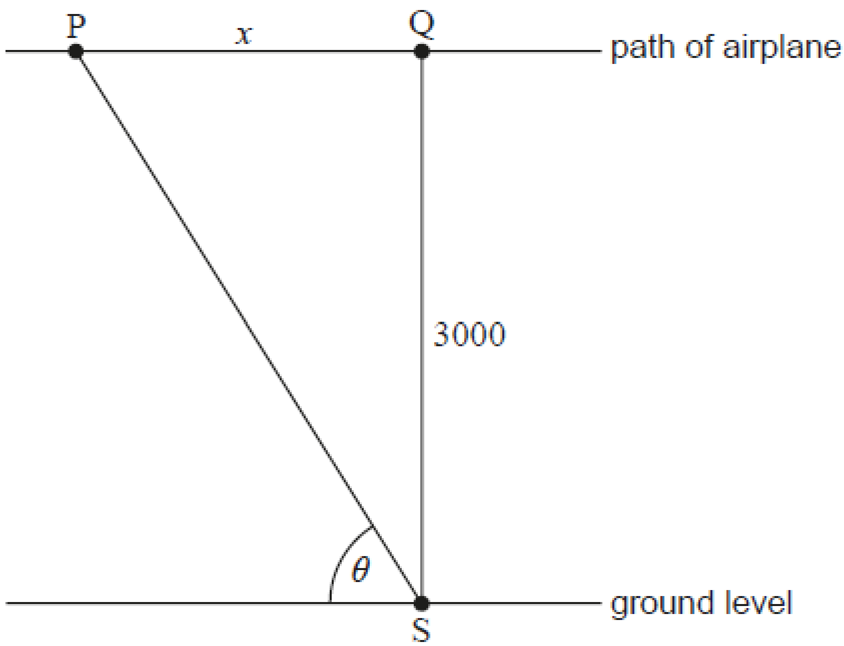
(c) Find the equation of this curve. [4]

(d) Write down f as a piecewise function. [1]

23. [Maximum mark: 9] 23M.1.AHL.TZ1.17

An airplane, P, is flying at a constant altitude of $3000\,\mathrm{m}$ at a speed of $250\,\mathrm{m\,s^{-1}}$. Its path passes over a tracking station, S, at ground level. Let Q be the point $3000\,\mathrm{m}$ directly above the tracking station.

At a particular time, T, as the airplane is flying towards Q, the angle of elevation, θ, of the airplane from S is increasing at a rate of 0.075 radians per second. The distance from Q to P is given by x.



(a) Use related rates to show that, at time T, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{10\,000}{3}$. [2]

(b) Find $x(\theta)$, x as a function of θ. [1]

(c) Find an expression for $\frac{\mathrm{d}x}{\mathrm{d}\theta}$ in terms of $\sin\theta$. [3]

(d) Hence find the horizontal distance from the station to the plane at time T. [3]

24. [Maximum mark: 5] 23M.1.AHL.TZ2.12

A spherical balloon is being inflated such that its volume is increasing at a rate of $15\ \mathrm{cm}^3\ \mathrm{s}^{-1}$.

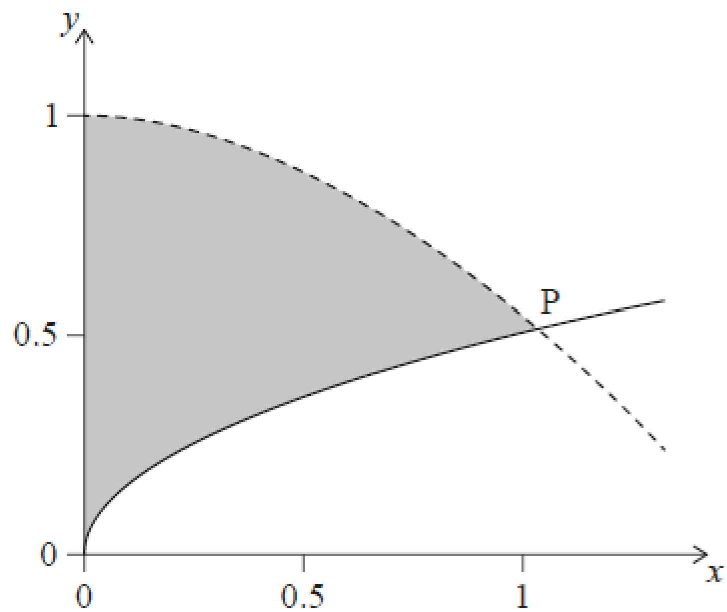
(a) Find the radius of the balloon when its volume is $288\pi\ \mathrm{cm}^3$. [2]

(b) Hence or otherwise, find the rate of change of the radius at this instant. [3]

25. [Maximum mark: 9] 23M.1.AHL.TZ2.16

The following diagram shows parts of the curves of $y = \cos x$ and $y = \frac{\sqrt{x}}{2}$.

P is the point of intersection of the two curves.



(a) Use your graphic display calculator to find the coordinates of P. [2]

The shaded region is rotated 360° **about the y-axis** to form a volume of revolution V.

(b) Express V as the sum of two definite integrals. [5]

(c) Hence find the value of V. [2]

26. [Maximum mark: 16] 23M.2.AHL.TZ2.3

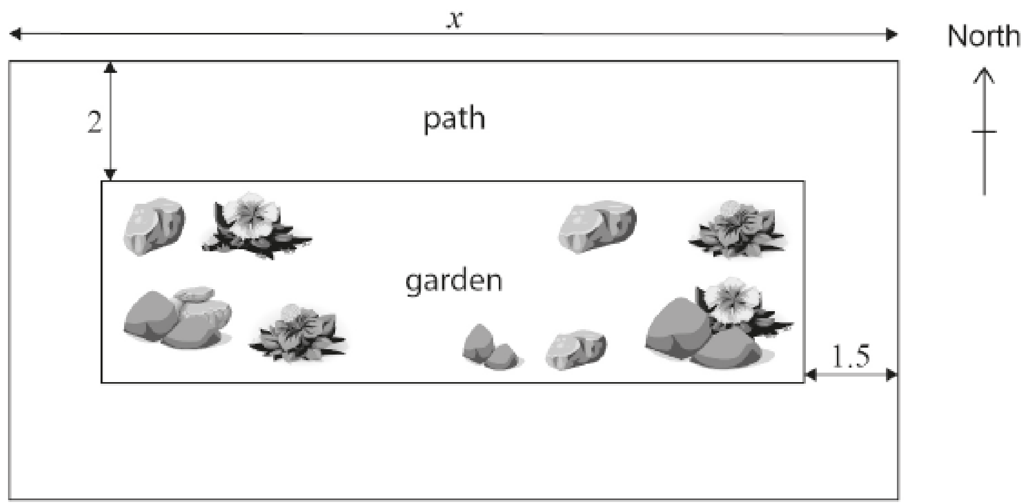
A particular park consists of a rectangular garden, of area $A\,\mathrm{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\mathrm{m}^2$.

The width of the path at the north and south side of the park is $2\,\mathrm{m}$.

The width of the path at the west and east side of the park is $1.5\,\mathrm{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale



(a) Show that $A = 1212 - 4x - \frac{3600}{x}$. [5]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\mathrm{m}^2$. [4]

(c) Find an expression for $\frac{\mathrm{d}A}{\mathrm{d}x}$. [3]

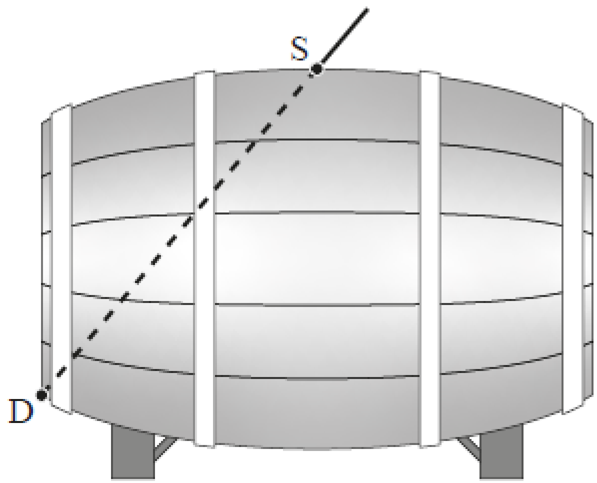
(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden. [2]

(e) Find the maximum possible area of the garden. [2]

27. [Maximum mark: 26] 23M.3.AHL.TZ1.1

In this question you will use a historic method of calculating the cost of a barrel of wine to determine which shape of barrel gives the best value for money.

In Austria in the 17th century, one method for measuring the volume of a barrel of wine, and hence determining its cost, was by inserting a straight stick into a hole in the side, as shown in the following diagram, and measuring the length SD. The longer the length, the greater the cost to the customer.



Let SD be d metres and the cost be C gulden (the local currency at the time). When the length of SD was 0.5 metres, the cost was 0.80 gulden.

(a) Given that C was directly proportional to d, find an equation for C in terms of d. [3]

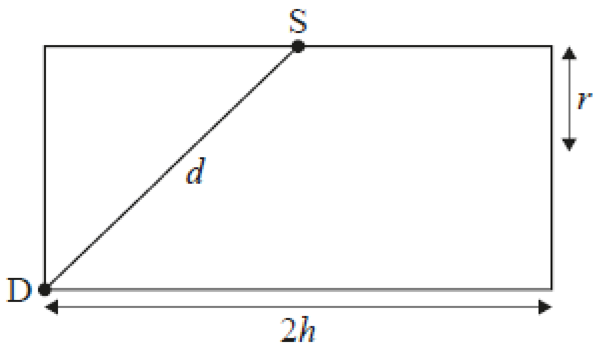
A particular barrel of wine cost 0.96 gulden.

(b) Show that $d = 0.6$. [1]

This method of determining the cost was noticed by a mathematician, Kepler, who decided to try to calculate the dimensions of a barrel which would give the maximum volume of wine for a given length SD.

Initially he modelled the barrel as a cylinder, with S at the midpoint of one side. He took the length of the cylinder as $2h$ metres and its radius as r metres, as shown in the following diagram of the cross-section.

diagram not to scale



(c) Find an expression for r^2 in terms of d and h. [3]

Let the volume of this barrel be $V\,\mathrm{m}^3$.

(d) Show that $V = \frac{\pi}{2}\left(d^2h - h^3\right)$. [2]

The remainder of this question considers the shape of barrel that gives the best value when $d = 0.6$.

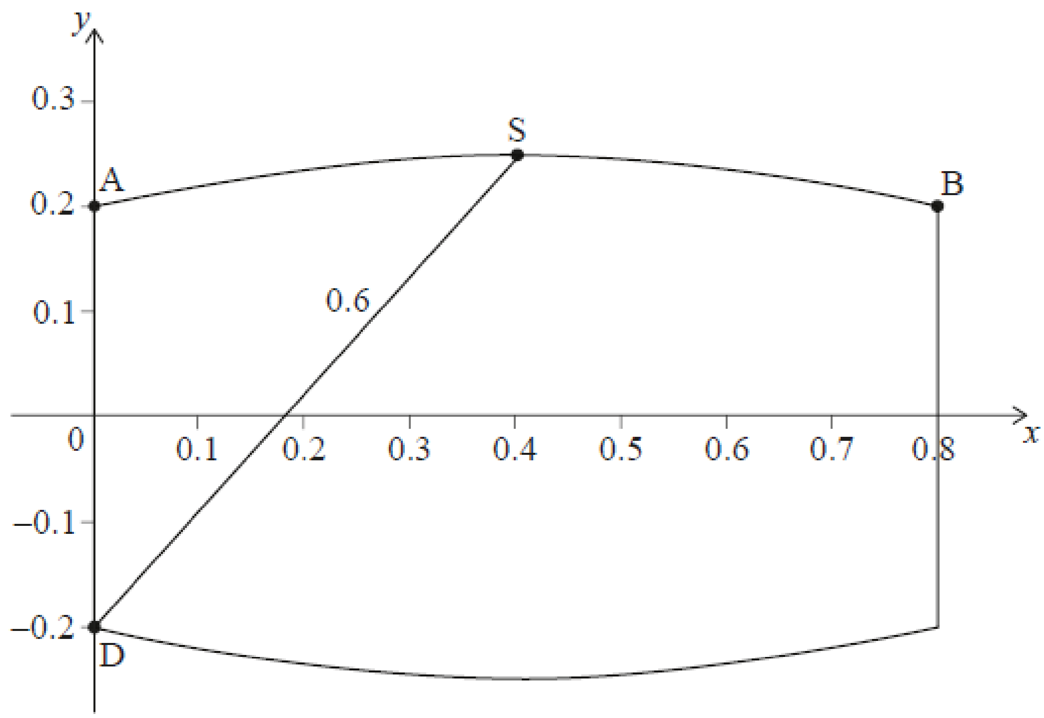
(e.i) Use the formula from part (d) to find the volume of this barrel when $h = 0.4$. [2]

(e.ii) Use differentiation to show that $h = \sqrt{0.12}$ when $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$. [3]

(e.iii) Given that this value of h maximizes the volume, find the largest possible volume of this barrel. [2]

Kepler then considered a non-cylindrical barrel whose base and lid are circles with radius $0.2\,\mathrm{m}$ and whose length is $0.8\,\mathrm{m}$.

He modelled the curved surface of this barrel by rotating a quadratic curve, ASB, with equation $y = ax^2 + bx + c, 0 \le x \le 0.8$, about the x-axis. The origin of the coordinate system is at the centre of one of the circular faces as shown in the following diagram. S is at the vertex of the quadratic curve and $\mathrm{SD} = 0.6$.



Kepler wished to find out if his barrel would give him more wine than any cylindrical barrel with $d = 0.6$.

The coordinates of A and B are $(0,\ 0.2)$ and $(0.8,\ 0.2)$ respectively.

(f) Find the equation of the quadratic curve, ASB. [6]

(g) Show that the volume of this barrel is greater than the maximum volume of any cylindrical barrel with $d = 0.6$. [3]

(h) State one assumption, not already given, that has been made in using these models to find the shape of the barrel that gives the best value. [1]

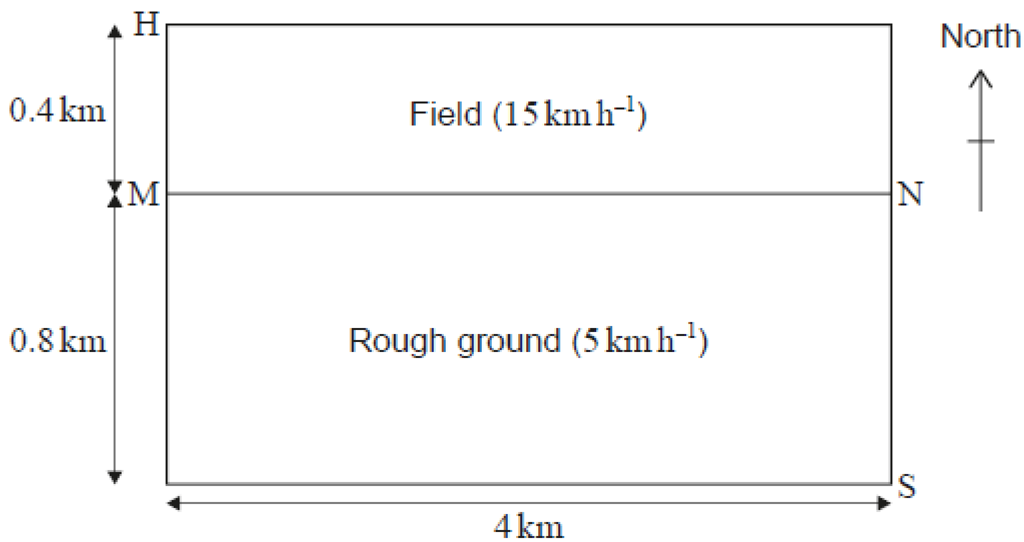
28. [Maximum mark: 26] 23M.3.AHL.TZ2.1

This question considers the optimal route between two points, separated by several regions where different speeds are possible.

Huw lives in a house, H, and he attends a school, S, where H and S are marked on the following diagram. The school is situated $1.2\ \mathrm{km}$ south and $4\ \mathrm{km}$ east of

Huw's house. There is a boundary $[\mathrm{MN}]$, going from west to east, $0.4\ \mathrm{km}$ south of his house. The land north of $[\mathrm{MN}]$ is a field over which Huw runs at 15 kilometres per hour ($\mathrm{km\ h^{-1}}$). The land south of $[\mathrm{MN}]$ is rough ground over which Huw walks at $5\ \mathrm{km\ h^{-1}}$. The two regions are shown in the following diagram.

diagram not to scale



(a) Huw travels in a straight line from H to S. Calculate the time that Huw takes to complete this journey. Give your answer correct to the nearest minute. [6]

(b) Huw realizes that his journey time could be reduced by taking a less direct route. He therefore defines a point P on $[\mathrm{MN}]$ that is x km east of M. Huw decides to run from H to P and then walk from P to S. Let $T(x)$ represent the time, in hours, taken by Huw to complete the journey along this route.

(b.i) Show that $T\left(x\right) = \frac{\sqrt{0.4^2+x^2}+3\sqrt{0.8^2+(4-x)^2}}{15}$. [3]

(b.ii) Sketch the graph of $y = T(x)$. [2]

(b.iii) Hence determine the value of x that minimizes $T(x)$. [1]

(b.iv) Find by how much Huw's journey time is reduced when he takes this optimal route, compared to travelling in a straight line from H to S. Give your answer correct to the nearest minute. [2]

(c.i) Determine an expression for the derivative $T\prime(x)$. [3]

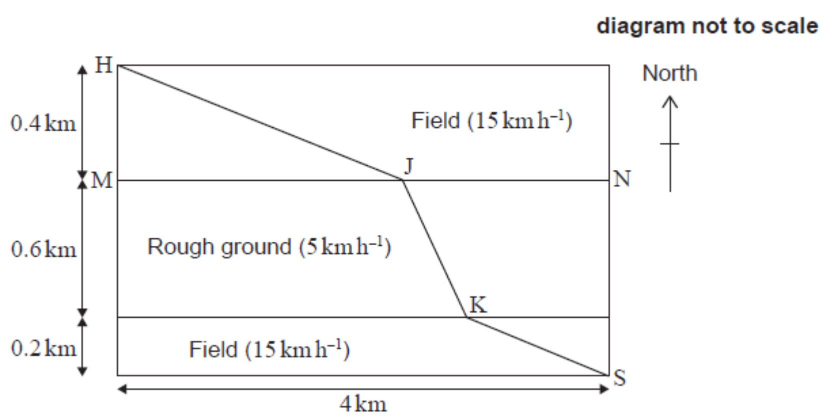
(c.ii) Hence show that $T(x)$ is minimized when

$$\frac{x}{\sqrt{0.16+x^2}} = \frac{3(4-x)}{\sqrt{0.64+(4-x)^2}}.$$ [1]

(c.iii) For the optimal route, verify that the equation in part (c)(ii) satisfies the following result:

$$\frac{\cos \mathrm{H\hat{P}M}}{\cos \mathrm{S\hat{P}N}} = \frac{\text{speed over field}}{\text{speed over rough ground}}.$$ [2]

(d) The owner of the rough ground converts the southern quarter into a field over which Huw can run at 15 km h^{-1}. The following diagram shows the optimal route, HJKS, in this new situation. You are given that $[\mathrm{HJ}]$ is parallel to $[\mathrm{KS}]$.



Using a similar result to that given in part (c)(iii), at the point J, determine MJ. [6]

29. [Maximum mark: 9] 22N.1.AHL.TZ0.14

A particle moves such that its velocity, v metres per second, at time t seconds, is given by $v = t\ \sin\left(t^2\right)$.

(a) Find an expression for the acceleration of the particle. [2]

(b) Hence, or otherwise, find its greatest acceleration for $0 \leq t \leq 8$. [2]

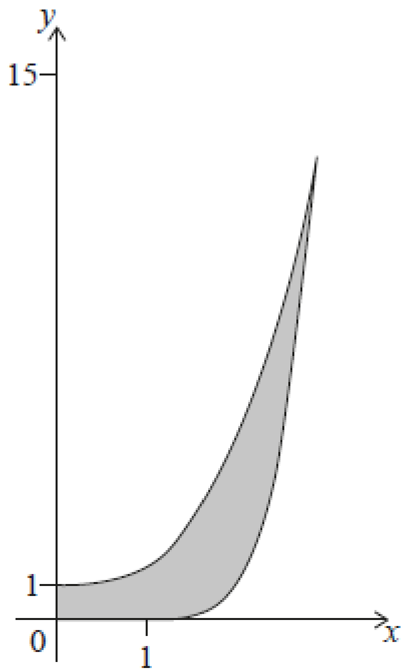
The particle starts at the origin.

(c) Find an expression for the displacement of the particle. [3]

(d) Hence show that the particle never has a negative displacement. [2]

30. [Maximum mark: 13] 22N.2.AHL.TZ0.5

Adesh is designing a glass. The glass has an inner surface and an outer surface. Part of the cross section of his design is shown in the following graph, where the shaded region represents the glass. The two surfaces meet at the top of the glass. 1 unit represents $1\,\mathrm{cm}$.



The inner surface is modelled by $f(x) = \frac{1}{2}x^3 + 1$ for $0 \le x \le p$.

The outer surface is modelled by $g(x)\begin{cases} 0 & \text{for } 0 \le x < 1 \\ (x-1)^4 & \text{for } 1 \le x \le p \end{cases}$.

(a) Find the value of p. [2]

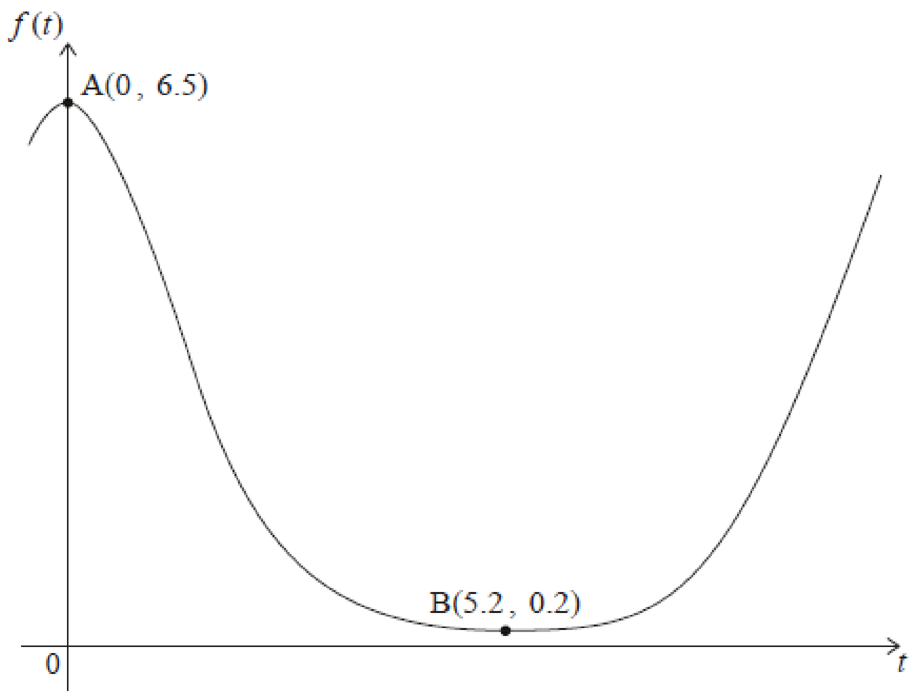
The glass design is finished by rotating the shaded region in the diagram through 360° about the y-axis.

(b) Find the volume of liquid that can be contained inside the finished glass. [5]

(c) Find the volume of the region between the two surfaces of the finished glass. [6]

31. [Maximum mark: 8] 22M.1.AHL.TZ1.17

A function f is of the form $f(t) = p\mathrm{e}^{q\cos(rt)},\ p,\ q,\ r \in \mathbb{R}^+$. Part of the graph of f is shown.



The points A and B have coordinates $\mathrm{A}(0,\ 6.5)$ and $\mathrm{B}(5.2,\ 0.2)$, and lie on f.

The point A is a local maximum and the point B is a local minimum.

Find the value of p, of q and of r. [8]

32. [Maximum mark: 7] 22M.1.AHL.TZ1.16

The wind chill index W is a measure of the temperature, in $^{\circ}\mathrm{C}$, felt when taking into account the effect of the wind.

When Frieda arrives at the top of a hill, the relationship between the wind chill index and the speed of the wind v in kilometres per hour $\left(\mathrm{km\,h^{-1}}\right)$ is given by the equation

$$W = 19.34 - 7.405v^{0.16}$$

(a) Find an expression for $\frac{\mathrm{d}W}{\mathrm{d}v}$. [2]

(b) When Frieda arrives at the top of a hill, the speed of the wind is 10 kilometres per hour and increasing at a rate of $5\,\text{km}\,\text{h}^{-1}\,\text{minute}^{-1}$.

Find the rate of change of W at this time. [5]

33. [Maximum mark: 6] 22M.1.AHL.TZ1.8

Consider the curve $y = 2x(4 - \mathrm{e}^{x})$.

(a.i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

(a.ii) Find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. [2]

(b) The curve has a point of inflexion at $(a,\ b)$.

Find the value of a. [2]

34. [Maximum mark: 8] 22M.1.AHL.TZ1.14

(a.i) Expand $\left(\frac{1}{u} + 1\right)^2$. [1]

(a.ii) Find $\int \left(\frac{1}{(x+2)} + 1\right)^2 \mathrm{d}\,x$. [3]

The region bounded by $y = \frac{1}{(x+2)} + 1,\ x = 0,\ x = 2$ and the x-axis is rotated through 2π about the x-axis to form a solid.

(b) Find the volume of the solid formed. Give your answer in the form $\frac{\pi}{4}(a + b\ln(c))$, where $a,\ b,\ c \in \mathbb{Z}$. [4]

35. [Maximum mark: 4] 22M.1.AHL.TZ2.14

The shape of a vase is formed by rotating a curve about the y-axis.

The vase is $10\,\text{cm}$ high. The internal radius of the vase is measured at $2\,\text{cm}$ intervals along the height:

Height (cm)	Radius (cm)
0	4
2	6
4	8
6	7
8	3
10	5

Use the trapezoidal rule to estimate the volume of water that the vase can hold. [4]

36. [Maximum mark: 21] 22M.2.AHL.TZ2.6

At an archery tournament, a particular competition sees a ball launched into the air while an archer attempts to hit it with an arrow.

The path of the ball is modelled by the equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} u_x \\ u_y - 5t \end{pmatrix}$$

where x is the horizontal displacement from the archer and y is the vertical displacement from the ground, both measured in metres, and t is the time, in seconds, since the ball was launched.

- u_x is the horizontal component of the initial velocity
- u_y is the vertical component of the initial velocity.

In this question both the ball and the arrow are modelled as single points. The ball is launched with an initial velocity such that $u_x = 8$ and $u_y = 10$.

(a.i) Find the initial speed of the ball. [2]

(a.ii) Find the angle of elevation of the ball as it is launched. [2]

(b) Find the maximum height reached by the ball. [3]

(c) Assuming that the ground is horizontal and the ball is not hit by the arrow, find the x coordinate of the point where the ball lands. [3]

(d) For the path of the ball, find an expression for y in terms of x. [3]

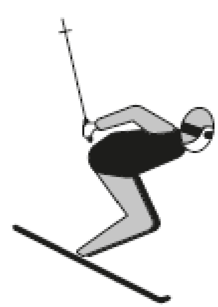
An archer releases an arrow from the point $(0,\ 2)$. The arrow is modelled as travelling in a straight line, in the same plane as the ball, with speed $60\,\mathrm{m\,s}^{-1}$ and an angle of elevation of 10°.

(e) Determine the two positions where the path of the arrow intersects the path of the ball. [4]

(f) Determine the time when the arrow should be released to hit the ball before the ball reaches its maximum height. [4]

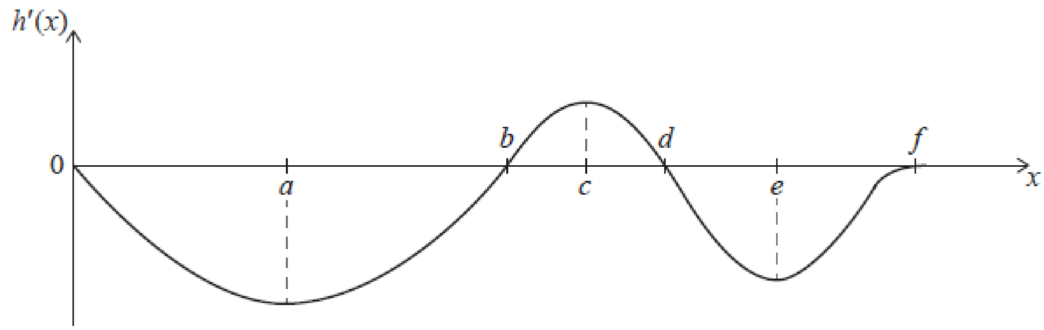
37. [Maximum mark: 5] 21N.1.AHL.TZ0.8

Juri skis from the top of a hill to a finishing point at the bottom of the hill. She takes the shortest route, heading directly to the finishing point (F).



Let $h(x)$ define the height of the hill above F at a horizontal distance x from the starting point at the top of the hill.

The graph of the **derivative** of $h(x)$ is shown below. The graph of $h\prime(x)$ has local minima and maxima when x is equal to $a,\ c$ and e. The graph of $h\prime(x)$ intersects the x-axis when x is equal to $b,\ d$, and f.

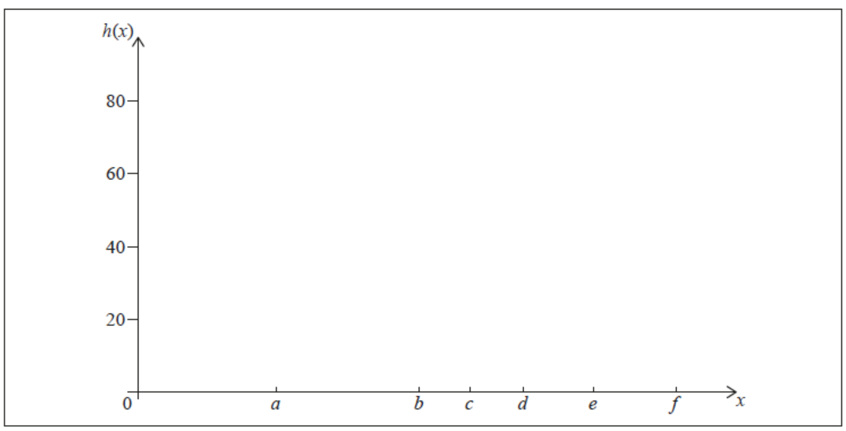


(a.i) Identify the x value of the point where $|h\prime(x)|$ has its maximum value. [1]

(a.ii) Interpret this point in the given context. [1]

(b) Juri starts at a height of 60 metres and finishes at F, where $x = f$.

Sketch a possible diagram of the hill on the following pair of coordinate axes.



[3]

38. [Maximum mark: 7] 21N.1.AHL.TZ0.17

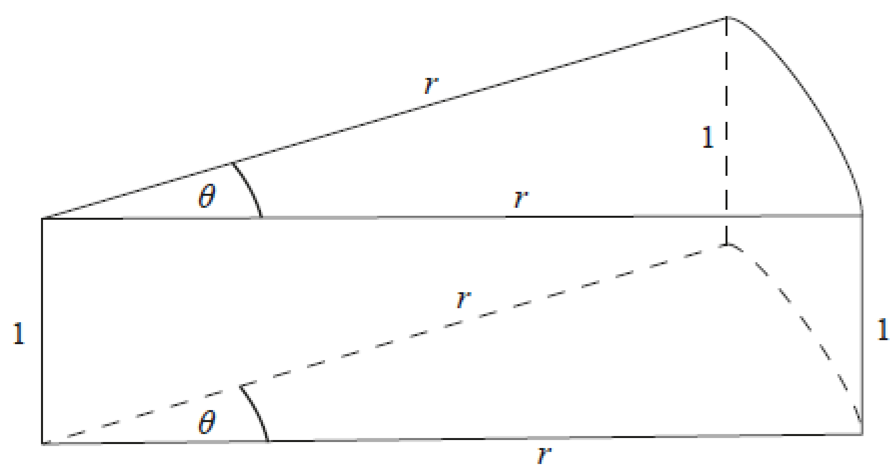
The sides of a bowl are formed by rotating the curve $y = 6\ \ln\ x,\ 0 \le y \le 9$, about the y-axis, where x and y are measured in centimetres. The bowl contains water to a height of h cm.

(a) Show that the volume of water, V, in terms of h is $V = 3\pi\left(e^{\frac{h}{3}} - 1\right)$. [5]

(b) Hence find the maximum capacity of the bowl in cm^3. [2]

39. [Maximum mark: 9] 21N.1.AHL.TZ0.15

The following diagram shows a frame that is made from wire. The total length of wire is equal to 15 cm. The frame is made up of two identical sectors of a circle that are parallel to each other. The sectors have angle θ radians and radius r cm. They are connected by 1 cm lengths of wire perpendicular to the sectors. This is shown in the diagram below.



(a) Show that $r = \frac{6}{2+\theta}$. [2]

The faces of the frame are covered by paper to enclose a volume, V.

(b.i) Find an expression for V in terms of θ. [2]

(b.ii) Find the expression $\frac{dV}{d\theta}$. [3]

(b.iii) Solve algebraically $\frac{dV}{d\theta} = 0$ to find the value of θ that will maximize the volume, V.

[2]

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