Kinematics 3 [83 marks]

1. [Maximum mark: 7] 22M.1.AHL.TZ2.16

The position vector of a particle, P, relative to a fixed origin O at time t is given by

$$\overrightarrow{\mathrm{OP}} = \begin{pmatrix} \sin\left(t^2\right) \\ \cos\left(t^2\right) \end{pmatrix}.$$

(a) Find the velocity vector of P. [2]

Markscheme

attempt at chain rule ***(M1)***

$$\left(\boldsymbol{v} = \frac{\mathrm{d}OP}{\mathrm{d}t} =\right) \begin{pmatrix} 2t\cos t^2 \\ -2t\sin t^2 \end{pmatrix}$$ ***A1***

[2 marks]

(b) Show that the acceleration vector of P is never parallel to the position vector of P. [5]

Markscheme

attempt at product rule ***(M1)***

$$\boldsymbol{a} = \begin{pmatrix} 2\cos t^2 - 4t^2\sin t^2 \\ -2\sin t^2 - 4t^2\cos t^2 \end{pmatrix}$$ ***A1***

METHOD 1

let $S = \sin t^2$ and $C = \cos t^2$

finding $\cos\theta$ using

$$\boldsymbol{a}\cdot\overrightarrow{\mathrm{OP}} = 2SC - 4t^2S^2 - 2SC - 4t^2C^2 = -4t^2$$ ***M1***

$$\left|\overrightarrow{\mathrm{OP}}\right| = 1$$

$$|\boldsymbol{a}| = \sqrt{\left(2C - 4t^2S\right)^2 + \left(-2S - 4t^2C\right)^2}$$

$$= \sqrt{4+16t^4} > 4t^2$$

if θ is the angle between them, then

$$\cos\theta = -\frac{4t^2}{\sqrt{4+16t^4}}$$ ***A1***

so $-1 < \cos\theta < 0$ therefore the vectors are never parallel ***R1***

METHOD 2

solve

$$\begin{pmatrix} 2\cos t^2 - 4t^2 \sin t^2 \\ -2\sin t^2 - 4t^2\cos t^2 \end{pmatrix} = k\begin{pmatrix} \sin t^2 \\ \cos t^2 \end{pmatrix}$$ ***M1***

then

$$(k =)\frac{2\cos t^2 - 4t^2\sin t^2}{\sin t^2} = \frac{-2\sin t^2 - 4t^2\cos t^2}{\cos t^2}$$

Note: Condone candidates not excluding the division by zero case here. Some might go straight to the next line.

$$2\cos^2 t^2 - 4t^2\cos t^2\sin t^2 = -2\sin^2 t^2 - 4t^2\cos t^2\sin t^2$$

$$2\cos^2 t^2 + 2\sin^2 t^2 = 0$$

$$2 = 0$$ ***A1***

this is never true so the two vectors are never parallel ***R1***

METHOD 3

embedding vectors in a 3d space and taking the cross product: ***M1***

$$\begin{pmatrix} \sin t^2 \\ \cos t^2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2\cos t^2 - 4t^2\sin t^2 \\ -2\sin t^2 - 4t^2\cos t^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2\sin^2 t^2 - 4t^2\cos t^2\sin t^2 - 2\cos^2 t^2 + 4t^2\cos t^2\sin t^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$$ ***A1***

since the cross product is never zero, the two vectors are never parallel ***R1***

[5 marks]

2. [Maximum mark: 18] 21M.2.AHL.TZ1.6

An ice-skater is skating such that her position vector when viewed from above at time t seconds can be modelled by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\mathrm{e}^{bt}\cos t \\ a\mathrm{e}^{bt}\sin t \end{pmatrix}$$

with respect to a rectangular coordinate system from a point O, where the non-zero constants a and b can be determined. All distances are in metres.

(a) Find the velocity vector at time t. [3]

Markscheme

use of product rule *(M1)*

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}abe^{bt}\cos t - ae^{bt}\sin t\\ abe^{bt}\sin t + ae^{bt}\cos t\end{pmatrix}$$ ***A1A1***

[3 marks]

(b) Show that the magnitude of the velocity of the ice-skater at time t is given by

$a\mathrm{e}^{bt}\sqrt{(1+b^2)}$. [4]

Markscheme

$|\boldsymbol{v}|^2 = \dot{x}^2 + \dot{y}^2 = \left[abe^{bt}\cos t - ae^{bt}\sin t\right]^2 + \left[abe^{bt}\sin t + ae^{bt}\cos t\right]^2$ ***M1***

Note: It is more likely that an expression for $|\boldsymbol{v}|$ is seen.
$\sqrt{\dot{x}^2+\dot{y}^2}$ is not sufficient to award the ***M1***, their part (a) must be substituted.

$= \left[a^2\sin^2 t - 2a^2b\sin t\cos t + a^2b^2\cos^2 t + a^2\cos^2 t + 2a^2b\sin t\cos t + a^2b^2\sin^2 t\right]\mathrm{e}^{2bt}$ ***A1***

use of $\sin^2 t + \cos^2 t = 1$ within a factorized expression that leads to the final answer ***M1***

$= a^2(b^2+1)\mathrm{e}^{2bt}$ ***A1***

magnitude of velocity is $a\mathrm{e}^{bt}\sqrt{(1+b^2)}$ ***AG***

[4 marks]

At time $t=0$, the displacement of the ice-skater is given by $\begin{pmatrix}5\\0\end{pmatrix}$ and the velocity of the ice-skater is given by $\begin{pmatrix}-3.5\\5\end{pmatrix}$.

(c) Find the value of a and the value of b. [3]

Markscheme

when $t=0,\ \ ae^{bt}\cos t = 5$

$a = 5$ ***A1***

$abe^{bt}\cos t - ae^{bt}\sin t = -3.5$ ***(M1)***

$b = -0.7$ ***A1***

Note: Use of $a\mathrm{e}^{bt}\sqrt{(1+b^2)}$ result from part (b) is an alternative approach.

[3 marks]

(d) Find the magnitude of the velocity of the ice-skater when $t = 2$. [2]

Markscheme

$5\,\mathrm{e}^{-0.7\times 2}\sqrt{\left(1+(-0.7)^2\right)}$ ***(M1)***

$1.51\ \ (1.50504\ldots)$ ***A1***

[2 marks]

(e) At a point P, the ice-skater is skating parallel to the y-axis for the first time.

Find OP. [6]

Markscheme

$\dot{x} = 0$ ***(M1)***

$a\,\mathrm{e}^{bt}(b\cos t - \sin t) = 0$

$\tan t = b$

$t = 2.53\ \ (2.53086\ldots)$ ***(A1)***

correct substitution of their t to find x or y ***(M1)***
$x = -0.697\ \ (-0.696591\ldots)$ **and** $y = 0.488\ \ (0.487614\ldots)$ ***(A1)***

use of Pythagoras / distance formula ***(M1)***
$\mathrm{OP} = 0.850\ \mathrm{m}\ \ (0.850297\ldots)$ ***A1***

[6 marks]

3. [Maximum mark: 20] 22N.2.AHL.TZ0.7

The position vector of a particle at time t is given by $\boldsymbol{r} = 3\cos(3t)\boldsymbol{i} + 4\sin(3t)\boldsymbol{j}$. Displacement is measured in metres and time is measured in seconds.

(a.i) Find an expression for the velocity of the particle at time t. [2]

Markscheme

use of chain rule ***(M1)***

$\boldsymbol{v} = -9\sin(3t)\boldsymbol{i} + 12\cos(3t)\boldsymbol{j}$ ***A1***

Note: Award ***(M1)*** for at least one correct term seen but condone omission of $\boldsymbol{i}$ or $\boldsymbol{j}$.

[2 marks]

(a.ii) Hence find the speed when $t = 3$. [2]

Markscheme

$|\boldsymbol{v}| = \sqrt{(-9\sin(9))^2 + (12\cos(9))^2}$ ***(M1)***

$= 11.5\,\text{ms}^{-1}\ \ (11.5455\ldots)$ ***A1***

[2 marks]

(b.i) Find an expression for the acceleration of the particle at time t. [1]

Markscheme

$\boldsymbol{a} = -27\cos(3t)\boldsymbol{i} - 36\sin(3t)\boldsymbol{j}$ ***A1***

[1 mark]

(b.ii) Hence show that the acceleration is always directed towards the origin. [3]

Markscheme

$\boldsymbol{a} = -9(3\cos(3t)\boldsymbol{i} - 4\sin(3t)\boldsymbol{j})$ ***M1***

$\boldsymbol{a} = -9\boldsymbol{r}$ (where $\boldsymbol{r}$ is a position vector from the origin) ***A1***

$\boldsymbol{a}$ is in opposite direction to the position vector ***R1***

hence $\boldsymbol{a}$ is always directed towards the origin ***AG***

[3 marks]

The position vector of a second particle is given by $\boldsymbol{r} = -4\sin(4t)\boldsymbol{i} + 3\cos(4t)\boldsymbol{j}$.

(c) For $0 \le t \le 10$, find the time when the two particles are closest to each other. [5]

Markscheme

relative position $\boldsymbol{d} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ ***(M1)***

distance between particles $= |\boldsymbol{d}|\ \ (= |\boldsymbol{r}_2 - \boldsymbol{r}_1|)$ ***(M1)***

$|\boldsymbol{d}| = \sqrt{(-4\sin(4t) - 3\cos(3t))^2 + (3\cos(4t) - 4\sin(3t))^2}$ ***(A1)***

minimum value of $|\boldsymbol{d}|$ when $t = 4.71\,(\text{s})\ \left(4.71238\ldots,\ \frac{3\pi}{2}\right)$ ***(M1)A1***

[5 marks]

At time k, where $0 < k < 1.5$, the second particle is moving parallel to the first particle.

(d.i) Find the value of k. [5]

Markscheme

for 2nd particle, $\boldsymbol{v} = -16\cos(4t)\boldsymbol{i} - 12\sin(4t)\boldsymbol{j}$ ***(A1)***

EITHER

consider the gradient of either $\boldsymbol{v}$ ***(M1)***

$m_1 = \frac{12\cos(3t)}{9\sin(3t)}$ and $m_2 = \frac{12\sin(4t)}{16\cos(4t)}$ ***(A1)***

attempt to solve $m_1 = m_2$ ***(M1)***

OR

vectors are parallel therefore one is a multiple of the other, $\boldsymbol{v}_2 = l\boldsymbol{v}_1$ ***(M1)***

$(l =)\ \frac{16\cos(4t)}{9\sin(3t)} = -\frac{\sin(4t)}{\cos(3t)}$ ***(A1)***

attempt to solve ***(M1)***

THEN

$t = 1.30\,(\mathrm{s})\ \ (1.30135\ldots)$ ***A1***

[5 marks]

(d.ii) At time k, show that the two particles are moving in the opposite direction. [2]

Markscheme

EITHER

at $t = 1.30$, $\boldsymbol{v}_1 = 6.22\boldsymbol{i} - 8.68\boldsymbol{j}$ and $\boldsymbol{v}_2 = -7.57\boldsymbol{i} + 10.6\boldsymbol{j}$ ***A1***

OR

$l = -1.22$ (following second method in part (d)(i)) ***A1***

THEN

$\boldsymbol{v}_2$ is a negative multiple of $\boldsymbol{v}_1$ $(\boldsymbol{v}_2 = -1.22\boldsymbol{v}_1)$ ***A1***

the two particles are moving in the opposite direction ***AG***

[2 marks]

4. [Maximum mark: 21] 22M.2.AHL.TZ2.6

At an archery tournament, a particular competition sees a ball launched into the air while an archer attempts to hit it with an arrow.

The path of the ball is modelled by the equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} u_x \\ u_y - 5t \end{pmatrix}$$

where x is the horizontal displacement from the archer and y is the vertical displacement from the ground, both measured in metres, and t is the time, in seconds, since the ball was launched.

- u_x is the horizontal component of the initial velocity
- u_y is the vertical component of the initial velocity.

In this question both the ball and the arrow are modelled as single points. The ball is launched with an initial velocity such that $u_x = 8$ and $u_y = 10$.

(a.i) Find the initial speed of the ball. [2]

Markscheme

$\sqrt{10^2 + 8^2}$ ***(M1)***

$= 12.8 \ \left(12.8062\ldots,\ \sqrt{164}\right)\ \left(\text{ms}^{-1}\right)$ ***A1***

[2 marks]

(a.ii) Find the angle of elevation of the ball as it is launched. [2]

Markscheme

$\tan^{-1}\left(\frac{10}{8}\right)$ ***(M1)***

$= 0.896$ **OR** 51.3 $(0.896055\ldots$ **OR** $51.3401\ldots^\circ)$ ***A1***

Note: Accept 0.897 or 51.4 from use of $\arcsin\left(\frac{10}{12.8}\right)$.

[2 marks]

(b) Find the maximum height reached by the ball. [3]

Markscheme

$y = t(10 - 5t)$ ***(M1)***

Note: The ***M1*** might be implied by a correct graph or use of the correct equation.

METHOD 1 – graphical Method

sketch graph ***(M1)***

Note: The ***M1*** might be implied by correct graph or correct maximum (eg $t = 1$).

max occurs when $y = 5\,\mathrm{m}$ ***A1***

METHOD 2 – calculus

differentiating and equating to zero ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = 10 - 10t = 0$

$t = 1$

$y(= 1(10 - 5)) = 5\,\mathrm{m}$ ***A1***

METHOD 3 – symmetry

line of symmetry is $t = 1$ ***(M1)***

$y(= 1(10 - 5)) = 5\,\mathrm{m}$ ***A1***

[3 marks]

(c) Assuming that the ground is horizontal and the ball is not hit by the arrow, find the x coordinate of the point where the ball lands. [3]

Markscheme

attempt to solve $t(10 - 5t) = 0$ ***(M1)***

$t = 2$ (or $t = 0$) ***(A1)***

$x\ (= 5 + 8 \times 2) =\ 21\,\mathrm{m}$ ***A1***

Note: Do not award the final ***A1*** if $x = 5$ is also seen.

[3 marks]

(d) For the path of the ball, find an expression for y in terms of x. [3]

Markscheme

METHOD 1

$t = \frac{x-5}{8}$ ***M1A1***

$y = \left(\frac{x-5}{8}\right)\left(10 - 5 \times \frac{x-5}{8}\right)$ ***A1***

METHOD 2

$y = k(x-5)(x-21)$ ***A1***

when $x = 13,\ y = 5$ so $k = \frac{5}{(13-5)(13-21)} = -\frac{5}{64}$ ***M1A1***

$\left(y = -\frac{5}{64}(x-5)(x-21)\right)$

METHOD 3

if $y = ax^2 + bx + c$

$0 = 25a + 5b + c$
$5 = 169a + 13b + c$
$0 = 441a + 21b + c$ ***M1A1***

solving simultaneously, $a = -\frac{5}{64},\ b = \frac{130}{64},\ c = -\frac{525}{64}$ ***A1***

$(y = -\frac{5}{64}x^2 + \frac{130}{64}x - \frac{525}{64})$

METHOD 4

use quadratic regression on $(5,\ 0),\ (13,\ 5),\ (21,\ 0)$ ***M1A1***

$y = -\frac{5}{64}x^2 + \frac{130}{64}x - \frac{525}{64}$ ***A1***

Note: Question asks for expression; condone omission of "$y =$".

[3 marks]

An archer releases an arrow from the point $(0,\ 2)$. The arrow is modelled as travelling in a straight line, in the same plane as the ball, with speed $60\,\mathrm{m\,s^{-1}}$ and an angle of elevation of 10°.

(e) Determine the two positions where the path of the arrow intersects the path of the ball. [4]

Markscheme

trajectory of arrow is $y = x\tan 10 + 2$ ***(A1)***

intersecting $y = x\tan 10 + 2$ and their answer to (d) ***(M1)***

$(8.66,\ 3.53)\ \ ((8.65705\ldots,\ 3.52647\ldots))$ ***A1***

$(15.1,\ 4.66)\quad ((15.0859\ldots,\ 4.66006\ldots))$ ***A1***

[4 marks]

(f) Determine the time when the arrow should be released to hit the ball before the ball reaches its maximum height. [4]

Markscheme

when $x_{\text{target}} = 8.65705\ldots,\ \ t_{\text{target}} = \frac{8.65705\ldots - 5}{8} = 0.457132\ldots\text{ s}$ ***(A1)***

attempt to find the distance from point of release to intersection ***(M1)***

$\sqrt{8.65705\ldots^2 + (3.52647\ldots - 2)^2}\ \ (= 8.79060\ldots\text{ m})$

time for arrow to get there is $\frac{8.79060\ldots}{60} = 0.146510\ldots\text{ s}$ ***(A1)***

so the arrow should be released when

$t = 0.311\,(\text{s})\ \ (0.310622\ldots\,(\text{s}))$ ***A1***

[4 marks]

5. [Maximum mark: 17] EXN.2.AHL.TZ0.7

A ball is attached to the end of a string and spun horizontally. Its position relative to a given point, O, at time t seconds, $t \geq 0$, is given by the equation

$\boldsymbol{r} = \begin{pmatrix} 1.5\cos\left(0.1t^2\right) \\ 1.5\sin\left(0.1t^2\right) \end{pmatrix}$ where all displacements are in metres.

(a) Show that the ball is moving in a circle with its centre at O and state the radius of the circle. [4]

Markscheme

*This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$|\boldsymbol{r}| = \sqrt{1.5^2\cos^2\left(0.1t^2\right) + 1.5^2\sin^2\left(0.1t^2\right)}$ **M1**

$= 1.5$ as $\sin^2\theta + \cos^2\theta = 1$ **R1**

Note: use of the identity needs to be explicitly stated.

Hence moves in a circle as displacement from a fixed point is constant. **R1**

Radius $= 1.5\,\text{m}$ **A1**

[4 marks]

(b.i) Find an expression for the velocity of the ball at time t. [2]

Markscheme

$$\boldsymbol{v} = \begin{pmatrix} -0.3t \sin\left(0.1t^2\right) \\ 0.3t \cos\left(0.1t^2\right) \end{pmatrix}$$ **M1A1**

Note: M1 is for an attempt to differentiate each term

[2 marks]

(b.ii) Hence show that the velocity of the ball is always perpendicular to the position vector of the ball. [2]

Markscheme

$$\boldsymbol{v} \bullet \boldsymbol{r} = \begin{pmatrix} 1.5 \cos\left(0.1t^2\right) \\ 1.5 \sin\left(0.1t^2\right) \end{pmatrix} \bullet \begin{pmatrix} -0.3t \sin\left(0.1t^2\right) \\ 0.3t \cos\left(0.1t^2\right) \end{pmatrix}$$ **M1**

Note: M1 is for an attempt to find $\boldsymbol{v} \bullet \boldsymbol{r}$

$$= 1.5 \cos\left(0.1t^2\right) \times \left(-0.3t \sin\left(0.1t^2\right)\right) + 1.5 \sin\left(0.1t^2\right) \times 0.3t \sin\left(0.1t^2\right) = 0$$ **A1**

Hence velocity and position vector are perpendicular. **AG**

[2 marks]

(c.i) Find an expression for the acceleration of the ball at time t. [3]

Markscheme

$$\boldsymbol{a} = \begin{pmatrix} -0.3 \sin\left(0.1t^2\right) - 0.06t^2 \cos\left(0.1t^2\right) \\ 0.3 \cos\left(0.1t^2\right) - 0.06t^2 \sin\left(0.1t^2\right) \end{pmatrix}$$ **M1A1A1**

[3 marks]

The string breaks when the magnitude of the ball's acceleration exceeds $20\,\text{ms}^{-2}$.

(c.ii) Find the value of t at the instant the string breaks. [3]

Markscheme

$\left(-0.3 \sin \left(0.1t^2\right) - 0.06t^2 \cos \left(0.1t^2\right)\right)^2 + \left(0.3 \cos \left(0.1t^2\right) - 0.06t^2 \sin \left(0.1t^2\right)\right)^2 = 400$ **(M1)** **(A1)**

Note: M1 is for an attempt to equate the magnitude of the acceleration to 20.

$t = 18.3 \ (18.256\ldots)\ (\mathrm{s})$ **A1**

[3 marks]

(c.iii) How many complete revolutions has the ball completed from $t = 0$ to the instant at which the string breaks? [3]

Markscheme

Angle turned through is $0.1 \times 18.256^2 =$ **M1**

$= 33.329\ldots$ **A1**

$\frac{33.329}{2\pi}$ **M1**

$\frac{33.329}{2\pi} = 5.30\ldots$

5 complete revolutions **A1**

[4 marks]