Kinematics 3 [83 marks]

1. [Maximum mark: 7] 22M.1.AHL.TZ2.16

The position vector of a particle, P, relative to a fixed origin O at time t is given by

$$\overrightarrow{\mathrm{OP}} = \begin{pmatrix} \sin\left(t^2\right) \\ \cos\left(t^2\right) \end{pmatrix}.$$

(a) Find the velocity vector of P. [2]

(b) Show that the acceleration vector of P is never parallel to the position vector of P. [5]

2. [Maximum mark: 18] 21M.2.AHL.TZ1.6

An ice-skater is skating such that her position vector when viewed from above at time t seconds can be modelled by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\,\mathrm{e}^{bt}\cos t \\ a\,\mathrm{e}^{bt}\sin t \end{pmatrix}$$

with respect to a rectangular coordinate system from a point O, where the non-zero constants a and b can be determined. All distances are in metres.

(a) Find the velocity vector at time t. [3]

(b) Show that the magnitude of the velocity of the ice-skater at time t is given by

$$a\,\mathrm{e}^{bt}\sqrt{(1+b^2)}.$$ [4]

At time $t = 0$, the displacement of the ice-skater is given by $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and the velocity of the ice-skater is given by $\begin{pmatrix} -3.5 \\ 5 \end{pmatrix}$.

(c) Find the value of a and the value of b. [3]

(d) Find the magnitude of the velocity of the ice-skater when $t = 2$. [2]

(e) At a point P, the ice-skater is skating parallel to the y-axis for the first time.

Find OP. [6]

3. [Maximum mark: 20] 22N.2.AHL.TZ0.7

The position vector of a particle at time t is given by $\boldsymbol{r} = 3\cos(3t)\boldsymbol{i} + 4\sin(3t)\boldsymbol{j}$. Displacement is measured in metres and time is measured in seconds.

(a.i) Find an expression for the velocity of the particle at time t. [2]

(a.ii) Hence find the speed when $t = 3$. [2]

(b.i) Find an expression for the acceleration of the particle at time t. [1]

(b.ii) Hence show that the acceleration is always directed towards the origin. [3]

The position vector of a second particle is given by $\boldsymbol{r} = -4\sin(4t)\boldsymbol{i} + 3\cos(4t)\boldsymbol{j}$.

(c) For $0 \leq t \leq 10$, find the time when the two particles are closest to each other. [5]

At time k, where $0 < k < 1.5$, the second particle is moving parallel to the first particle.

(d.i) Find the value of k. [5]

(d.ii) At time k, show that the two particles are moving in the opposite direction. [2]

4. [Maximum mark: 21] 22M.2.AHL.TZ2.6

At an archery tournament, a particular competition sees a ball launched into the air while an archer attempts to hit it with an arrow.

The path of the ball is modelled by the equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} u_x \\ u_y - 5t \end{pmatrix}$$

where x is the horizontal displacement from the archer and y is the vertical displacement from the ground, both measured in metres, and t is the time, in seconds, since the ball was launched.

- u_x is the horizontal component of the initial velocity
- u_y is the vertical component of the initial velocity.

In this question both the ball and the arrow are modelled as single points. The ball is launched with an initial velocity such that $u_x = 8$ and $u_y = 10$.

(a.i) Find the initial speed of the ball. [2]

(a.ii) Find the angle of elevation of the ball as it is launched. [2]

(b) Find the maximum height reached by the ball. [3]

(c) Assuming that the ground is horizontal and the ball is not hit by the arrow, find the x coordinate of the point where the ball lands.

[3]

(d) For the path of the ball, find an expression for y in terms of x. [3]

An archer releases an arrow from the point $(0,\ 2)$. The arrow is modelled as travelling in a straight line, in the same plane as the ball, with speed $60\,\text{m}\,\text{s}^{-1}$ and an angle of elevation of 10°.

(e) Determine the two positions where the path of the arrow intersects the path of the ball. [4]

(f) Determine the time when the arrow should be released to hit the ball before the ball reaches its maximum height. [4]

5. [Maximum mark: 17] EXN.2.AHL.TZ0.7

A ball is attached to the end of a string and spun horizontally. Its position relative to a given point, O, at time t seconds, $t \geq 0$, is given by the equation

$\boldsymbol{r} = \begin{pmatrix} 1.5\cos\left(0.1t^2\right) \\ 1.5\sin\left(0.1t^2\right) \end{pmatrix}$ where all displacements are in metres.

(a) Show that the ball is moving in a circle with its centre at O and state the radius of the circle. [4]

(b.i) Find an expression for the velocity of the ball at time t. [2]

(b.ii) Hence show that the velocity of the ball is always perpendicular to the position vector of the ball. [2]

(c.i) Find an expression for the acceleration of the ball at time t. [3]

The string breaks when the magnitude of the ball's acceleration exceeds $20\,\text{m}\text{s}^{-2}$.

(c.ii) Find the value of t at the instant the string breaks. [3]

(c.iii) How many complete revolutions has the ball completed from $t = 0$ to the instant at which the string breaks? [3]