3 Solve for x using the null factor law:

a $x^2(x+5)=0$
b $4(5-x)^2=0$
c $-3(3x-1)^2=0$
d $-6(x-5)(3x+2)^2=0$
e $x(x+1)(x-2)=0$
f $3(x+2)(x+4)(2x-1)=0$

4 Solve, if possible, using the null factor law:

a $\frac{a}{b}=0$
b $\frac{3xy}{z}=0$
c $\frac{2}{xy}=0$
d $-\frac{x}{2y}=0$

C QUADRATIC EQUATIONS

A **quadratic equation** is an equation which can be written in the form $ax^2+bx+c=0$ where a, b, and c are constants, $a \neq 0$.

We have seen that a linear equation such as $2x+3=11$ will usually have *one* solution. In contrast, a quadratic equation may have *two*, *one*, or *zero* solutions.

Here are some quadratic equations which show the truth of this statement:

Equation	$ax^2+bx+c=0$ *form*	*Solutions*	*Number of solutions*
$(x+2)(x-2)=0$	$x^2+0x-4=0$	$x=2$ or $x=-2$	**two**
$(x-2)^2=0$	$x^2-4x+4=0$	$x=2$	**one**
$x^2=-4$	$x^2+0x+4=0$	none as x^2 is always $\geqslant 0$	**zero**

To solve quadratic equations we have the following methods to choose from:

- rewrite the quadratic into **factored form** then use the **null factor law**
- rewrite the quadratic into **completed square form** then solve $(x-h)^2=k$
- use the **quadratic formula**
- use **technology**.

SOLVING BY FACTORISATION

Step 1: If necessary, rearrange the equation so one side is zero.
Step 2: Fully factorise the other side.
Step 3: Use the null factor law: If $ab=0$ then $a=0$ or $b=0$.
Step 4: Solve the resulting linear equations.

Example 5 **Self Tutor**

Solve for x: **a** $3x^2+5x=0$ **b** $x^2=5x+6$

a
$$3x^2+5x=0$$
$$\therefore\ x(3x+5)=0$$
$$\therefore\ x=0 \text{ or } 3x+5=0$$
$$\therefore\ x=0 \text{ or } x=-\tfrac{5}{3}$$

b
$$x^2=5x+6$$
$$\therefore\ x^2-5x-6=0$$
$$\therefore\ (x-6)(x+1)=0$$
$$\therefore\ x=6 \text{ or } -1$$

Example 6 **Self Tutor**

Solve for x:

a $4x^2 + 1 = 4x$ **b** $6x^2 = 11x + 10$

a
$$4x^2 + 1 = 4x$$
$$\therefore\ 4x^2 - 4x + 1 = 0$$
$$\therefore\ (2x-1)^2 = 0$$
$$\therefore\ x = \tfrac{1}{2}$$

b
$$6x^2 = 11x + 10$$
$$\therefore\ 6x^2 - 11x - 10 = 0$$
$$\therefore\ (2x-5)(3x+2) = 0$$
$$\therefore\ x = \tfrac{5}{2} \text{ or } -\tfrac{2}{3}$$

Caution:

- Do not be tempted to divide both sides by an expression involving x.
  If you do this then you may lose one of the solutions.
  For example, consider $x^2 = 5x$.

  Correct solution
  $$x^2 = 5x$$
  $$\therefore\ x^2 - 5x = 0$$
  $$\therefore\ x(x-5) = 0$$
  $$\therefore\ x = 0 \text{ or } 5$$

  Incorrect solution
  $$x^2 = 5x$$
  $$\therefore\ \frac{x^{\not 2}}{\not x} = \frac{5\not x}{\not x}$$
  $$\therefore\ x = 5$$

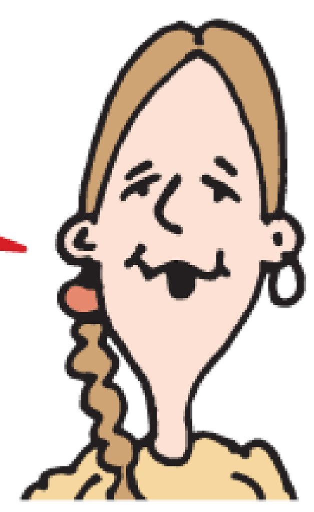
We cannot divide by 0. In dividing both sides by x, we assume $x \neq 0$. For this reason, the solution $x = 0$ is lost.

- Be careful when taking square roots of both sides of an equation.
  If you do this then you may lose one of the solutions.
  For example, consider $(2x-7)^2 = (x+1)^2$.

  Correct solution
  $$(2x-7)^2 = (x+1)^2$$
  $$\therefore\ (2x-7)^2 - (x+1)^2 = 0$$
  $$\therefore\ (2x-7+x+1)(2x-7-x-1) = 0$$
  $$\therefore\ (3x-6)(x-8) = 0$$
  $$\therefore\ x = 2 \text{ or } 8$$

  Incorrect solution
  $$(2x-7)^2 = (x+1)^2$$
  $$\therefore\ 2x - 7 = x + 1$$
  $$\therefore\ x = 8$$

If $a^2 = b^2$ then $a = \pm b$. If we take the square root of both sides, we consider only the case $a = b$. The solution from $a = -b$ is lost.



EXERCISE 4C.1

1 Solve for x:

a $4x^2 + 7x = 0$ **b** $6x^2 + 2x = 0$ **c** $3x^2 - 7x = 0$

d $2x^2 - 11x = 0$ **e** $3x^2 = 8x$ **f** $9x = 6x^2$

2 Solve for x:

a $x^2 - 5x + 6 = 0$ **b** $x^2 - 2x + 1 = 0$ **c** $x^2 + 2x - 8 = 0$

d $x^2 + 7x + 12 = 0$ **e** $x^2 = 2x + 8$ **f** $x^2 + 21 = 10x$

g $9 + x^2 = 6x$ **h** $x^2 + x = 12$ **i** $x^2 + 8x = 33$

j $3x^2 + 9x = 12$ **k** $4x = 70 - 2x^2$ **l** $50 - 5x^2 = -15x$

3 Solve for x:

a $9x^2 - 12x + 4 = 0$ **b** $2x^2 - 13x - 7 = 0$ **c** $3x^2 = 16x + 12$

d $3x^2 + 5x = 2$ **e** $2x^2 + 3 = 5x$ **f** $3x^2 + 8x + 4 = 0$

g $3x^2 = 10x + 8$ **h** $4x^2 + 4x = 3$ **i** $4x^2 = 11x + 3$

j $12x^2 = 11x + 15$ **k** $7x^2 + 6x = 1$ **l** $15x^2 + 2x = 56$

Example 7 **Self Tutor**

Solve for x: $3x + \frac{2}{x} = -7$

$$3x + \frac{2}{x} = -7$$

$\therefore\ 3x^2 + 2 = -7x$ {multiplying both sides by x}

$\therefore\ 3x^2 + 7x + 2 = 0$ {making the RHS $= 0$}

$\therefore\ (x+2)(3x+1) = 0$ {factorising}

$\therefore\ x = -2$ or $-\frac{1}{3}$

4 Solve for x:

a $(x+1)^2 = 2x^2 - 5x + 11$ **b** $(x+2)(1-x) = -4$

c $5 - 4x^2 = 3(2x+1) + 2$ **d** $x + \frac{2}{x} = 3$

e $2x - \frac{1}{x} = -1$ **f** $\frac{x+3}{1-x} = -\frac{9}{x}$

g $(x+3)(2-x) = 4$ **h** $(x-4)(x+2) = 16$

i $(x-5)(x+3) = 20$ **j** $(4x-5)(4x-3) = 143$

LEARNING ALGEBRA

SOLVING BY "COMPLETING THE SQUARE"

As you would be aware by now, not all quadratics factorise easily.

For example, $x^2 + 4x + 1$ cannot be factorised by easily identifying factors. In particular, we cannot write $x^2 + 4x + 1$ in the form $(x-a)(x-b)$ where a and b are rational numbers.

An alternative method is to rewrite the equation in the form $(x-h)^2 = k$. We refer to this process as "completing the square".

Start with the quadratic equation in the form $ax^2 + bx + c = 0$.

Step 1: If $a \neq 1$, divide both sides by a.

Step 2: Rearrange the equation so that only the constant is on the RHS.

Step 3: Add to both sides $\left(\frac{\text{coefficient of } x}{2}\right)^2$.

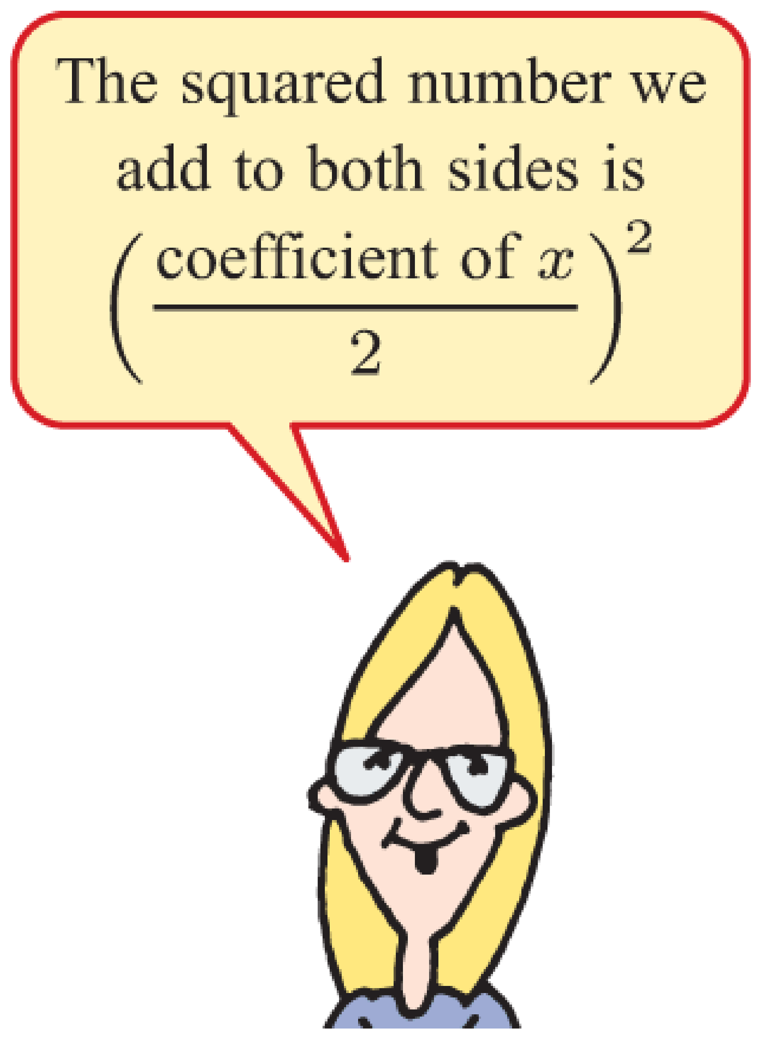
Step 4: Factorise the LHS.

Step 5: Use the rule: If $X^2 = k$ then $X = \pm\sqrt{k}$.

Example 8 — Self Tutor

Solve exactly for x: $x^2 + 4x + 1 = 0$

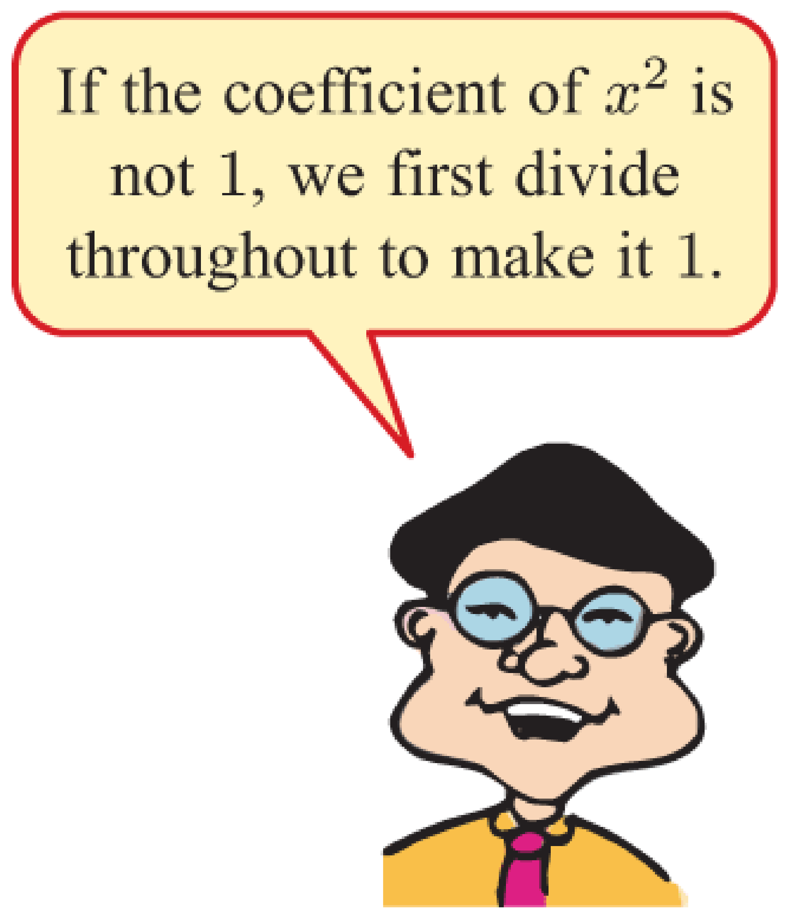
$$\begin{aligned} x^2 + 4x + 1 &= 0 \\ \therefore\ x^2 + 4x &= -1 \quad &&\{\text{writing the constant on the RHS}\} \\ \therefore\ x^2 + 4x + 2^2 &= -1 + 2^2 \quad &&\{\text{completing the square}\} \\ \therefore\ (x+2)^2 &= 3 \quad &&\{\text{factorising the LHS}\} \\ \therefore\ x + 2 &= \pm\sqrt{3} \\ \therefore\ x &= -2 \pm \sqrt{3} \end{aligned}$$



Example 9 — Self Tutor

Solve exactly for x: $-3x^2 + 12x + 5 = 0$

$$\begin{aligned} -3x^2 + 12x + 5 &= 0 \\ \therefore\ x^2 - 4x - \tfrac{5}{3} &= 0 \quad &&\{\text{dividing both sides by } -3\} \\ \therefore\ x^2 - 4x &= \tfrac{5}{3} \quad &&\{\text{writing the constant on the RHS}\} \\ \therefore\ x^2 - 4x + (-2)^2 &= \tfrac{5}{3} + (-2)^2 \quad &&\{\text{completing the square}\} \\ \therefore\ (x-2)^2 &= \tfrac{17}{3} \quad &&\{\text{factorising the LHS}\} \\ \therefore\ x - 2 &= \pm\sqrt{\tfrac{17}{3}} \\ \therefore\ x &= 2 \pm \sqrt{\tfrac{17}{3}} \end{aligned}$$



EXERCISE 4C.2

1 Solve exactly by completing the square:

- **a** $x^2 - 4x + 1 = 0$
- **b** $x^2 + 6x + 2 = 0$
- **c** $x^2 - 14x + 46 = 0$
- **d** $x^2 = 4x + 3$
- **e** $x^2 + 6x + 7 = 0$
- **f** $x^2 = 2x + 6$
- **g** $x^2 + 6x = 2$
- **h** $x^2 + 10 = 8x$
- **i** $x^2 + 6x = -11$

2 Solve exactly by completing the square:

- **a** $2x^2 + 4x + 1 = 0$
- **b** $2x^2 - 10x + 3 = 0$
- **c** $3x^2 + 12x + 5 = 0$
- **d** $3x^2 = 6x + 4$
- **e** $5x^2 - 15x + 2 = 0$
- **f** $4x^2 + 4x = 5$

3 Solve for x:

- **a** $3x - \dfrac{2}{x} = 4$
- **b** $1 - \dfrac{1}{x} = -5x$
- **c** $3 + \dfrac{1}{x^2} = -\dfrac{5}{x}$

4 Suppose $ax^2 + bx + c = 0$ where a, b, and c are constants, $a \neq 0$. Solve for x by completing the square.

LEARNING ALGEBRA

THE QUADRATIC FORMULA

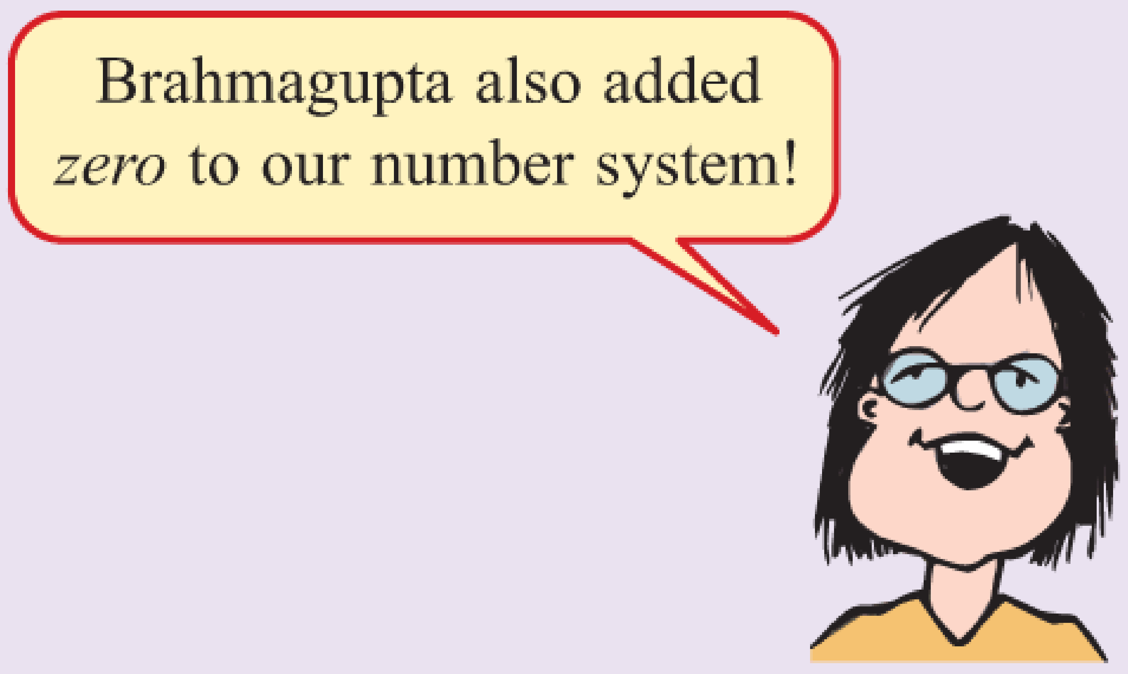
HISTORICAL NOTE — THE QUADRATIC FORMULA

Thousands of years ago, people knew how to calculate the area of a rectangular shape given its side lengths. When they wanted to find the side lengths necessary to give a certain area, however, they ended up with a quadratic equation which they needed to solve.

The first known solution of a quadratic equation is written on the Berlin Papyrus from the Middle Kingdom (2160 - 1700 BC) in Egypt. By 400 BC, the Babylonians were using the method of "completing the square".

Pythagoras and **Euclid** both used geometric methods to explore the problem. Pythagoras noted that the square root was not always an integer, but he refused to accept that irrational solutions existed. Euclid also discovered that the square root was not always rational, but concluded that irrational numbers *did* exist.

A major jump forward was made in India around 700 AD, when Hindu mathematician **Brahmagupta** devised a general (but incomplete) solution for the quadratic equation $ax^2 + bx = c$ which was equivalent to $x = \frac{\sqrt{4ac + b^2} - b}{2a}$. Taking into account the sign of c, this is one of the two solutions we know today.

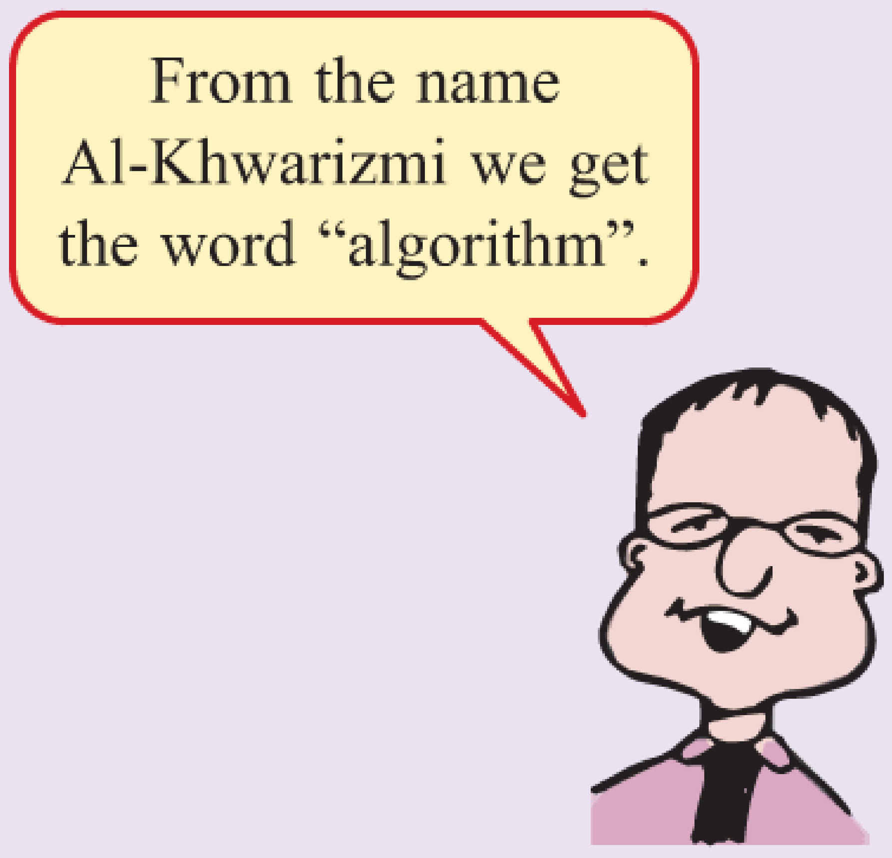


The final, complete solution as we know it today first came around 1100 AD, by another Hindu mathematician called **Bhaskhara**. He was the first to recognise that any positive number has two square roots, which could be negative or irrational. In fact, the quadratic formula is known in some countries today as "Bhaskhara's Formula".

While the Indians had knowledge of the quadratic formula even at this early stage, it took somewhat longer for the quadratic formula to arrive in Europe.

Around 820 AD, the Islamic mathematician **Muhammad bin Musa Al-Khwarizmi**, who was familiar with the work of Brahmagupta, recognised that for a quadratic equation to have real solutions, the value $b^2 - 4ac$ could not be negative.

Al-Khwarizmi's work was brought to Europe by the Jewish mathematician and astronomer **Abraham bar Hiyya** (also known as Savasorda) who lived in Barcelona around 1100 AD.



By 1545, **Girolamo Cardano** had blended the algebra of Al-Khwarizmi with Euclidean geometry. His work allowed for the existence of roots which are not real, as well as negative and irrational roots.

At the end of the 16th Century the mathematical notation and symbolism was introduced by **François Viète** in France.

In 1637, when **René Descartes** published *La Géométrie*, the quadratic formula adopted the form we see today.

In many cases, factorising a quadratic or completing the square can be long or difficult. We can instead use the **quadratic formula**:

If $ax^2 + bx + c = 0$, $a \neq 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Proof:

If $ax^2 + bx + c = 0$, $a \neq 0$

then $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ {dividing each term by a, as $a \neq 0$}

$\therefore\ x^2 + \frac{b}{a}x = -\frac{c}{a}$

$\therefore\ x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$ {completing the square}

$\therefore\ \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ {factorising}

$\therefore\ x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

$\therefore\ x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 10 Self Tutor

Solve for x:

a $x^2 - 2x - 6 = 0$

b $2x^2 + 3x - 6 = 0$

a $x^2 - 2x - 6 = 0$ has $a = 1$, $b = -2$, $c = -6$

$\therefore\ x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-6)}}{2(1)}$

$\therefore\ x = \dfrac{2 \pm \sqrt{28}}{2}$

$\therefore\ x = \dfrac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7}$

b $2x^2 + 3x - 6 = 0$ has $a = 2$, $b = 3$, $c = -6$

$\therefore\ x = \dfrac{-3 \pm \sqrt{3^2 - 4(2)(-6)}}{2(2)}$

$\therefore\ x = \dfrac{-3 \pm \sqrt{57}}{4}$

EXERCISE 4C.3

1 Use the quadratic formula to solve exactly for x:

a $x^2 - 4x - 3 = 0$
b $x^2 + 6x + 7 = 0$
c $x^2 + 1 = 4x$
d $x^2 + 4x = 1$
e $x^2 - 4x + 2 = 0$
f $2x^2 - 2x - 3 = 0$
g $3x^2 - 5x - 1 = 0$
h $-x^2 + 4x + 6 = 0$
i $-2x^2 + 7x - 2 = 0$

2 Rearrange the following equations so they are written in the form $ax^2 + bx + c = 0$, then use the quadratic formula to solve exactly for x.

a $(x+2)(x-1) = 2 - 3x$
b $(2x+1)^2 = 3 - x$
c $(x-2)^2 = 1 + x$
d $(3x+1)^2 = -2x$
e $(x+3)(2x+1) = 9$
f $(2x+3)(2x-3) = x$
g $\dfrac{x-1}{2-x} = 2x + 1$
h $x - \dfrac{1}{x} = 1$
i $2x - \dfrac{1}{x} = 3$

THE DISCRIMINANT OF A QUADRATIC

We can determine how many real solutions a quadratic equation has, without actually solving the equation.

In the quadratic formula, the quantity $b^2 - 4ac$ under the square root sign is called the **discriminant**.

The symbol **delta** Δ is used to represent the discriminant, so $\Delta = b^2 - 4ac$.

The quadratic formula becomes $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ where Δ replaces $b^2 - 4ac$.

- If $\Delta > 0$, $\sqrt{\Delta}$ is a positive real number, so there are **two distinct real roots**
$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{\Delta}}{2a}.$$
- If $\Delta = 0$, $x = \dfrac{-b}{2a}$ is the **only solution**, which we call a **repeated root**.
- If $\Delta < 0$, $\sqrt{\Delta}$ is not a real number and so there are **no real roots**.
- If a, b, and c are rational and Δ is a **square** then the equation has two rational roots which can be found by factorisation.

Example 11 — Self Tutor

Use the discriminant to determine the nature of the roots of:

a $2x^2 - 2x + 3 = 0$ **b** $3x^2 - 4x - 2 = 0$

a $\Delta = b^2 - 4ac$
$= (-2)^2 - 4(2)(3)$
$= -20$
Since $\Delta < 0$, there are no real roots.

b $\Delta = b^2 - 4ac$
$= (-4)^2 - 4(3)(-2)$
$= 40$
Since $\Delta > 0$, but 40 is not a square, there are 2 distinct irrational roots.

EXERCISE 4C.4

1 Consider the quadratic equation $x^2 - 7x + 9 = 0$.
 a Find the discriminant.
 b Hence state the nature of the roots of the equation.
 c Check your answer to **b** by solving the equation.

2 Consider the quadratic equation $4x^2 - 4x + 1 = 0$.
 a Find the discriminant.
 b Hence state the nature of the roots of the equation.
 c Check your answer to **b** by solving the equation.

3 **a** Without using the discriminant, explain why the equation $x^2 + 5 = 0$ has no real roots.
 b Check that $\Delta < 0$ for this equation.

4 Using the discriminant only, state the nature of the solutions of:
 a $x^2 + 7x - 3 = 0$ **b** $x^2 - 3x + 2 = 0$ **c** $3x^2 + 2x - 1 = 0$
 d $5x^2 + 4x - 3 = 0$ **e** $x^2 + x + 5 = 0$ **f** $16x^2 - 8x + 1 = 0$

5 Using the discriminant only, determine which of the following quadratic equations have rational roots which can be found by factorisation.

a $6x^2 - 5x - 6 = 0$ **b** $2x^2 - 7x - 5 = 0$ **c** $3x^2 + 4x + 1 = 0$

d $6x^2 - 47x - 8 = 0$ **e** $4x^2 - 3x + 2 = 0$ **f** $8x^2 + 2x - 3 = 0$

Example 12 Self Tutor

Consider $x^2 - 2x + m = 0$. Find the discriminant Δ, and hence find the values of m for which the equation has:

a a repeated root **b** two distinct real roots **c** no real roots.

$x^2 - 2x + m = 0$ has $a = 1$, $b = -2$, and $c = m$

$$\therefore \Delta = b^2 - 4ac = (-2)^2 - 4(1)(m) = 4 - 4m$$

a For a repeated root

$\Delta = 0$

$\therefore 4 - 4m = 0$

$\therefore 4 = 4m$

$\therefore m = 1$

b For two distinct real roots

$\Delta > 0$

$\therefore 4 - 4m > 0$

$\therefore -4m > -4$

$\therefore m < 1$

c For no real roots

$\Delta < 0$

$\therefore 4 - 4m < 0$

$\therefore -4m < -4$

$\therefore m > 1$

6 For each of the following quadratic equations, find the discriminant Δ in simplest form. Hence find the values of m for which the equation has:

i a repeated root **ii** two distinct real roots **iii** no real roots.

a $x^2 + 4x + m = 0$ **b** $mx^2 + 3x + 2 = 0$, $m \neq 0$ **c** $mx^2 - 3x + 1 = 0$, $m \neq 0$

7 The quadratic equation $4x^2 + kx + (3 - k) = 0$ has a repeated root.
Find the possible values of k, and the repeated root in each case.

THE SUM AND PRODUCT OF THE ROOTS

If $ax^2 + bx + c = 0$ has roots α and β, then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Proof: Let α and β be the roots of $ax^2 + bx + c = 0$.

$$\therefore ax^2 + bx + c = a(x - \alpha)(x - \beta) = a(x^2 - [\alpha + \beta]x + \alpha\beta)$$

$$\therefore x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - [\alpha + \beta]x + \alpha\beta$$

Equating coefficients, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Example 13 Self Tutor

Find the sum and product of the roots of $25x^2 - 20x + 1 = 0$.
Check your answer by solving the quadratic.

If α and β are the roots then $\alpha+\beta=-\dfrac{b}{a}=\dfrac{20}{25}=\dfrac{4}{5}$

and $\alpha\beta=\dfrac{c}{a}=\dfrac{1}{25}$

Check: $25x^2-20x+1=0$ has roots

$$\frac{20\pm\sqrt{400-4(25)(1)}}{50}=\frac{20\pm\sqrt{300}}{50}=\frac{20\pm10\sqrt{3}}{50}=\frac{2\pm\sqrt{3}}{5}$$

These have sum $=\dfrac{2+\sqrt{3}}{5}+\dfrac{2-\sqrt{3}}{5}=\dfrac{4}{5}$ ✓

and product $=\left(\dfrac{2+\sqrt{3}}{5}\right)\left(\dfrac{2-\sqrt{3}}{5}\right)=\dfrac{4-3}{25}=\dfrac{1}{25}$ ✓

EXERCISE 4C.5

1 For each of the following quadratic equations:

- **i** Find the sum and product of the roots.
- **ii** Check your answer by solving the quadratic.

a $x^2+4x-21=0$
b $x^2-5x+5=0$
c $4x^2-12x+5=0$
d $3x^2-4x-2=0$

2 For the equation $kx^2-(1+k)x+(3k+2)=0$, the sum of the roots is twice their product. Find k and the two roots.

3 The quadratic equation $ax^2-6x+a-2=0$, $a\neq0$, has one root which is double the other.

a Let the roots be α and 2α. Hence find two equations involving α.
b Find a and the two roots of the quadratic equation.

4 The quadratic equation $kx^2+(k-8)x+(1-k)=0$, $k\neq0$, has one root which is two more than the other. Find k and the two roots.

Example 14 **Self Tutor**

The roots of the equation $4x^2+5x-1=0$ are α and β.
Find a quadratic equation with roots 3α and 3β.

If α and β are the roots of $4x^2+5x-1=0$, then $\alpha+\beta=-\frac{5}{4}$ and $\alpha\beta=-\frac{1}{4}$.

For the quadratic equation with roots 3α and 3β,

the sum of the roots $=3\alpha+3\beta$
$=3(\alpha+\beta)$
$=3\left(-\frac{5}{4}\right)$
$=-\frac{15}{4}$

and the product of the roots $=(3\alpha)(3\beta)$
$=9\alpha\beta$
$=9\left(-\frac{1}{4}\right)$
$=-\frac{9}{4}$

So, we have $-\dfrac{b}{a}=-\dfrac{15}{4}$ and $\dfrac{c}{a}=-\dfrac{9}{4}$.

The simplest solution is $a=4$, $b=15$, $c=-9$

$\therefore$ the quadratic equation is $4x^2+15x-9=0$.

5 Find a quadratic equation with roots:

a 3 and -5 **b** $2+\sqrt{3}$ and $2-\sqrt{3}$.

6 The roots of the equation $3x^2+2x-4=0$ are α and β.
Find a quadratic equation with roots:

a $-\alpha$ and $-\beta$ **b** 2α and 2β.

7 The roots of the equation $-2x^2+5x+1=0$ are α and β.

a Find a quadratic equation with roots $\alpha+2$ and $\beta+2$.

b Find *all* quadratic equations with roots $\frac{\alpha}{2}$ and $\frac{\beta}{2}$.

8 The roots of the equation $x^2-4x-11=0$ are α and β.
Find *all* quadratic equations with roots $\alpha\beta$ and $\alpha+\beta$.

9 The roots of $2x^2+5x-9=0$ are α and β.
Find *all* quadratic equations with roots α^2 and β^2.

10 The roots of the equation $x^2-6x+7=0$ are α and β.
Find a quadratic equation with roots $\alpha+\frac{1}{\beta}$ and $\beta+\frac{1}{\alpha}$.

11 The roots of $2x^2-3x-5=0$ are p and q.
Find *all* quadratic equations with roots p^2+q and q^2+p.

D SOLVING POLYNOMIAL EQUATIONS USING TECHNOLOGY

We have already seen that:

- a **linear** equation can be written in the form $ax+b=0$, $a\neq 0$
- a **quadratic** equation can be written in the form $ax^2+bx+c=0$, $a\neq 0$.

These are the first two members of a family called **polynomial equations**.

The next two members of the family are:

- a **cubic** equation which can be written in the form $ax^3+bx^2+cx+d=0$, $a\neq 0$
- a **quartic** equation which can be written in the form $ax^4+bx^3+cx^2+dx+e=0$, $a\neq 0$.

The highest power of x in a polynomial equation is called its **degree**.
If a polynomial equation has degree n then it may have up to n real solutions.

You can use your graphics calculator to solve polynomial equations. You may need to first rearrange them into polynomial form.

Some calculator models will show exact solutions as well as numerical approximations.

GRAPHICS CALCULATOR INSTRUCTIONS

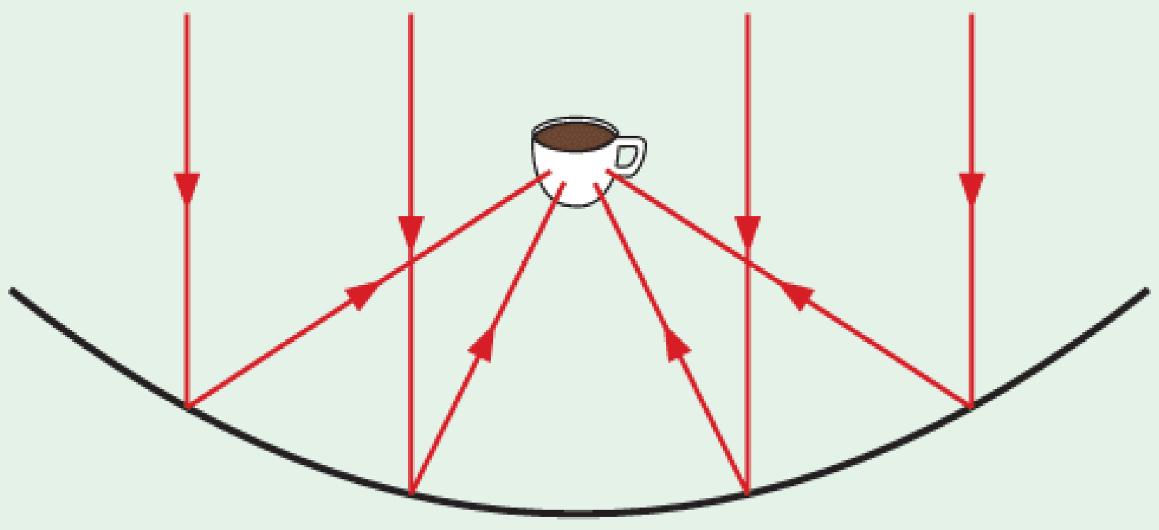
Chapter 14

Quadratic functions

Contents:

OPENING PROBLEM

Energy-conscious Misha wants to use solar energy to heat his cup of coffee. He has decided to build a reflecting surface to focus the sun's light on the cup.



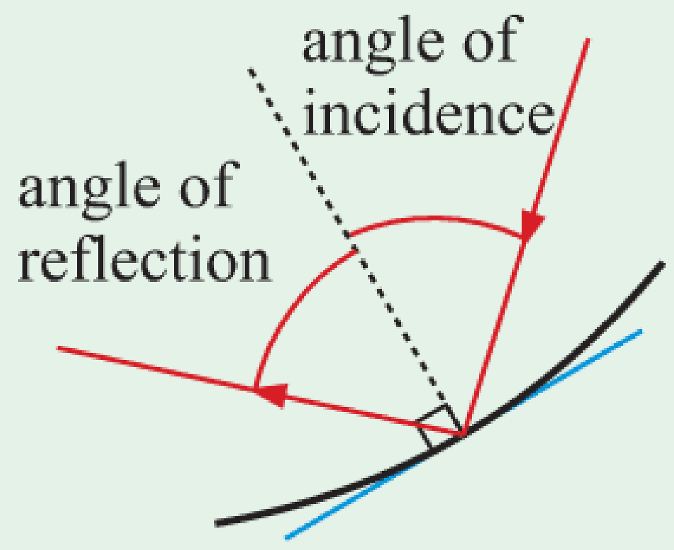
He understands that the sun's rays will arrive parallel, and that each ray will bounce off the surface according to the law of reflection:

angle of incidence = angle of reflection

Things to think about:

a What *shape* should the surface have?

b Can we write a *formula* which defines the shape of the surface?

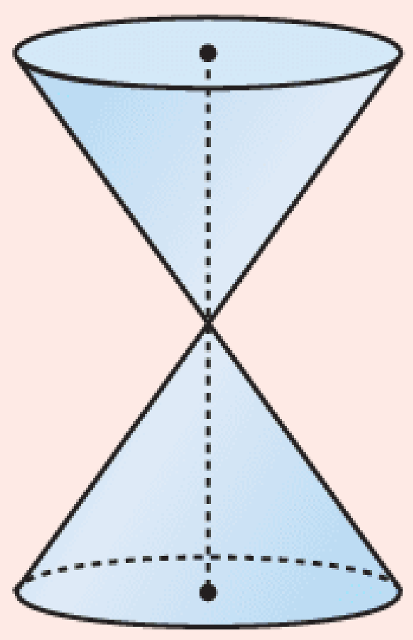


In this Chapter we will study **quadratic functions** and investigate their graphs which are called **parabolas**. There are many examples of parabolas in everyday life, including water fountains, bridges, and radio telescopes.

We will see how the curve Misha needs in the **Opening Problem** is actually a parabola, and how the **Opening Problem** relates to the geometric definition of a parabola.

ACTIVITY 1 CONIC SECTIONS

A cone is *right-circular* if its apex is directly above the centre of the base.



Suppose we have two right-circular cones, and we place one upside-down on the first. Now suppose the cones are infinitely tall.

We call the resulting shape a **double inverted right-circular cone**.

When a double inverted right-circular cone is cut by a plane, 7 possible intersections may result, called **conic sections**:

- a point
- a line
- a line-pair
- a circle
- an ellipse
- a parabola
- a hyperbola

Click on the icon to explore the conic sections.

DEMO

You should observe how the parabola results when cutting the cone parallel to its slant edge.

A QUADRATIC FUNCTIONS

A **quadratic function** is a relationship between two variables x and y which can be written in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.

For any value of x, the corresponding value of y can be found by substitution.

Example 1 Self Tutor

Determine whether the given point satisfies the quadratic function:

a $y = 3x^2 + 2x$ $(2, 16)$

b $y = -x^2 - 2x + 1$ $(-3, 1)$

a When $x = 2$,
$y = 3(2)^2 + 2(2)$
$= 12 + 4$
$= 16$
$\therefore$ $(2, 16)$ satisfies the function $y = 3x^2 + 2x$.

b When $x = -3$,
$y = -(-3)^2 - 2(-3) + 1$
$= -9 + 6 + 1$
$= -2$
$\therefore$ $(-3, 1)$ does not satisfy the function $y = -x^2 - 2x + 1$.

When we substitute a value for y into a quadratic function, we are left with a quadratic equation. Solving the quadratic equation gives us the values of x corresponding to that y-value. There may be 0, 1, or 2 solutions.

Example 2 Self Tutor

If $y = x^2 - 2x + 3$, find the value(s) of x when:

a $y = 2$

b $y = 18$.

a If $y = 2$ then
$x^2 - 2x + 3 = 2$
$\therefore$ $x^2 - 2x + 1 = 0$
$\therefore$ $(x - 1)^2 = 0$
$\therefore$ $x = 1$

b If $y = 18$ then
$x^2 - 2x + 3 = 18$
$\therefore$ $x^2 - 2x - 15 = 0$
$\therefore$ $(x - 5)(x + 3) = 0$
$\therefore$ $x = -3$ or 5

EXERCISE 14A

1 Copy and complete each table of values:

a $y = x^2 - 3x + 1$

x	-2	-1	0	1	2
y					

b $y = x^2 + 2x - 5$

x	-2	-1	0	1	2
y					

c $y = 2x^2 - x + 3$

x	-4	-2	0	2	4
y					

d $y = -3x^2 + 2x + 4$

x	-4	-2	0	2	4
y					

2 Determine whether the given point satisfies the quadratic function:

a $y = 2x^2 + 5$ (0, 4)
b $y = x^2 - 3x + 2$ (2, 0)
c $y = -x^2 + 2x - 5$ (−1, −8)
d $y = -2x^2 - x + 6$ (3, −15)
e $y = 3x^2 - 4x + 10$ (2, 10)
f $y = -\frac{1}{2}x^2 + 4x - 1$ (2, 5)

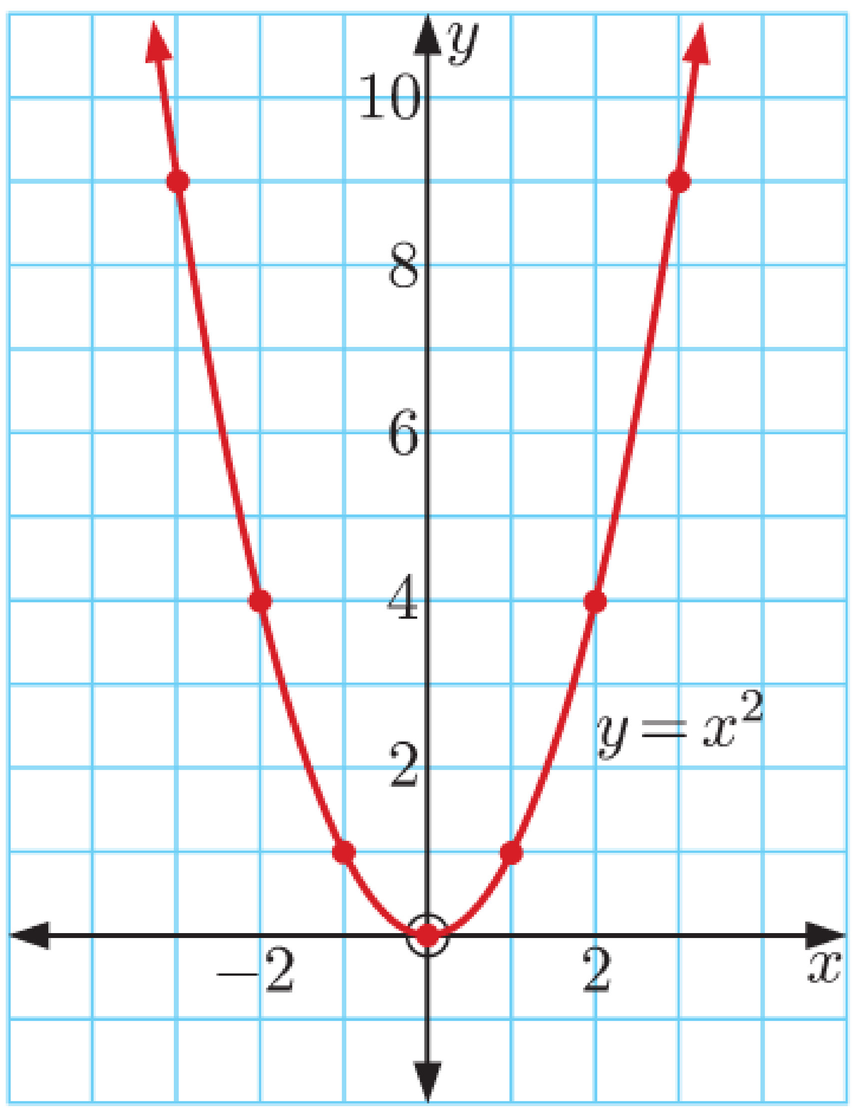
3 For each of the following quadratic functions, find the value(s) of x for the given value of y:

a $y = x^2 + 3x + 6$ when $y = 4$
b $y = x^2 - 4x + 7$ when $y = 3$
c $y = x^2 - 6x + 1$ when $y = -4$
d $y = 2x^2 + 5x + 1$ when $y = 4$
e $y = \frac{1}{2}x^2 + \frac{5}{2}x - 2$ when $y = 1$
f $y = -\frac{1}{2}x^2 + 2x - 1$ when $y = 2$

B GRAPHS OF QUADRATIC FUNCTIONS

The simplest quadratic function is $y = x^2$. Its graph can be drawn from a table of values.

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9



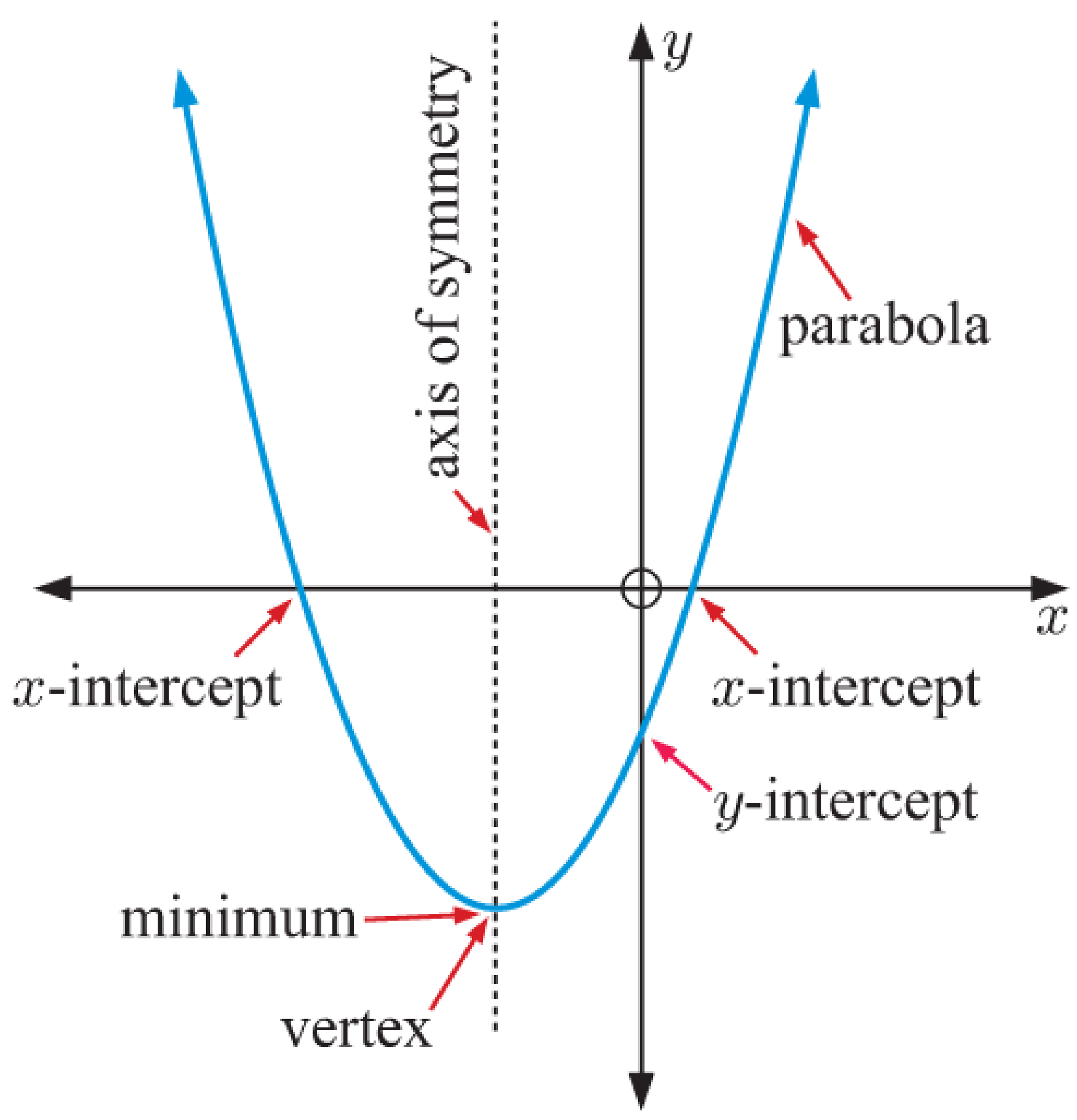
The graph of a quadratic function is called a **parabola**.

The point where the graph "turns" is called the **vertex**.

If the graph opens upwards, the vertex is the **minimum** or **minimum turning point**, and the graph is **concave upwards**.

If the graph opens downwards, the vertex is the **maximum** or **maximum turning point**, and the graph is **concave downwards**.

The vertical line that passes through the vertex is called the **axis of symmetry**. Every parabola is symmetrical about its axis of symmetry.



The value of y where the graph crosses the y-axis is the ***y*-intercept**.

The values of x (if they exist) where the graph crosses the x-axis are called the ***x*-intercepts**. They correspond to the **roots** of the quadratic equation $ax^2 + bx + c = 0$.

INVESTIGATION 1 GRAPHING $y = a(x-p)(x-q)$

In this Investigation we consider the properties of the graph of a quadratic stated in factored form. It is best done using a **graphing package** or **graphics calculator**.

GRAPHING PACKAGE

What to do:

1 **a** Use technology to help you to sketch:
$y = (x-1)(x-3)$, $y = 2(x-1)(x-3)$, $y = -(x-1)(x-3)$,
$y = -3(x-1)(x-3)$, and $y = -\frac{1}{2}(x-1)(x-3)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of a in $y = a(x-1)(x-3)$?

2 **a** Use technology to help you to sketch:
$y = 2(x-1)(x-4)$, $y = 2(x-3)(x-5)$, $y = 2(x+1)(x-2)$,
$y = 2x(x+5)$, and $y = 2(x+2)(x+4)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of p and q in $y = 2(x-p)(x-q)$?

3 **a** Use technology to help you to sketch:
$y = 2(x-1)^2$, $y = 2(x-3)^2$, $y = 2(x+2)^2$, and $y = 2x^2$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of p in $y = 2(x-p)^2$?

4 Copy and complete:

- If a quadratic has the form $y = a(x-p)(x-q)$ then it ...... the x-axis at ......
- If a quadratic has the form $y = a(x-p)^2$ then it ...... the x-axis at ......

INVESTIGATION 2 GRAPHING $y = a(x-h)^2 + k$

In this Investigation we consider the properties of the graph of a quadratic stated in completed square form. It is best done using a **graphing package** or **graphics calculator**.

GRAPHING PACKAGE

What to do:

1 **a** Use technology to help you to sketch:
$y = (x-3)^2 + 2$, $y = 2(x-3)^2 + 2$, $y = -2(x-3)^2 + 2$,
$y = -(x-3)^2 + 2$, and $y = -\frac{1}{3}(x-3)^2 + 2$

b Find the coordinates of the vertex for each function in **a**.

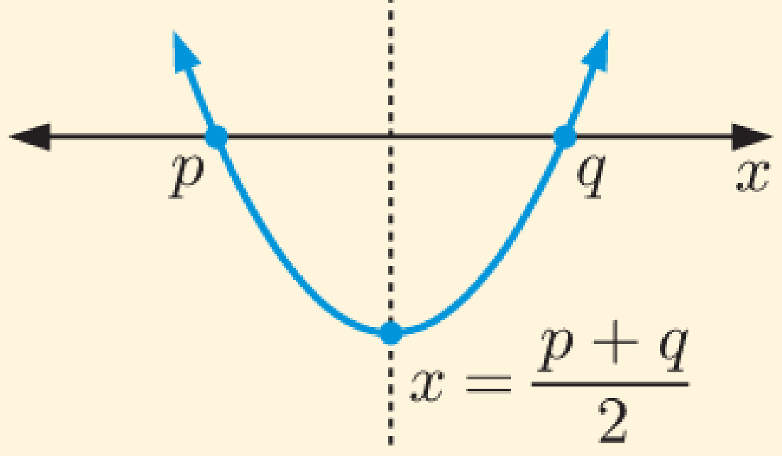
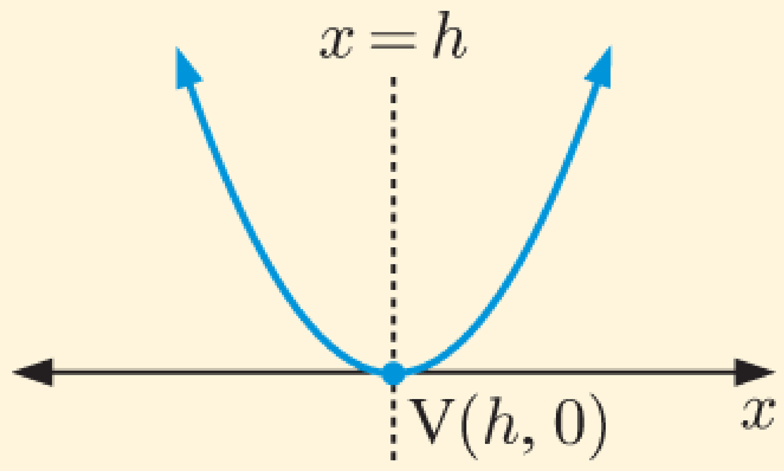
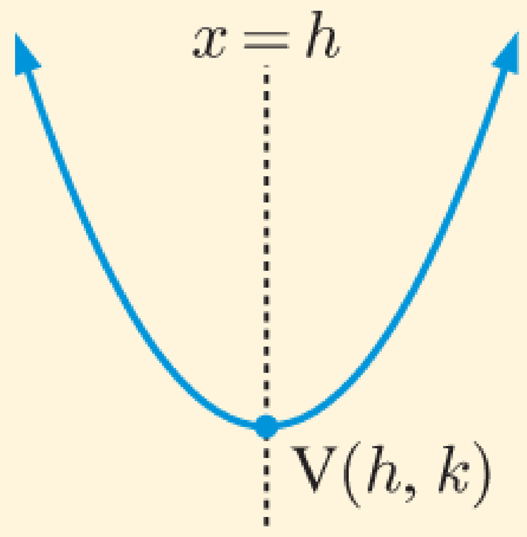
c What is the geometrical significance of a in $y = a(x-3)^2 + 2$?

2 **a** Use technology to help you to sketch:
$y = 2(x-1)^2 + 3$, $y = 2(x-2)^2 + 4$, $y = 2(x-3)^2 + 1$,
$y = 2(x+1)^2 + 4$, $y = 2(x+2)^2 - 5$, and $y = 2(x+3)^2 - 2$

b Find the coordinates of the vertex for each function in **a**.

c What is the geometrical significance of h and k in $y = 2(x-h)^2 + k$?

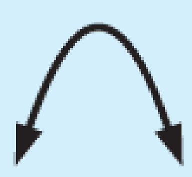
3 Copy and complete:
If a quadratic has the form $y = a(x-h)^2 + k$ then its vertex has coordinates ......

Quadratic form, $a \neq 0$	*Graph*	*Facts*
$y = a(x-p)(x-q)$ where $p, q \in \mathbb{R}$		• x-intercepts are p and q • axis of symmetry is $x = \frac{p+q}{2}$ • vertex has x-coordinate $\frac{p+q}{2}$
$y = a(x-h)^2$ where $h \in \mathbb{R}$		• touches x-axis at h • axis of symmetry is $x = h$ • vertex is $(h, 0)$
$y = a(x-h)^2 + k$ where $h, k \in \mathbb{R}$		• axis of symmetry is $x = h$ • vertex is (h, k)

You should have found that a, the coefficient of x^2, controls the width of the graph and whether it opens upwards or downwards.

For a quadratic function $y = ax^2 + bx + c$, $a \neq 0$:

- $a > 0$ produces the shape  called concave up.

  $a < 0$ produces the shape  called concave down.

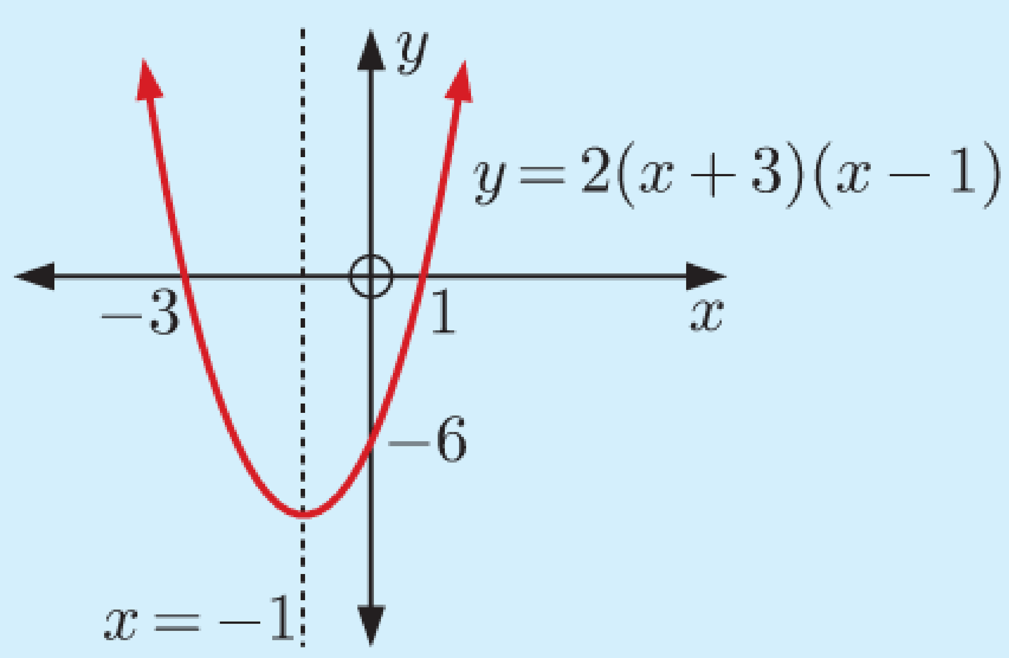
- If $-1 < a < 1$, $a \neq 0$ the graph is wider than $y = x^2$.
  If $a < -1$ or $a > 1$ the graph is narrower than $y = x^2$.

Example 3 **Self Tutor**

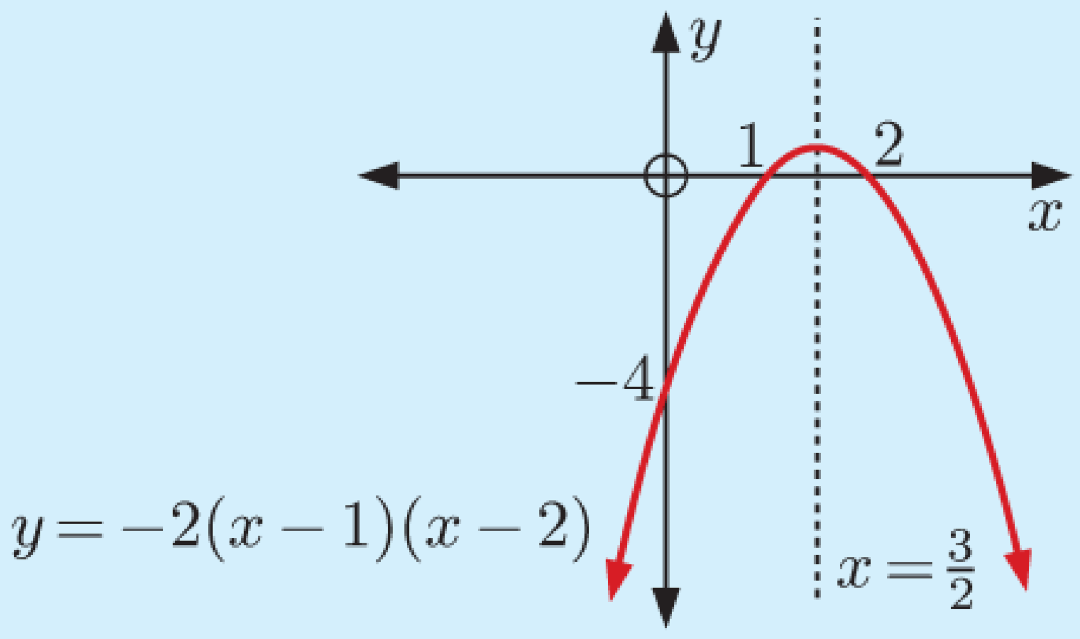
Sketch the graph using axes intercepts, and state the equation of the axis of symmetry:

a $y = 2(x+3)(x-1)$ **b** $y = -2(x-1)(x-2)$ **c** $y = \frac{1}{2}(x+2)^2$

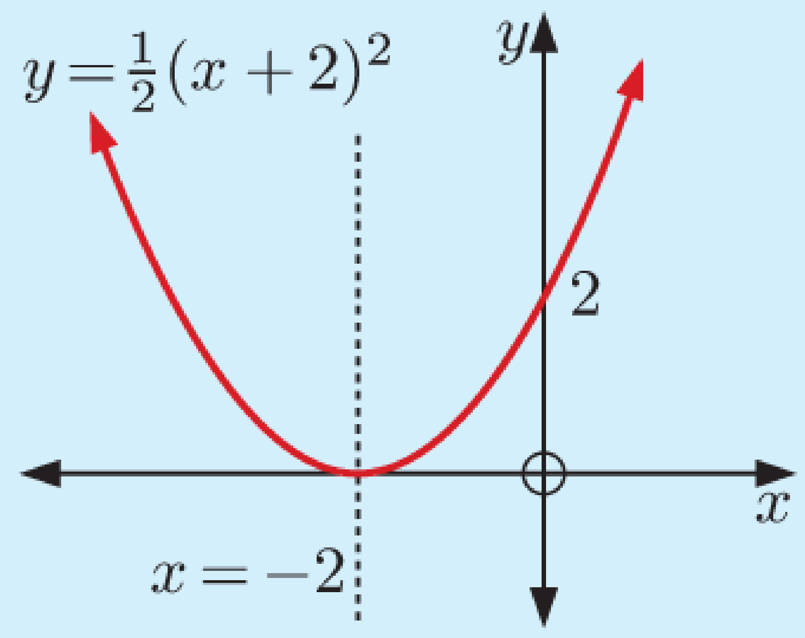
a $y = 2(x+3)(x-1)$
has x-intercepts -3, 1
When $x = 0$, $y = 2(3)(-1)$
$= -6$
$\therefore$ y-intercept is -6



b $y=-2(x-1)(x-2)$
has x-intercepts 1, 2
When $x=0$, $y=-2(-1)(-2)$
$=-4$
$\therefore$ y-intercept is -4



c $y=\frac{1}{2}(x+2)^2$
touches x-axis at -2
When $x=0$, $y=\frac{1}{2}(2)^2$
$=2$
$\therefore$ y-intercept is 2

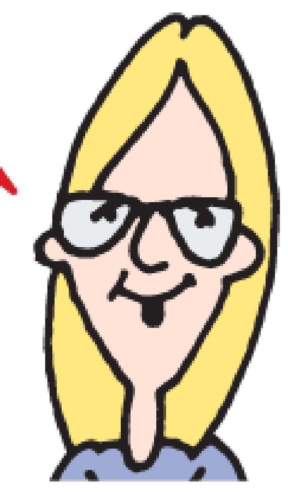


EXERCISE 14B.1

1 Sketch the graph using axes intercepts, and state the equation of the axis of symmetry:

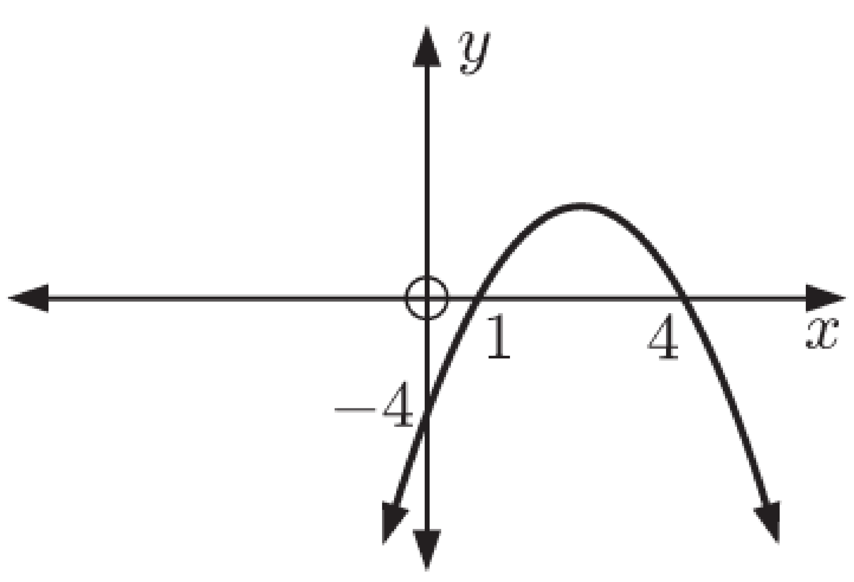
a $y=(x-4)(x+2)$ **b** $y=-(x-4)(x+2)$
c $y=2(x+3)(x+5)$ **d** $y=-3(x+1)(x+5)$
e $y=2(x+3)^2$ **f** $y=-\frac{1}{4}(x+2)^2$

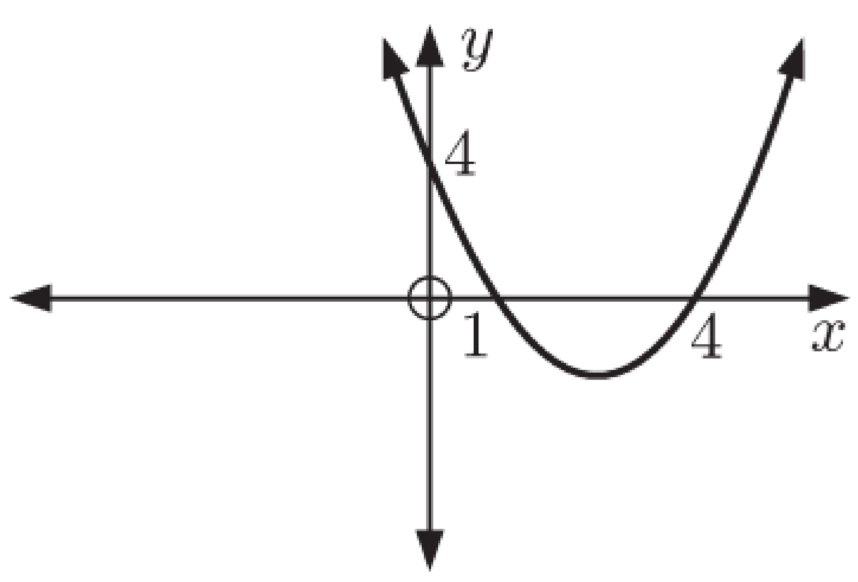
The axis of symmetry is midway between the x-intercepts.

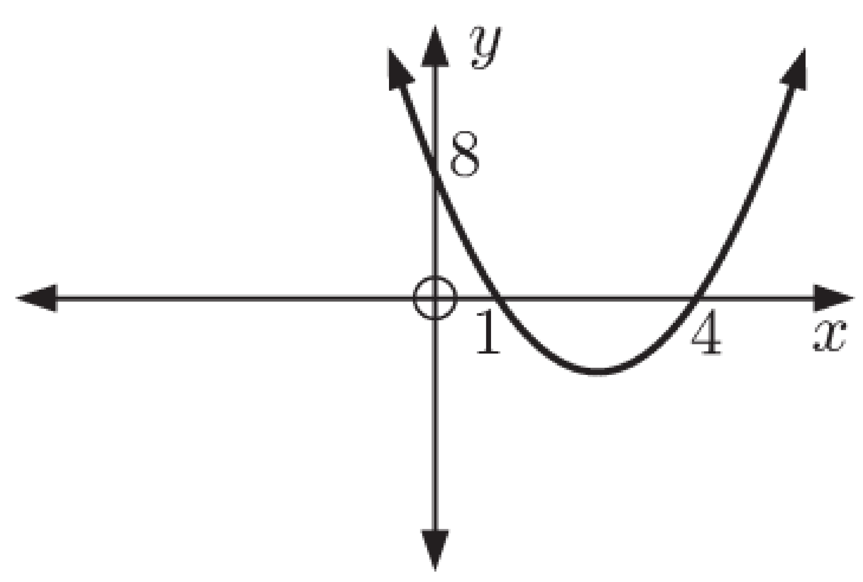


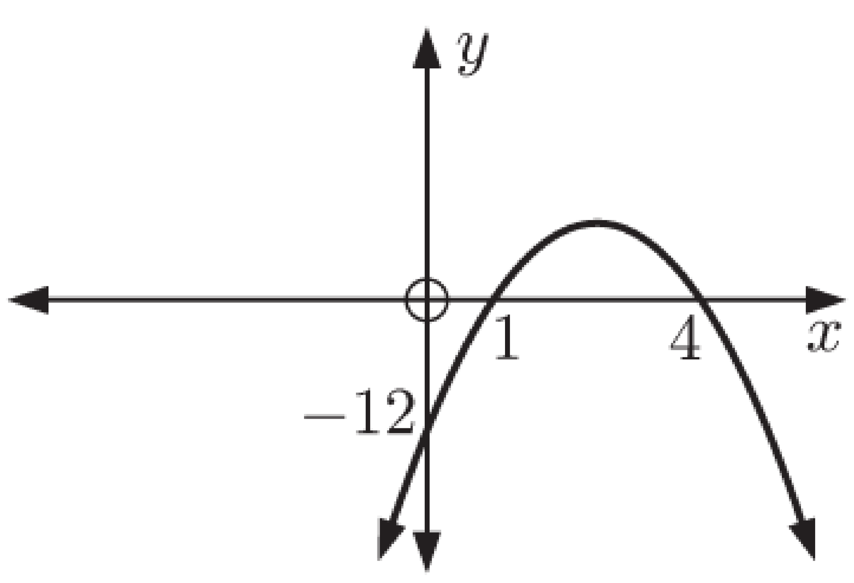
2 Match each quadratic function with its corresponding graph.

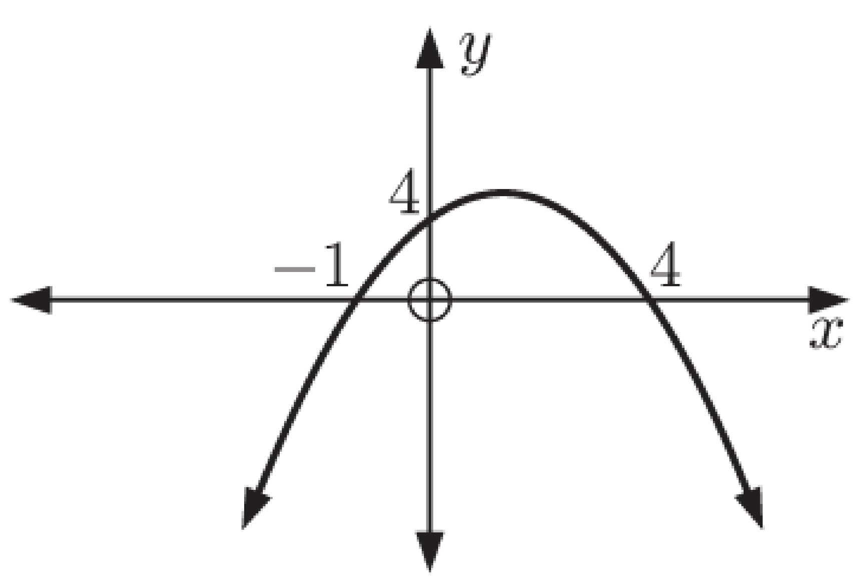
a $y=2(x-1)(x-4)$ **b** $y=-(x+1)(x-4)$ **c** $y=(x-1)(x-4)$
d $y=(x+1)(x-4)$ **e** $y=2(x+4)(x-1)$ **f** $y=-3(x+4)(x-1)$
g $y=2(x+1)(x+4)$ **h** $y=-(x-1)(x-4)$ **i** $y=-3(x-1)(x-4)$

A


B


C


D


E


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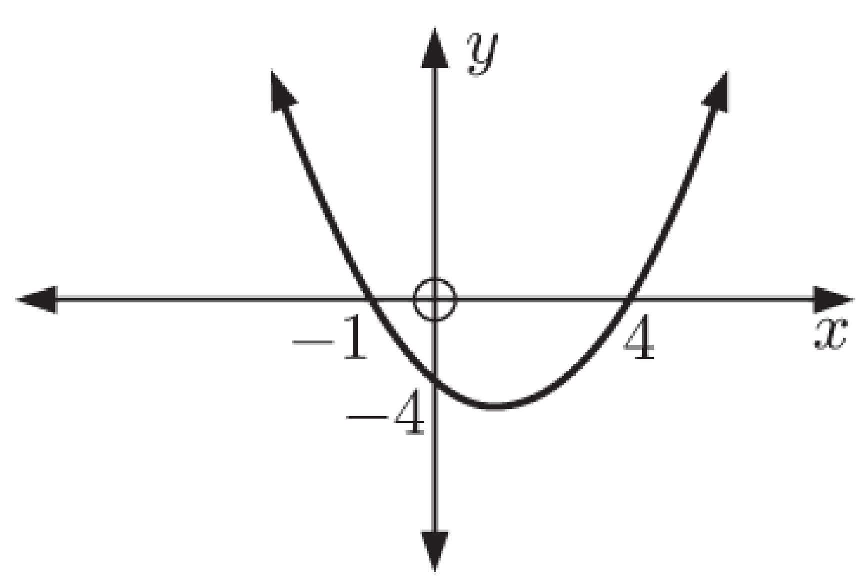

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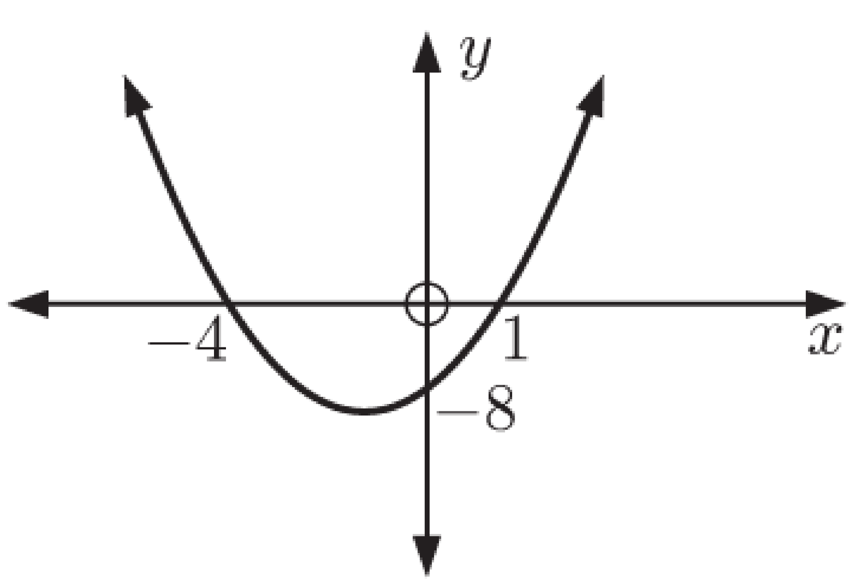
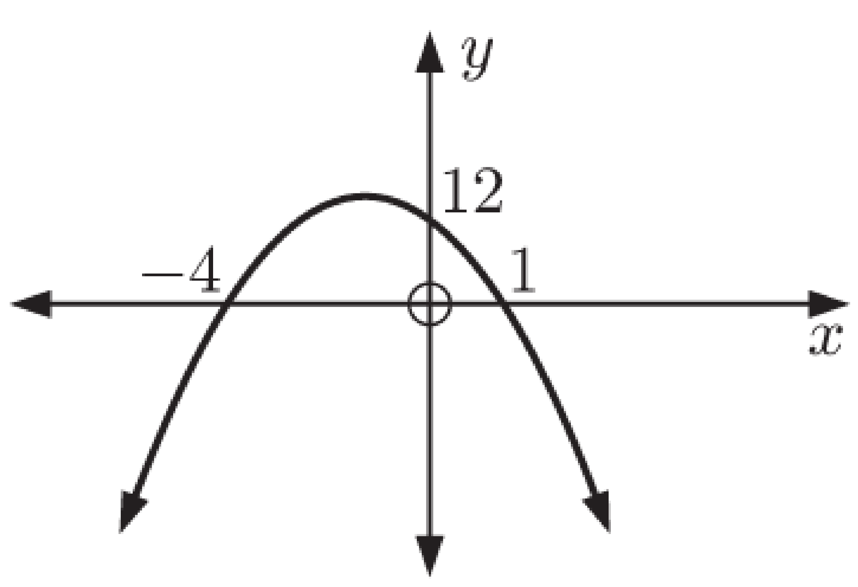
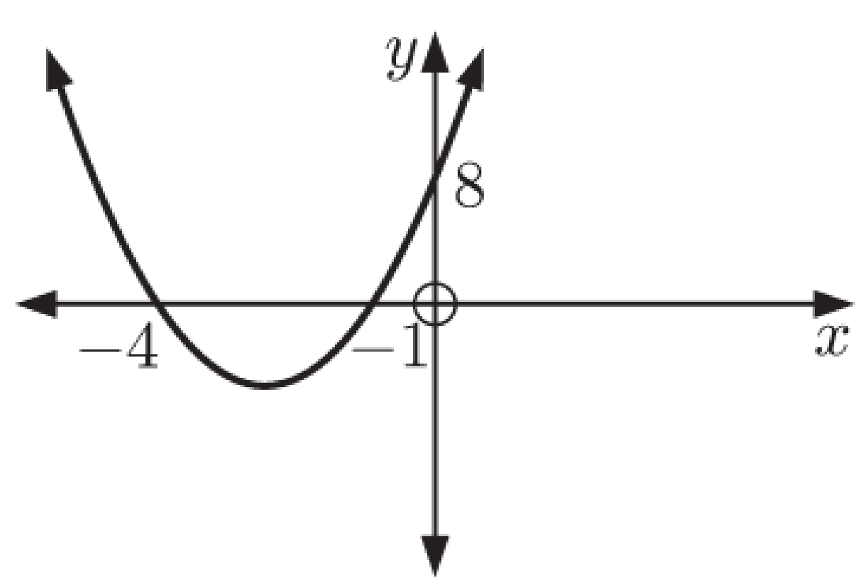


H


I


Example 4 — Self Tutor

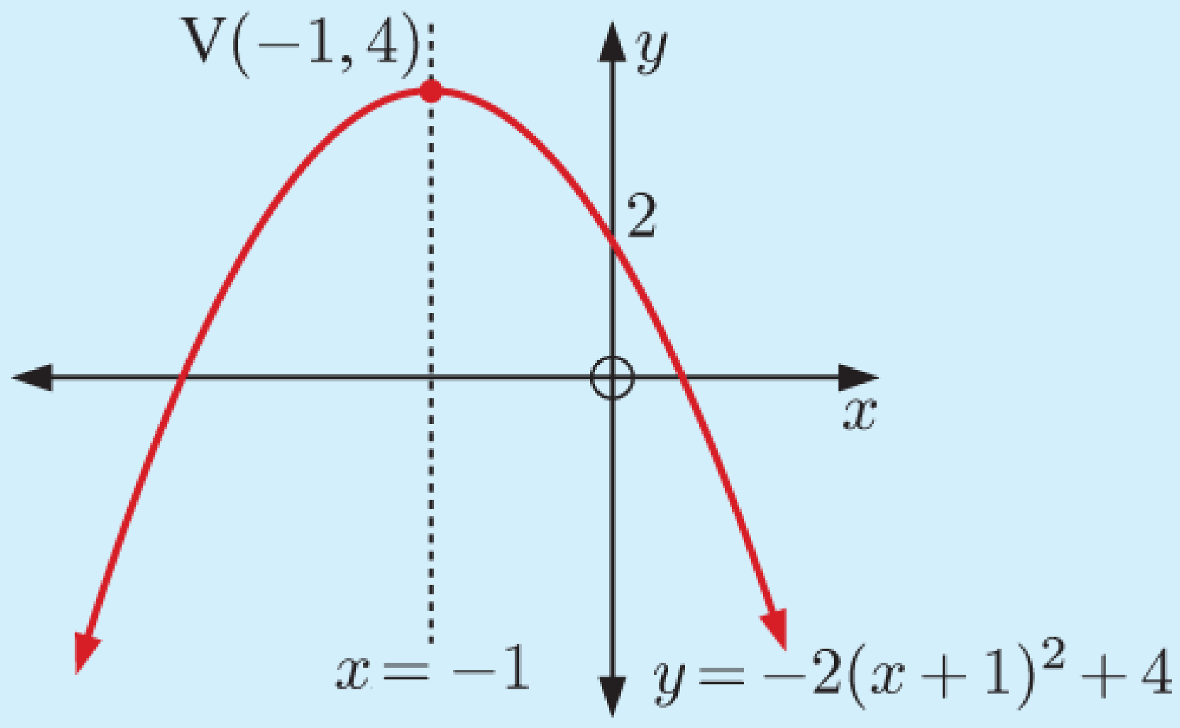
Use the vertex, axis of symmetry, and y-intercept to graph $y = -2(x+1)^2 + 4$.

The axis of symmetry is $x = -1$.

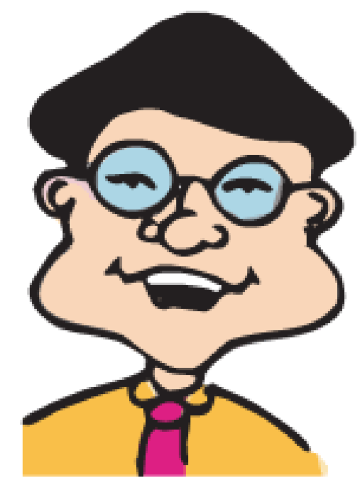
The vertex is $(-1, 4)$.

When $x = 0$, $y = -2(1)^2 + 4$
$= 2$

$a < 0$ so the shape is 



$y = a(x-h)^2 + k$ is called **completed square form**.



3 Use the vertex, axis of symmetry, and y-intercept to graph:

- **a** $y = (x-1)^2 + 3$
- **b** $y = (x+3)^2 - 4$
- **c** $y = -(x+4)^2 + 2$
- **d** $y = 2(x+2)^2 + 1$
- **e** $y = -2(x-1)^2 - 3$
- **f** $y = \frac{1}{3}(x+6)^2 - 1$
- **g** $y = \frac{1}{2}(x-3)^2 + 2$
- **h** $y = -\frac{1}{3}(x-1)^2 + 4$
- **i** $y = -\frac{1}{10}(x+2)^2 - 3$

SKETCHING GRAPHS BY "COMPLETING THE SQUARE"

If we wish to graph a quadratic given in general form $y = ax^2 + bx + c$, one approach is to use "**completing the square**" to convert it to the completed square form $y = a(x-h)^2 + k$. We can then read off the coordinates of the vertex (h, k).

Example 5 — Self Tutor

Write $y = x^2 + 4x + 3$ in the form $y = (x-h)^2 + k$ by "completing the square".
Hence sketch $y = x^2 + 4x + 3$, stating the coordinates of the vertex.

$y = x^2 + 4x + 3$

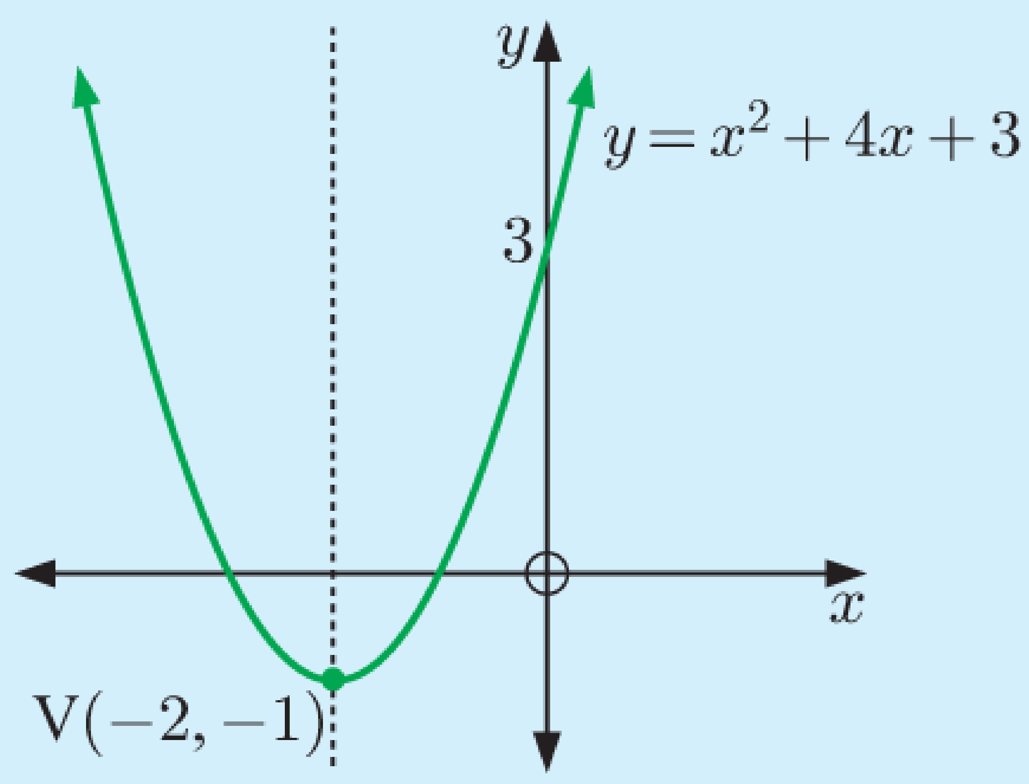
$\therefore \quad = x^2 + 4x + 2^2 + 3 - 2^2$

$\therefore \quad y = (x+2)^2 - 1$

So, the axis of symmetry is $x = -2$
and the vertex is $(-2, -1)$.

When $x = 0$, $y = 3$

$\therefore$ the y-intercept is 3.



EXERCISE 14B.2

1 Write the following quadratics in the form $y = (x-h)^2 + k$ by "completing the square".
Hence sketch each function, stating the coordinates of the vertex.

- **a** $y = x^2 - 2x + 3$
- **b** $y = x^2 + 4x - 2$
- **c** $y = x^2 - 4x$
- **d** $y = x^2 + 3x$
- **e** $y = x^2 + 5x - 2$
- **f** $y = x^2 - 3x + 2$
- **g** $y = x^2 - 6x + 5$
- **h** $y = x^2 + 8x - 2$
- **i** $y = x^2 - 5x + 1$

Example 6 **Self Tutor**

a Convert $y = 3x^2 - 4x + 1$ to the completed square form $y = a(x-h)^2 + k$.

b Hence write down the coordinates of the vertex, and sketch the quadratic.

a $y = 3x^2 - 4x + 1$

$= 3[x^2 - \frac{4}{3}x + \frac{1}{3}]$

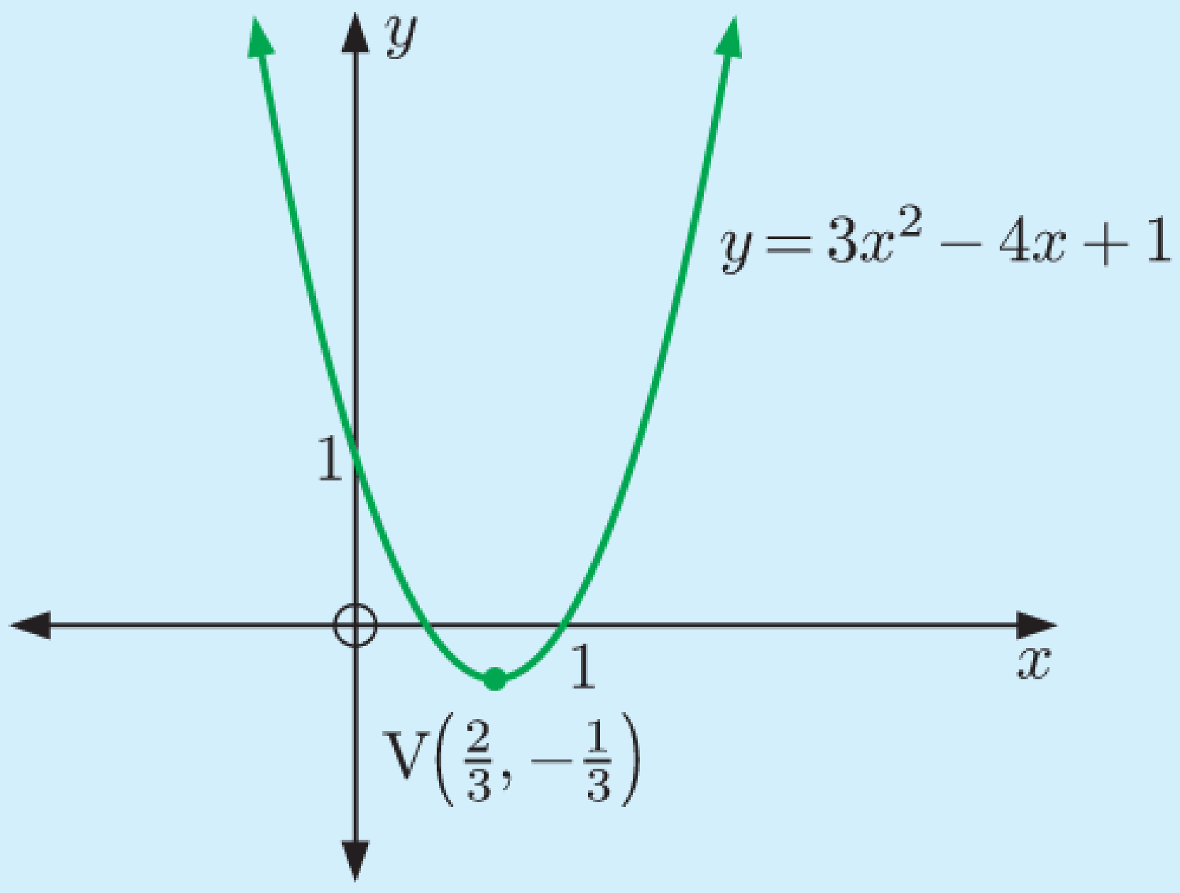
$= 3[x^2 - 2(\frac{2}{3})x + (\frac{2}{3})^2 + \frac{1}{3} - (\frac{2}{3})^2]$

$= 3[(x - \frac{2}{3})^2 + \frac{3}{9} - \frac{4}{9}]$

$= 3[(x - \frac{2}{3})^2 - \frac{1}{9}]$

$= 3(x - \frac{2}{3})^2 - \frac{1}{3}$

b The vertex is $(\frac{2}{3}, -\frac{1}{3})$ and the y-intercept is 1.



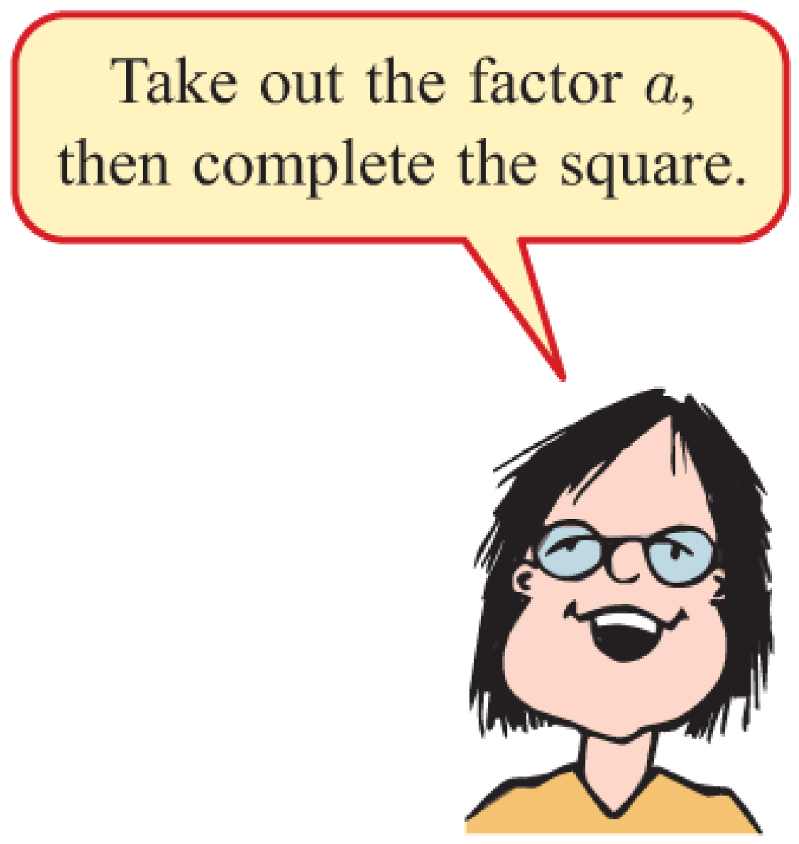
2 For each of the following quadratics:

i Write the quadratic in the completed square form $y = a(x-h)^2 + k$.

ii State the coordinates of the vertex.

iii Find the y-intercept.

iv Sketch the graph of the quadratic.



a $y = 2x^2 + 4x + 5$ **b** $y = 2x^2 - 8x + 3$

c $y = 2x^2 - 6x + 1$ **d** $y = 3x^2 - 6x + 5$

e $y = -x^2 + 4x + 2$ **f** $y = -2x^2 - 5x + 3$

g $y = -\frac{1}{3}x^2 + 2x - 3$ **h** $y = \frac{1}{2}x^2 + 3x - 4$

SKETCHING QUADRATICS IN THE GENERAL FORM $y = ax^2 + bx + c$

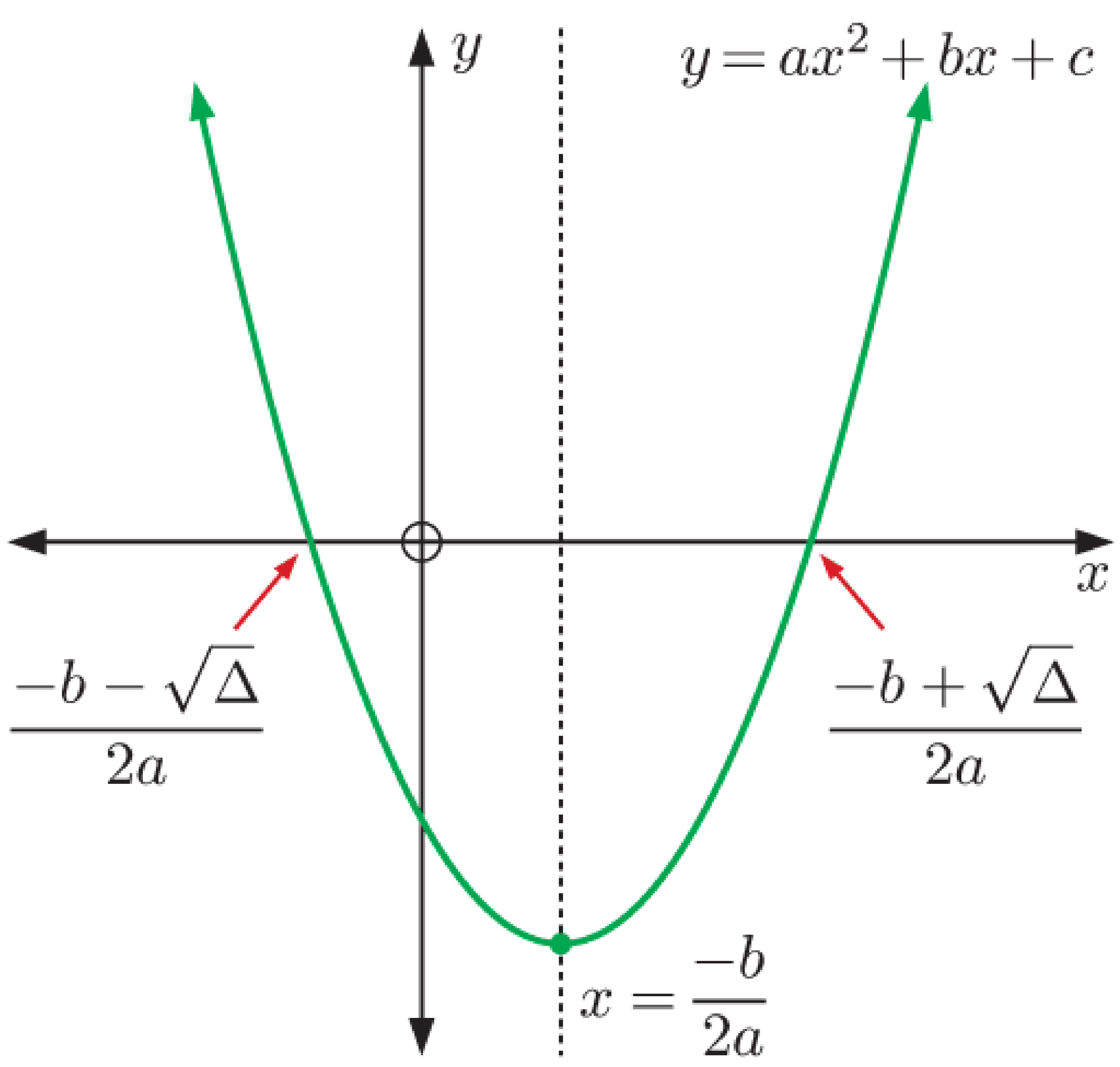
We now consider a method of graphing quadratics of the form $y = ax^2 + bx + c$ directly, without having to first convert them to a different form.

We know that the quadratic equation $ax^2 + bx + c = 0$ has solutions $\frac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$.

If $\Delta \geqslant 0$, these are the x-intercepts of the graph of the quadratic function $y = ax^2 + bx + c$.

The average of the values is $\frac{-b}{2a}$, so we conclude that:

- the axis of symmetry is $x = \frac{-b}{2a}$
- the vertex of the quadratic has x-coordinate $\frac{-b}{2a}$.



To graph a quadratic of the form $y = ax^2 + bx + c$, we:

- Find the axis of symmetry $x = \dfrac{-b}{2a}$.
- Substitute this value to find the y-coordinate of the vertex.
- State the y-intercept c.
- Find the x-intercepts by solving $ax^2 + bx + c = 0$, either by factorisation or using the quadratic formula.
- Graph the quadratic using the information you have found.

Example 7 **Self Tutor**

Consider the quadratic $y = 2x^2 + 8x - 10$.

a Find the axis of symmetry.

b Find the coordinates of the vertex.

c Find the axes intercepts.

d Hence sketch the quadratic.

$y = 2x^2 + 8x - 10$ has $a = 2$, $b = 8$, and $c = -10$. Since $a > 0$, the shape is 

a $\dfrac{-b}{2a} = \dfrac{-8}{2(2)} = -2$

The axis of symmetry is $x = -2$.

b When $x = -2$,

$y = 2(-2)^2 + 8(-2) - 10$

$= -18$

The vertex is $(-2, -18)$.

c The y-intercept is -10.

When $y = 0$, $2x^2 + 8x - 10 = 0$

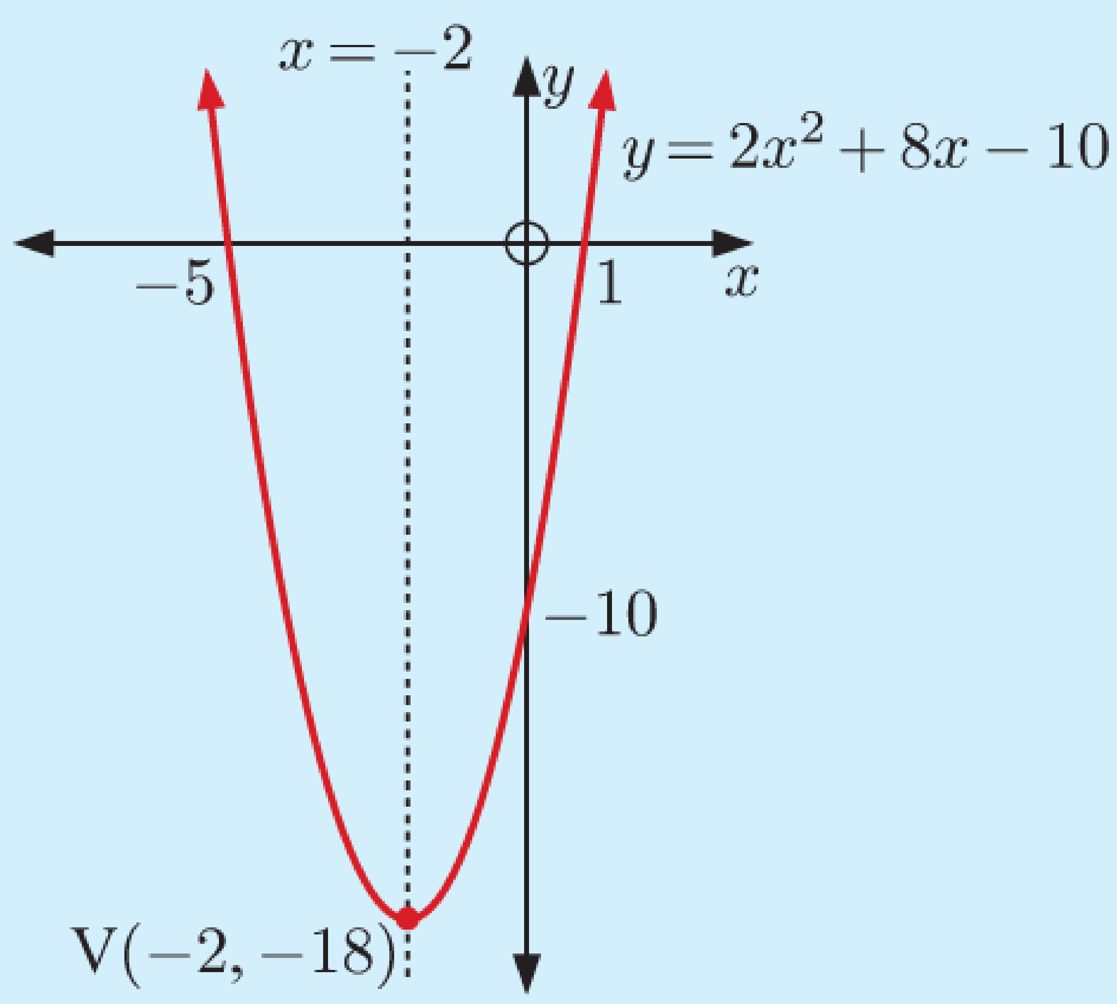
$\therefore\ 2(x^2 + 4x - 5) = 0$

$\therefore\ 2(x + 5)(x - 1) = 0$

$\therefore\ x = -5$ or 1

$\therefore$ the x-intercepts are -5 and 1.

d



EXERCISE 14B.3

1 For each of the following quadratics:

i Locate the turning point or vertex.

ii State whether the vertex is a minimum turning point or a maximum turning point.

a $y = x^2 - 4x + 2$

b $y = x^2 + 2x - 3$

c $y = 2x^2 + 4$

d $y = -3x^2 + 1$

e $y = 2x^2 + 8x - 7$

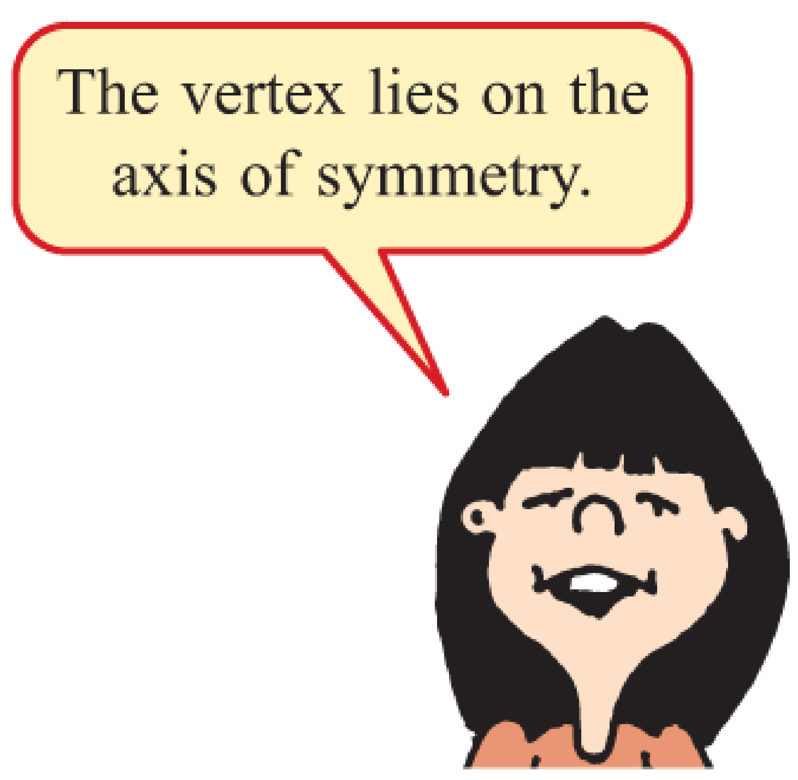
f $y = -x^2 - 4x - 9$

g $y = 2x^2 + 6x - 1$

h $y = 2x^2 - 10x + 3$

i $y = -\frac{1}{2}x^2 + x - 5$

j $y = \frac{1}{4}x^2 - 7x + 6$



2 For each of the following quadratics:

i State the axis of symmetry. **ii** Find the coordinates of the vertex.
iii Find the axes intercepts. **iv** Hence sketch the quadratic.

a $y = x^2 - 8x + 7$ **b** $y = -x^2 - 6x - 8$ **c** $y = 6x - x^2$
d $y = -x^2 + 3x - 2$ **e** $y = 2x^2 + 4x - 24$ **f** $y = -3x^2 + 4x - 1$
g $y = 2x^2 - 5x + 2$ **h** $y = 4x^2 - 8x - 5$ **i** $y = -\frac{1}{4}x^2 + 2x - 3$

3 For the quadratic function $y = ax^2 + bx + c$, suppose a and c remain constant, but b is allowed to vary. As b varies, the vertex of the quadratic changes.
Show that the path formed by the vertex is itself a quadratic function, and that the vertex of this quadratic function always lies on the y-axis.

DYNAMIC GEOMETRY PACKAGE

ACTIVITY 2

Click on the icon to run a card game for quadratic functions.

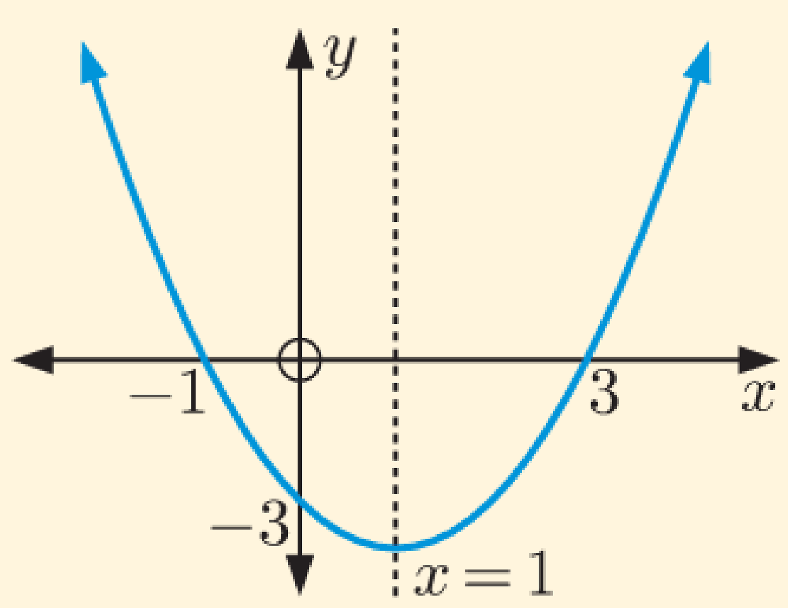
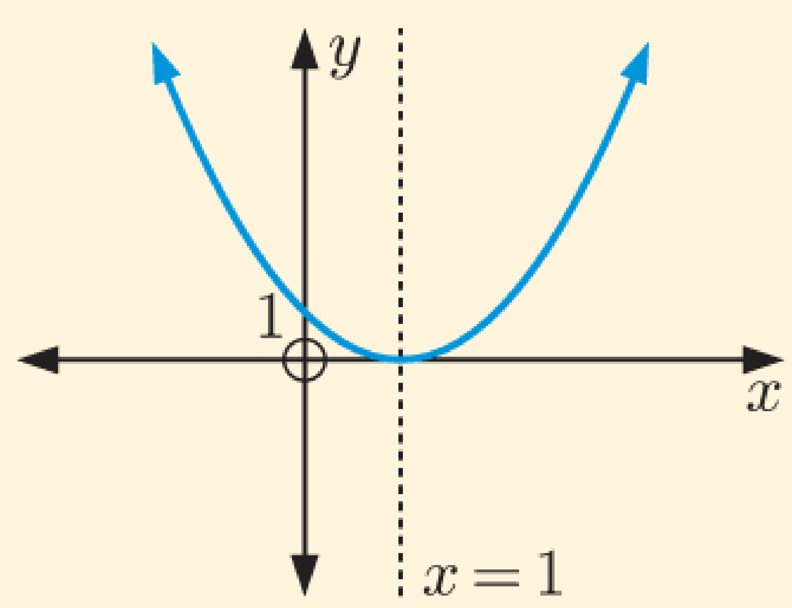
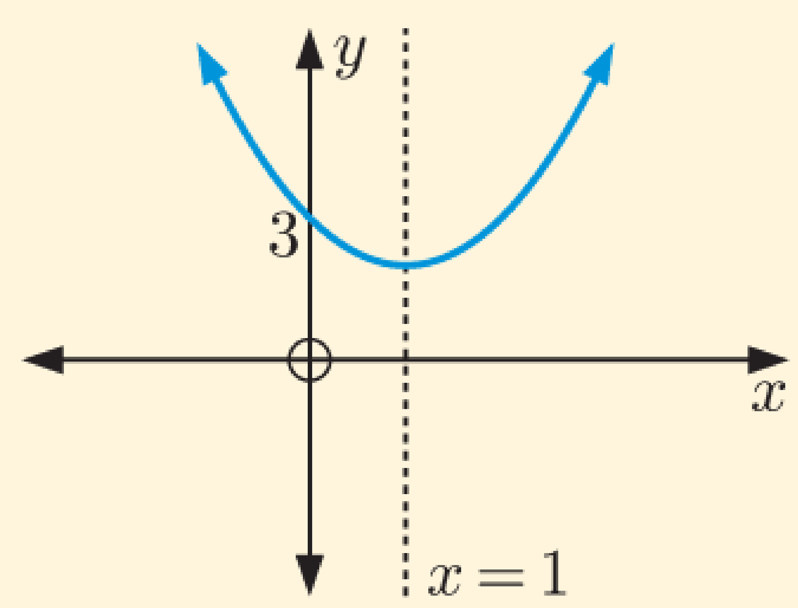
CARD GAME

C USING THE DISCRIMINANT

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $\Delta = b^2 - 4ac$.

We have used Δ to determine the number of real roots of the equation. If they exist, these roots correspond to zeros of the quadratic $y = ax^2 + bx + c$. Δ therefore tells us about the relationship between the graph of a quadratic function and the x-axis.

The graphs of $y = x^2 - 2x - 3$, $y = x^2 - 2x + 1$, and $y = x^2 - 2x + 3$ all have the same axis of symmetry, $x = 1$.

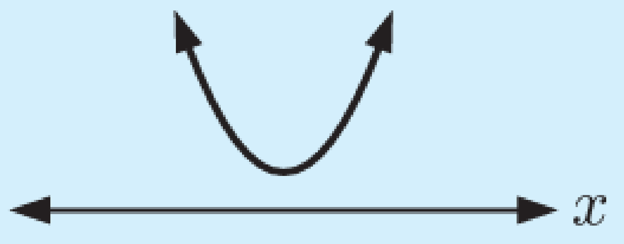
$y = x^2 - 2x - 3$	$y = x^2 - 2x + 1$	$y = x^2 - 2x + 3$
		
$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(-3)$ $= 16$ cuts the x-axis twice	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(1)$ $= 0$ touches the x-axis	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(3)$ $= -8$ does not cut the x-axis

For a quadratic function $y = ax^2 + bx + c$, we consider the discriminant $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, the graph cuts the x-axis twice.
- If $\Delta = 0$, the graph *touches* the x-axis.
- If $\Delta < 0$, the graph does not cut the x-axis.

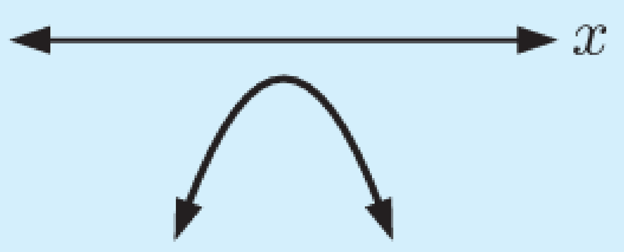
POSITIVE DEFINITE AND NEGATIVE DEFINITE QUADRATICS

Positive definite quadratics are quadratics which are positive for all values of x. So, $ax^2 + bx + c > 0$ for all $x \in \mathbb{R}$.



A quadratic is **positive definite** if and only if $a > 0$ and $\Delta < 0$.

Negative definite quadratics are quadratics which are negative for all values of x. So, $ax^2 + bx + c < 0$ for all $x \in \mathbb{R}$.



A quadratic is **negative definite** if and only if $a < 0$ and $\Delta < 0$.

Example 8 **Self Tutor**

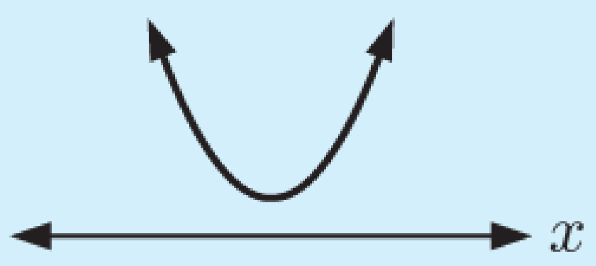
Use the discriminant to determine the relationship between the graph of each function and the x-axis:

a $y = x^2 + 3x + 4$ **b** $y = -2x^2 + 5x + 1$

a $a = 1,\ b = 3,\ c = 4$

$\therefore\ \Delta = b^2 - 4ac$
$= 9 - 4(1)(4)$
$= -7$

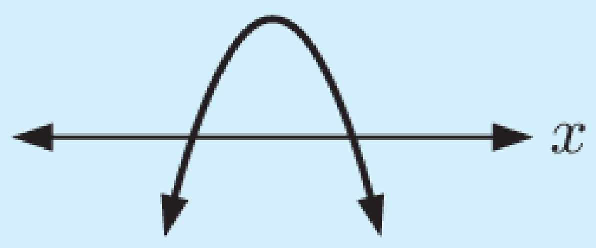
Since $\Delta < 0$, the graph does not cut the x-axis.
Since $a > 0$, the graph is concave up.
The graph is positive definite.



b $a = -2,\ b = 5,\ c = 1$

$\therefore\ \Delta = b^2 - 4ac$
$= 25 - 4(-2)(1)$
$= 33$

Since $\Delta > 0$, the graph cuts the x-axis twice.
Since $a < 0$, the graph is concave down.
The quadratic is neither positive definite nor negative definite.



EXERCISE 14C

1 Use the discriminant to determine the relationship between the graph of each function and the x-axis:

a $y = x^2 + x - 2$ **b** $y = x^2 - 4x + 1$ **c** $y = -x^2 - 3$
d $y = x^2 + 7x - 2$ **e** $y = x^2 + 8x + 16$ **f** $y = -2x^2 + 3x + 1$
g $y = 6x^2 + 5x - 4$ **h** $y = -x^2 + x + 6$ **i** $y = 9x^2 + 6x + 1$

2 Consider the graph of $y = 2x^2 - 5x + 1$.

a Describe the shape of the graph.
b Use the discriminant to show that the graph cuts the x-axis twice.
c Find the x-intercepts, rounding your answers to 2 decimal places.
d State the y-intercept.
e Hence sketch the function.

3 Consider the graph of $y = -x^2 + 4x - 7$.

a Use the discriminant to show that the graph does not cut the x-axis.

b Is the graph positive definite or negative definite? Explain your answer.

c Find the vertex and y-intercept.

d Hence sketch the function.

4 Show that:

a $2x^2 - 4x + 7$ is positive definite

b $-2x^2 + 3x - 4$ is negative definite

c $x^2 - 3x + 6 > 0$ for all x

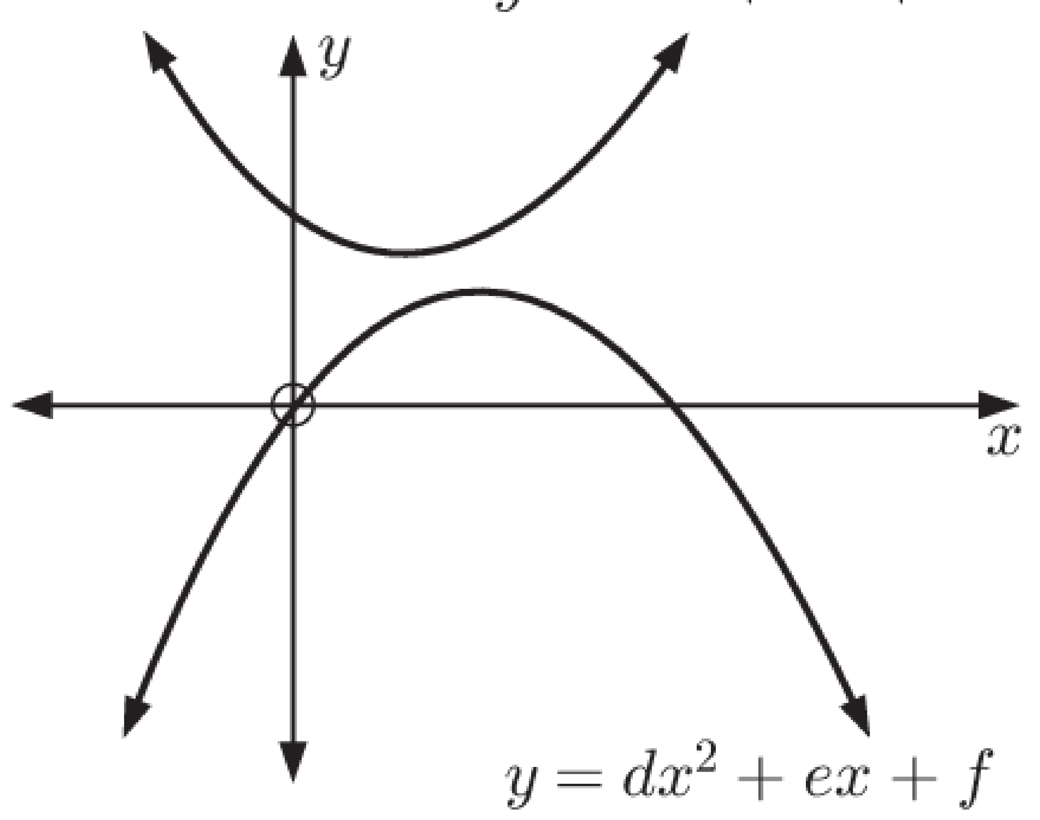
d $4x - x^2 - 6 < 0$ for all x.

5 Consider the graphs illustrated.

Let $y = ax^2 + bx + c$ have discriminant Δ_1, and $y = dx^2 + ex + f$ have discriminant Δ_2.

Copy and complete the following table by indicating whether each constant is positive, negative, or zero:

Constant	a	b	c	d	e	f	Δ_1	Δ_2
Sign								



Example 9 **Self Tutor**

Find the value(s) of k for which the function $y = x^2 - 6x + k$:

a cuts the x-axis twice **b** touches the x-axis **c** misses the x-axis.

$a = 1$, $b = -6$, $c = k$

$\therefore \Delta = b^2 - 4ac$

$= (-6)^2 - 4(1)(k)$

$= 36 - 4k$

a The graph cuts the x-axis twice if $\Delta > 0$.
$\therefore 36 - 4k > 0$
$\therefore 4k < 36$
$\therefore k < 9$

b The graph touches the x-axis twice if $\Delta = 0$.
$\therefore 36 - 4k = 0$
$\therefore k = 9$

c The graph does not cut the x-axis if $\Delta < 0$.
$\therefore 36 - 4k < 0$
$\therefore 4k > 36$
$\therefore k > 9$

6 For each quadratic function, find the value(s) of k for which the function:

i cuts the x-axis twice **ii** touches the x-axis **iii** misses the x-axis.

a $y = x^2 + 3x + k$ **b** $y = kx^2 - 4x + 1$ **c** $y = (k+1)x^2 - 2kx + (k-4)$

7 Explain why $3x^2 + kx - 1$ is never positive definite for any value of k.

8 Find the value of k such that $y = \frac{1}{2}x^2 + (k-2)x + k^2 + 4$ is *not* positive definite. What relationship does the graph have with the x-axis in this case?

9 b_1, c_1, b_2, and c_2 are real, non-zero numbers such that $b_1 b_2 = 2(c_1 + c_2)$. Show that at least one of the quadratics $y = x^2 + b_1 x + c_1$, and $y = x^2 + b_2 x + c_2$ cuts the x-axis twice.

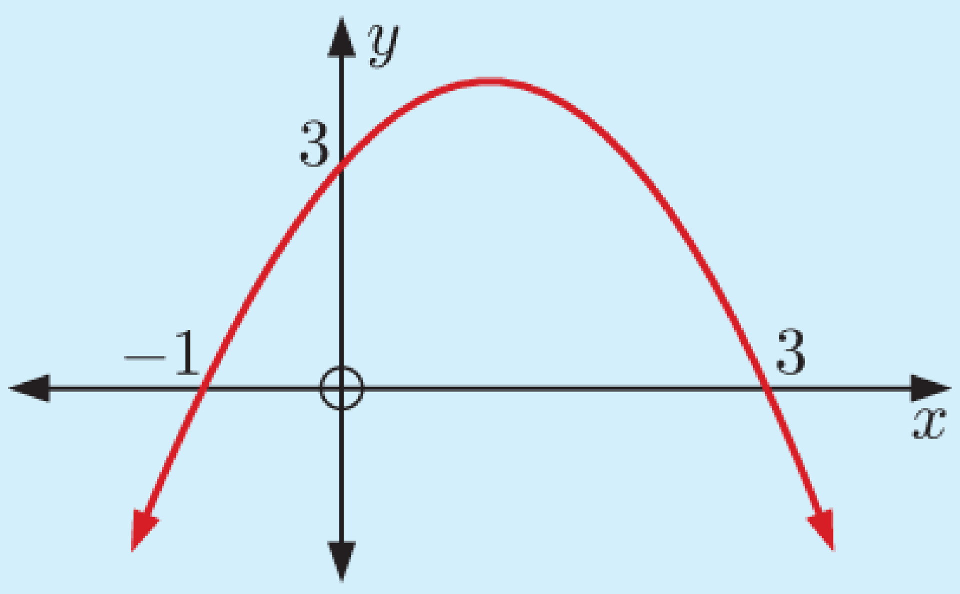
D FINDING A QUADRATIC FROM ITS GRAPH

If we are given sufficient information on or about a graph, we can determine the quadratic in whatever form is required.

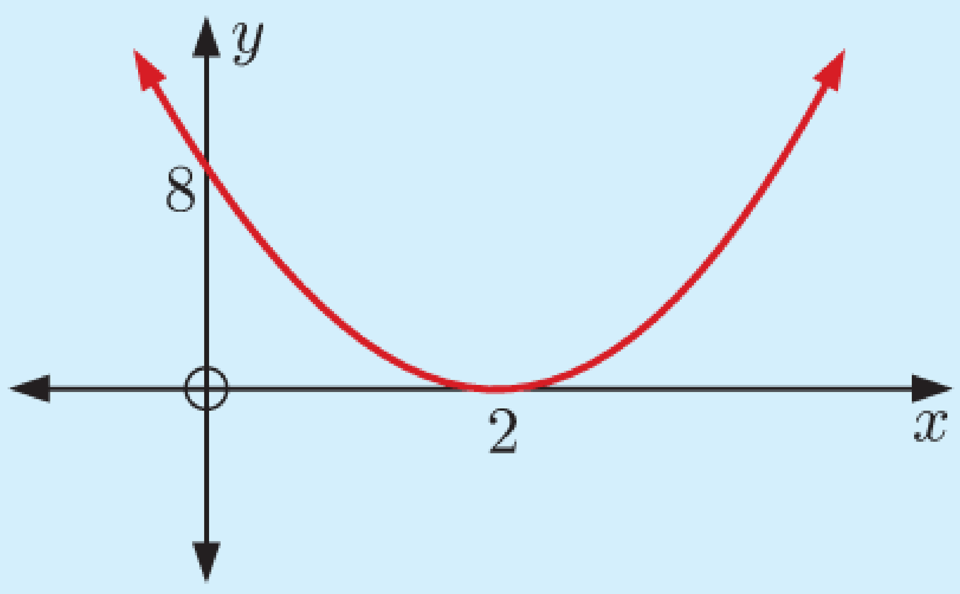
Example 10 Self Tutor

Find the equation of the quadratic with graph:

a



b



a Since the x-intercepts are -1 and 3, $y = a(x+1)(x-3)$.

The graph is concave down, so $a < 0$.

When $x = 0$, $y = 3$

$\therefore\ 3 = a(1)(-3)$

$\therefore\ a = -1$

The quadratic is $y = -(x+1)(x-3)$.

b The graph touches the x-axis at $x = 2$, so $y = a(x-2)^2$.

The graph is concave up, so $a > 0$.

When $x = 0$, $y = 8$

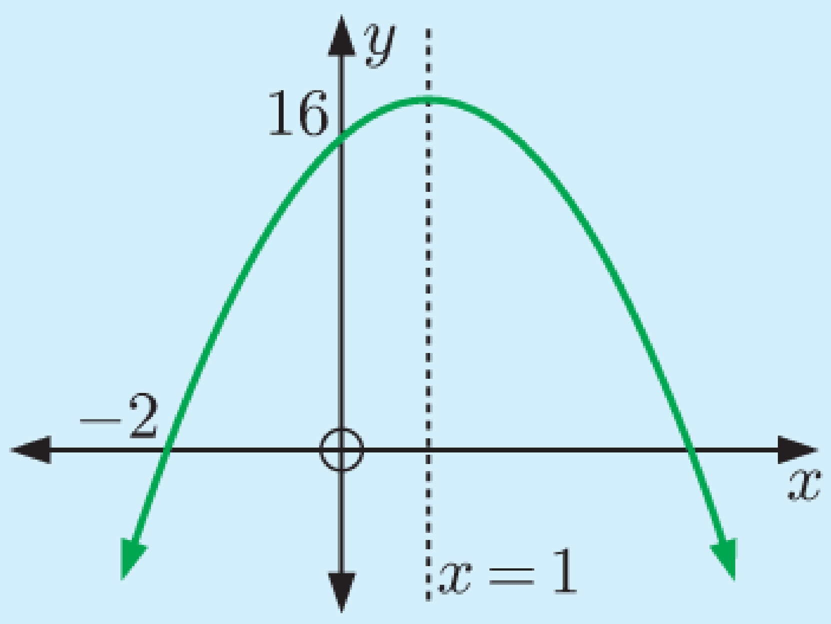
$\therefore\ 8 = a(-2)^2$

$\therefore\ a = 2$

The quadratic is $y = 2(x-2)^2$.

Example 11 Self Tutor

Find the equation of the quadratic with graph:



The axis of symmetry $x = 1$ lies midway between the x-intercepts.

$\therefore$ the other x-intercept is 4.

$\therefore$ the quadratic has the form

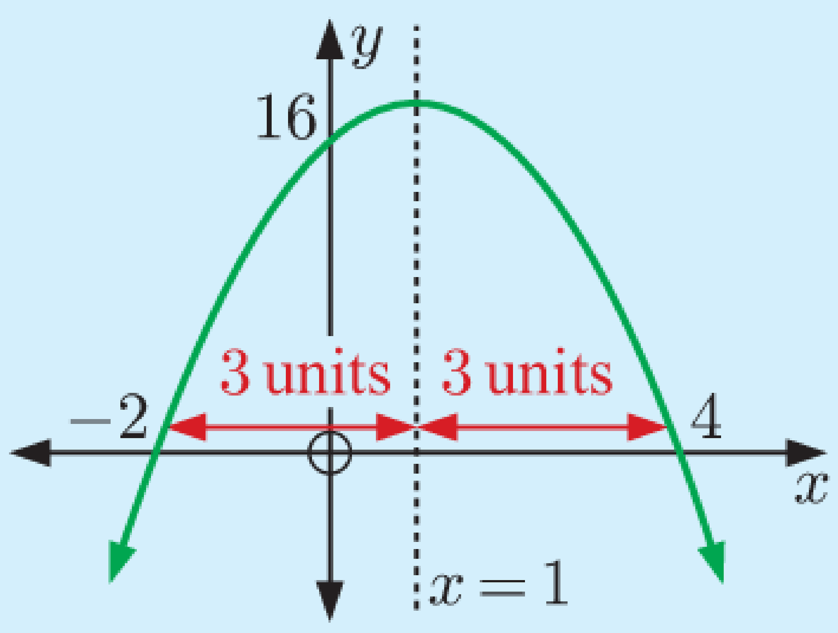
$y = a(x+2)(x-4)$ where $a < 0$

But when $x = 0$, $y = 16$

$\therefore\ 16 = a(2)(-4)$

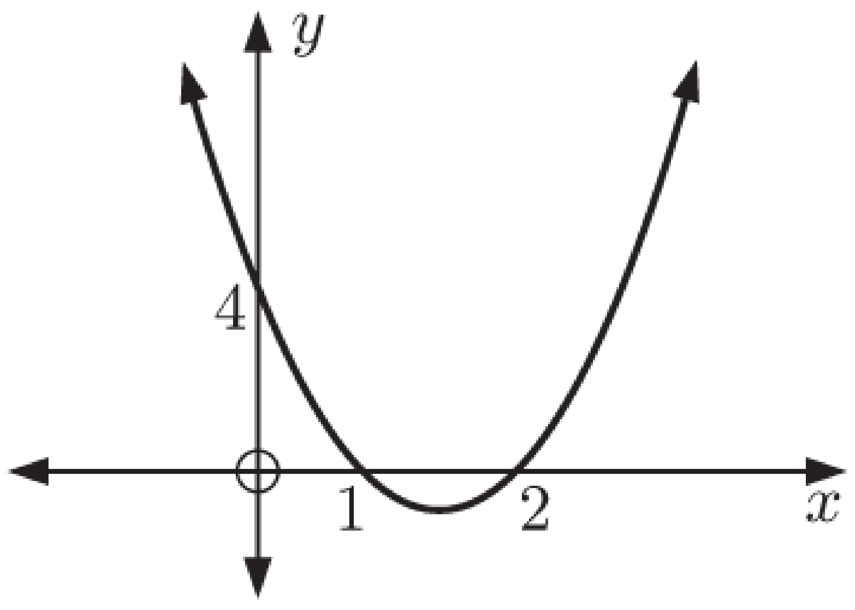
$\therefore\ a = -2$

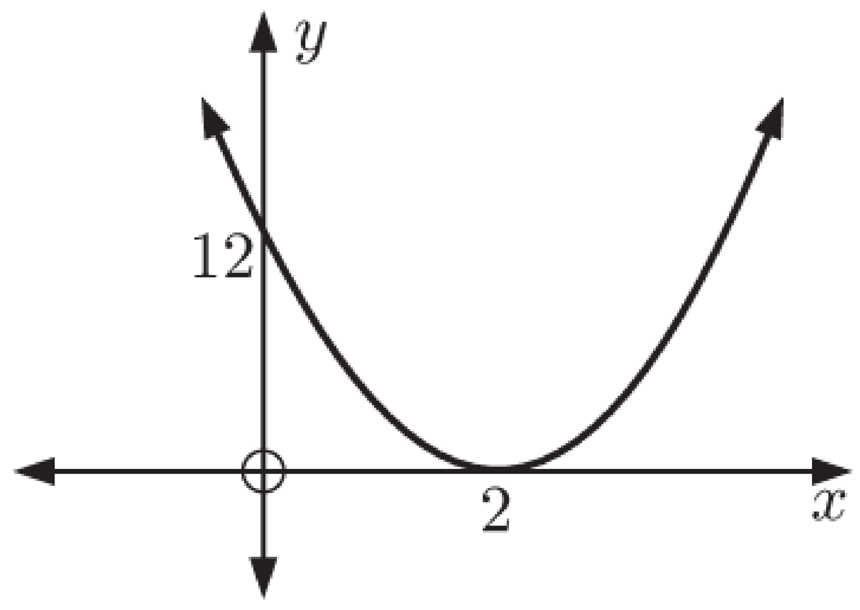
The quadratic is $y = -2(x+2)(x-4)$.

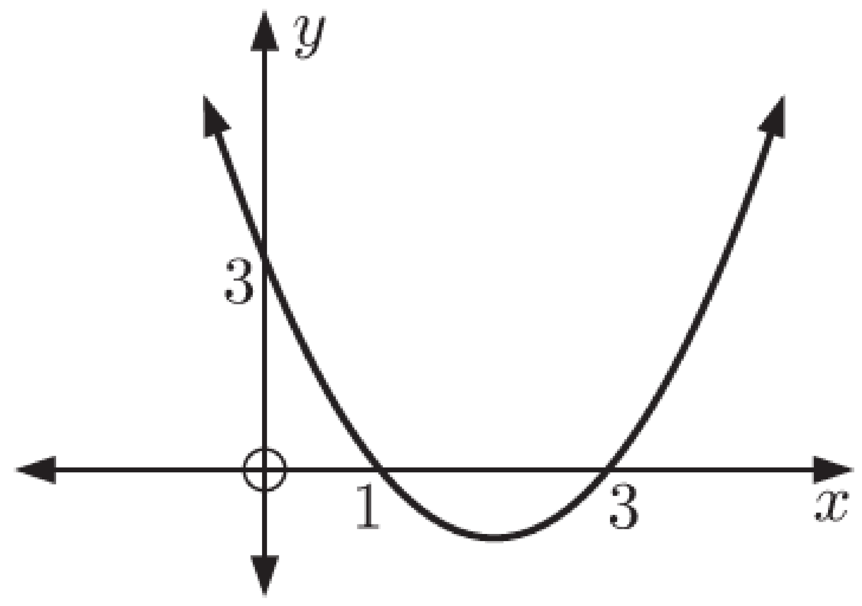


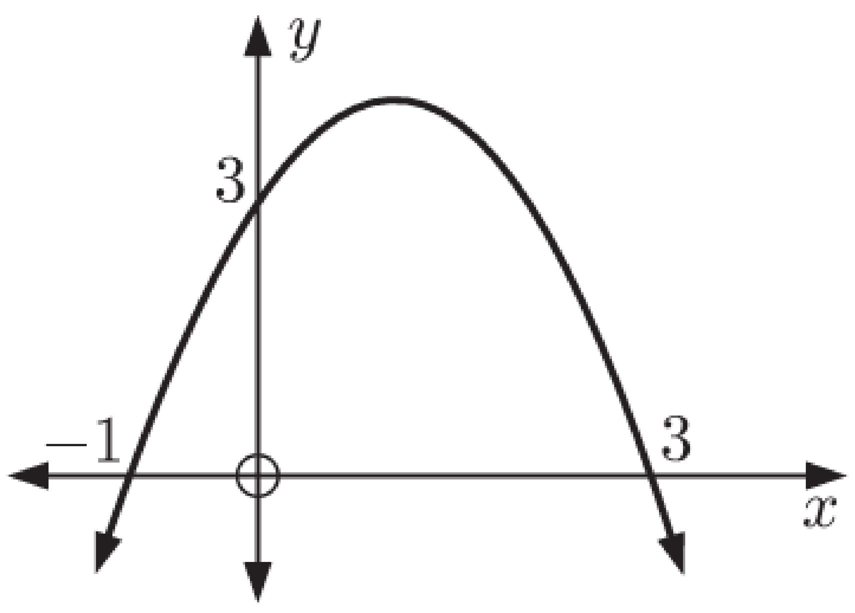
EXERCISE 14D

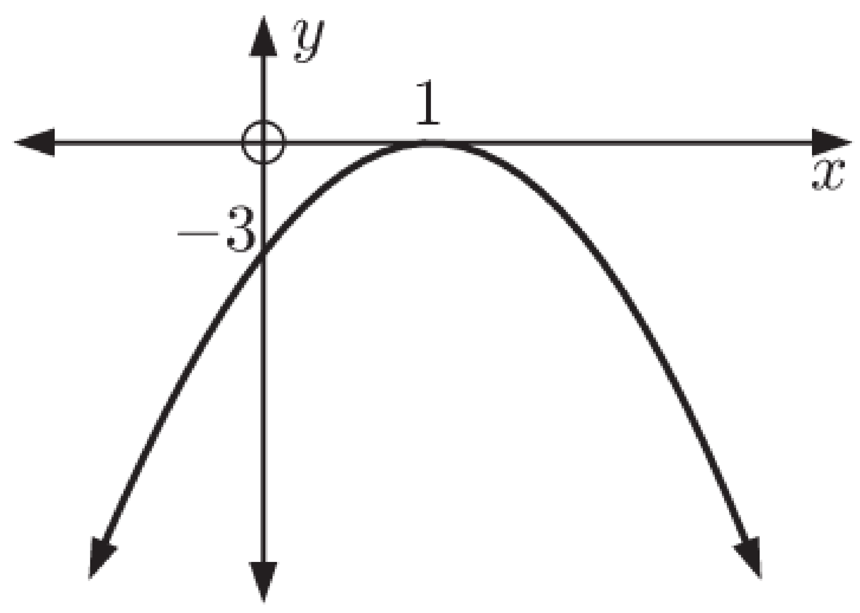
1 Find the equation of the quadratic with graph:

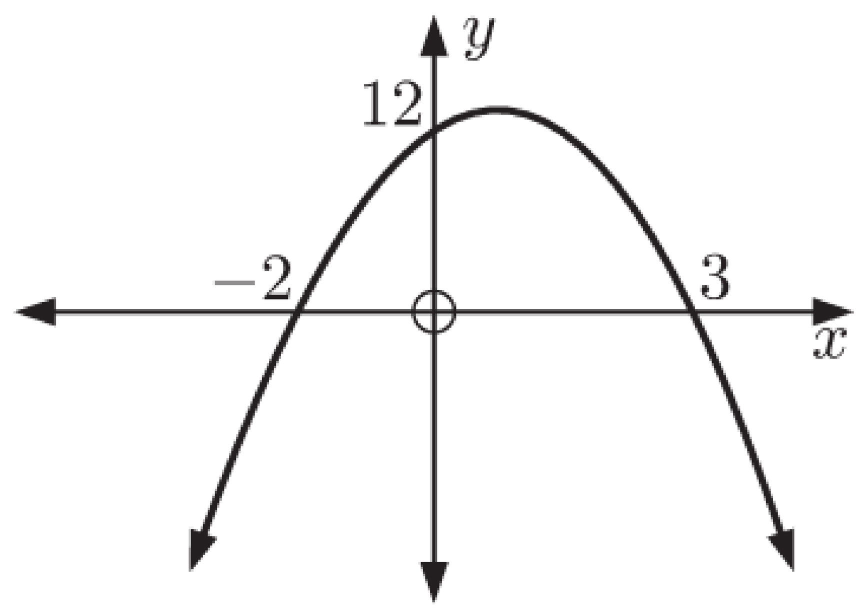
a


b


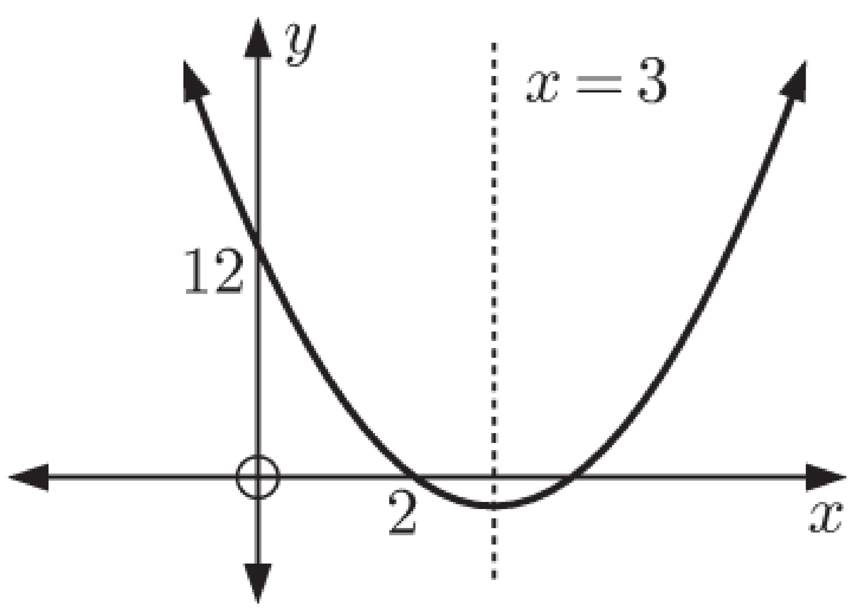
c


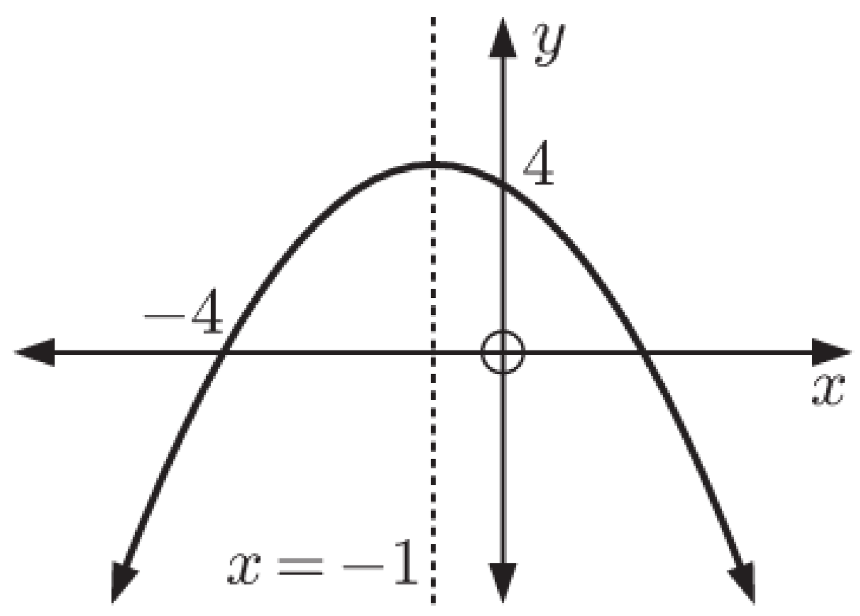
d


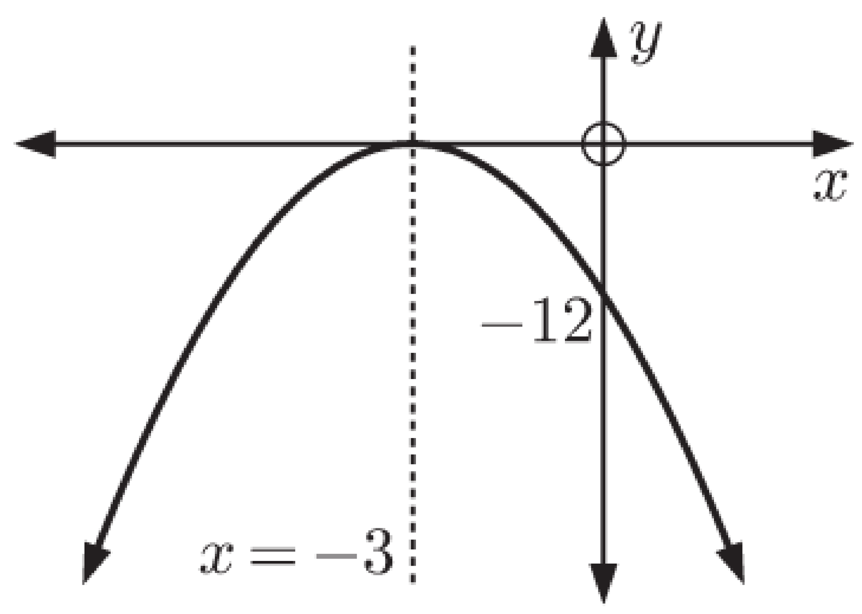
e


f


2 Find the equation of the quadratic with graph:

a


b


c


Example 12 **Self Tutor**

Find the equation of the quadratic whose graph cuts the x-axis at 4 and -3, and which passes through the point $(2, -20)$. Give your answer in the form $y = ax^2 + bx + c$.

Since the x-intercepts are 4 and -3, the quadratic has the form $y = a(x-4)(x+3)$, $a \neq 0$.

When $x = 2$, $y = -20$

$\therefore\ -20 = a(2-4)(2+3)$

$\therefore\ -20 = a(-2)(5)$

$\therefore\ a = 2$

The quadratic is $y = 2(x-4)(x+3)$

$= 2(x^2 - x - 12)$

$= 2x^2 - 2x - 24$

3 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph:

a cuts the x-axis at 5 and 1, and passes through $(2, -9)$

b cuts the x-axis at 2 and $-\frac{1}{2}$, and passes through $(3, -14)$

c touches the x-axis at 3 and passes through $(-2, -25)$

d touches the x-axis at -2 and passes through $(-1, 4)$

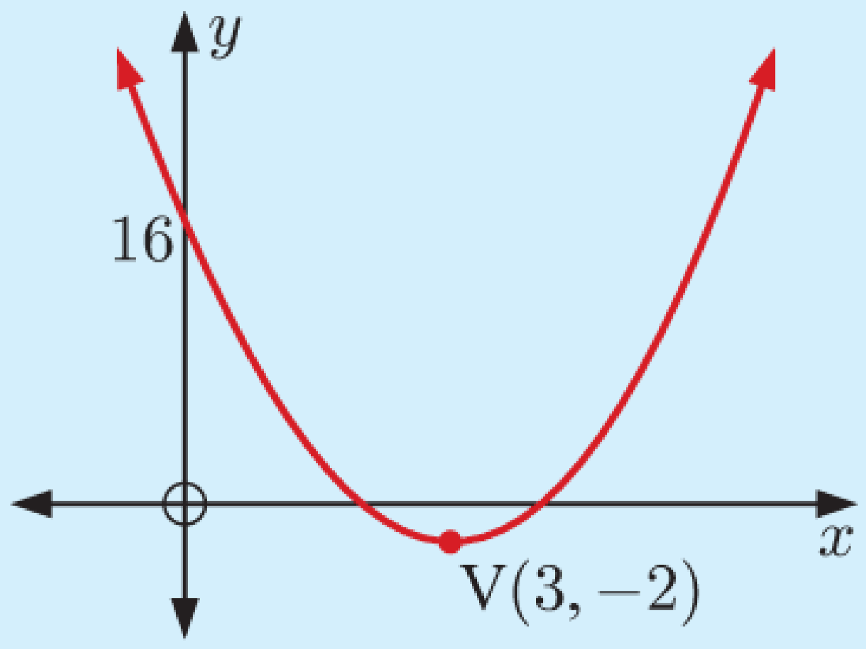
e cuts the x-axis at 3, passes through $(5, 12)$, and has axis of symmetry $x = 2$

f cuts the x-axis at 5, passes through $(2, 5)$, and has axis of symmetry $x = 1$.

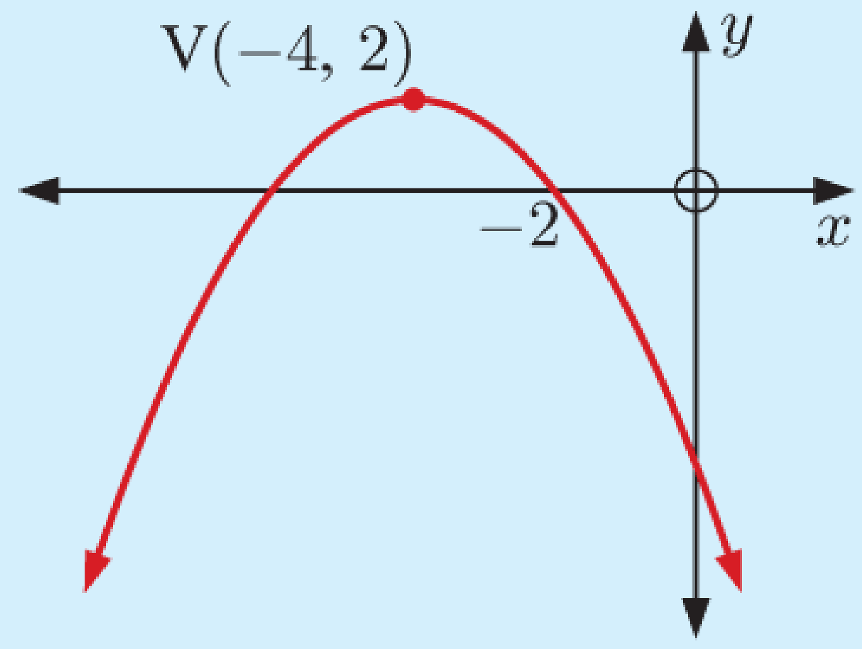
Example 13 **Self Tutor**

Find the equation of the quadratic with graph:

a



b



a Since the vertex is $(3, -2)$, the quadratic has the form $y = a(x-3)^2 - 2$ where $a > 0$.

When $x = 0$, $y = 16$

$\therefore\ 16 = a(-3)^2 - 2$

$\therefore\ 16 = 9a - 2$

$\therefore\ 18 = 9a$

$\therefore\ a = 2$

The quadratic is $y = 2(x-3)^2 - 2$.

b Since the vertex is $(-4, 2)$, the quadratic has the form $y = a(x+4)^2 + 2$ where $a < 0$.

When $x = -2$, $y = 0$

$\therefore\ 0 = a(2)^2 + 2$

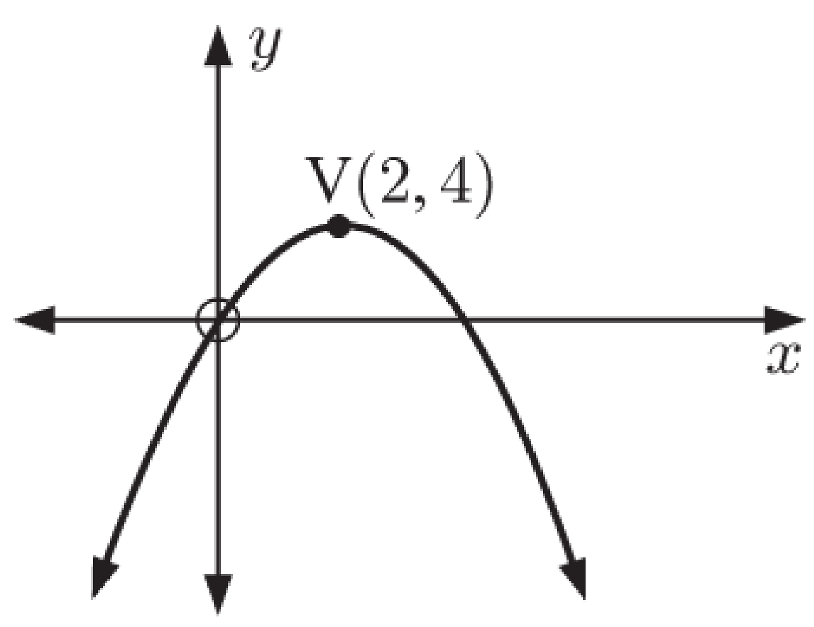
$\therefore\ 4a = -2$

$\therefore\ a = -\frac{1}{2}$

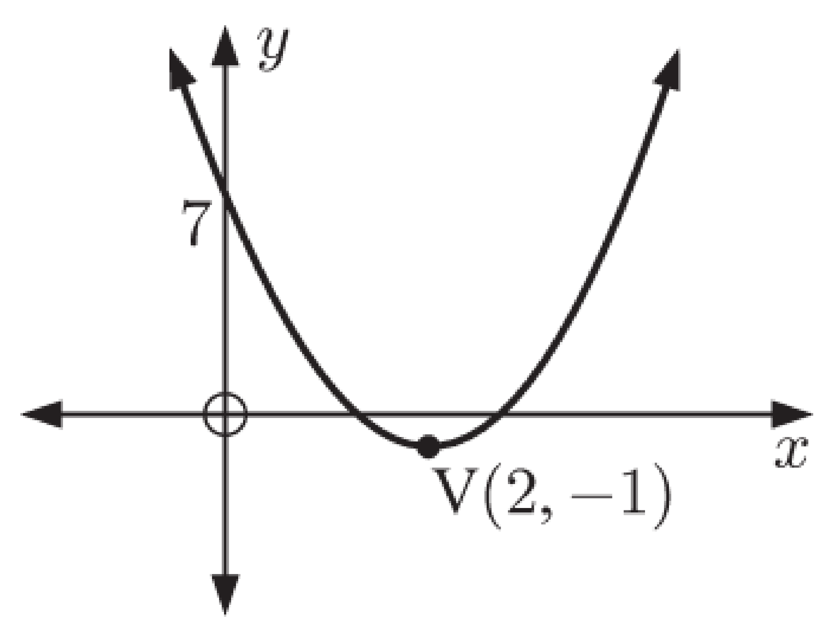
The quadratic is $y = -\frac{1}{2}(x+4)^2 + 2$.

4 If V is the vertex, find the equation of the quadratic with graph:

a



b



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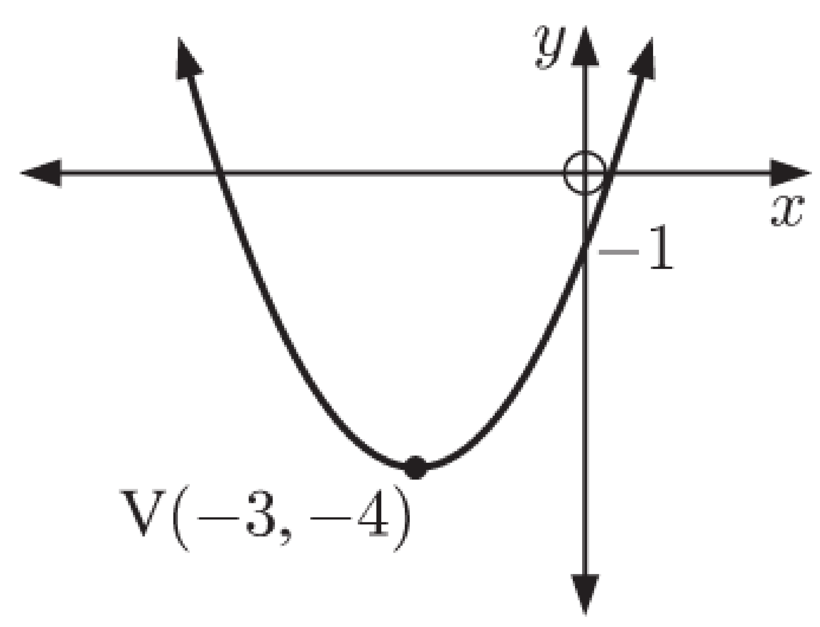

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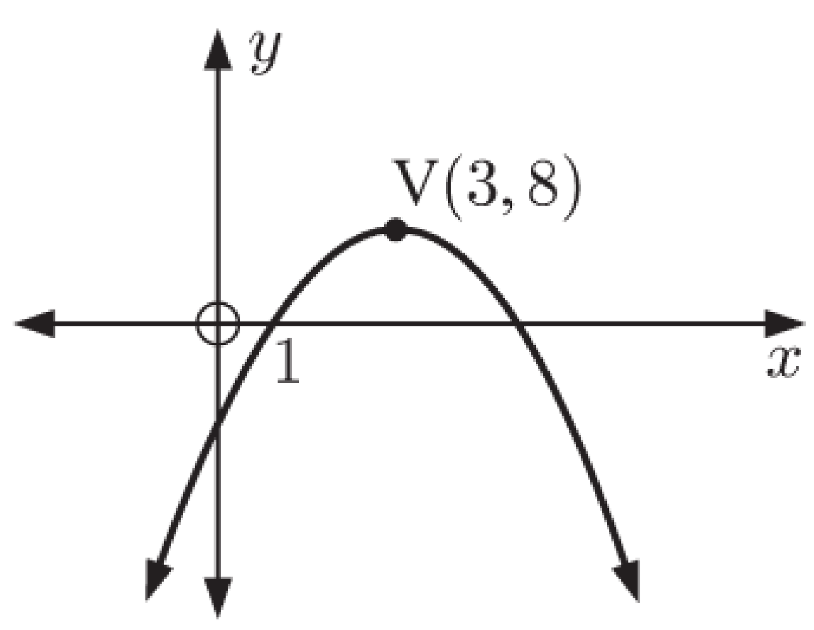

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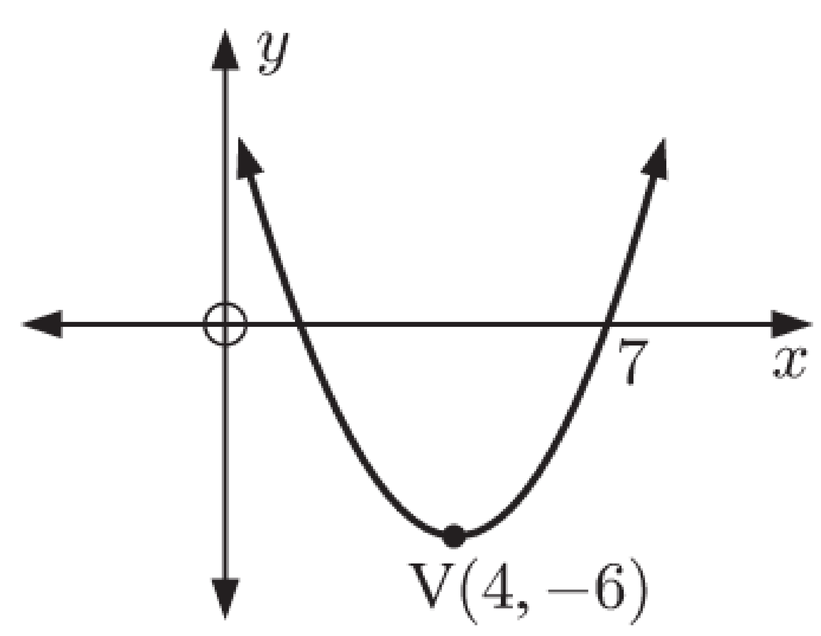

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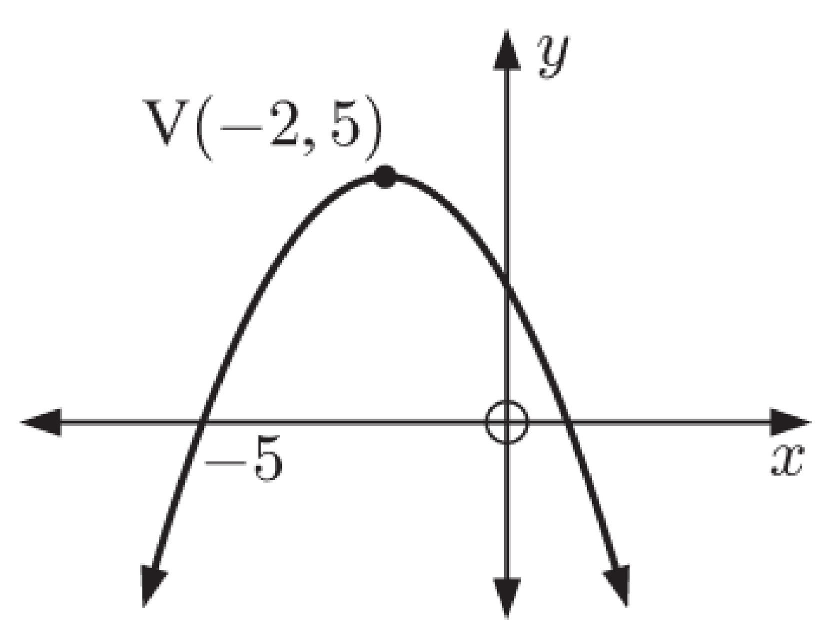

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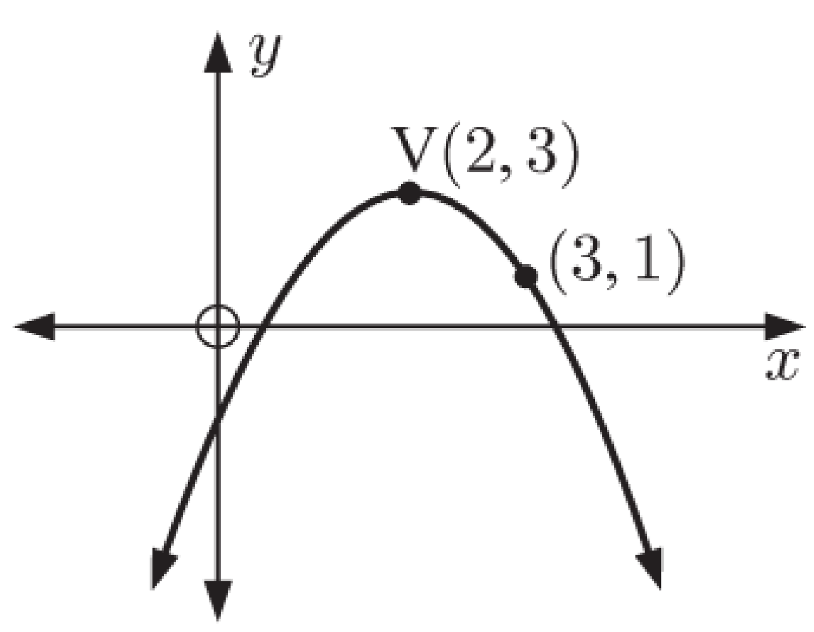

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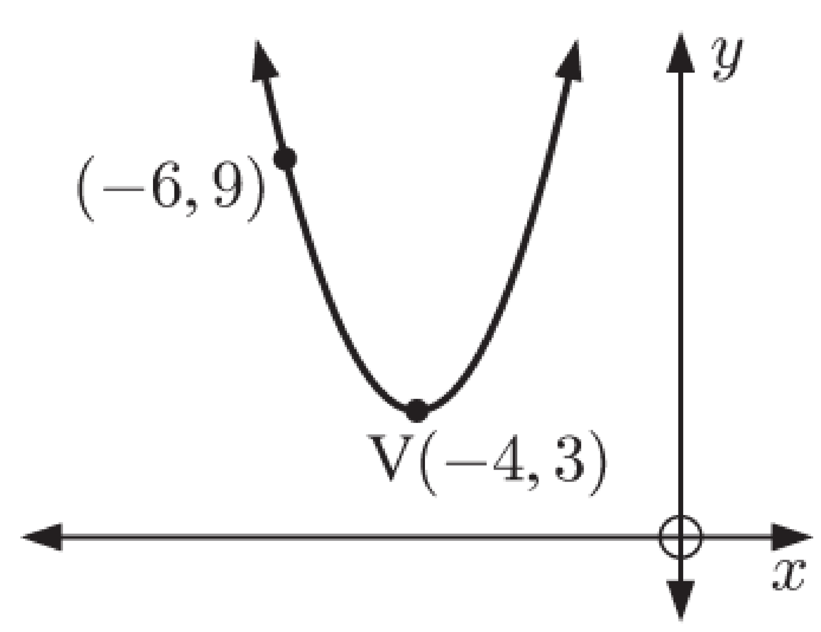
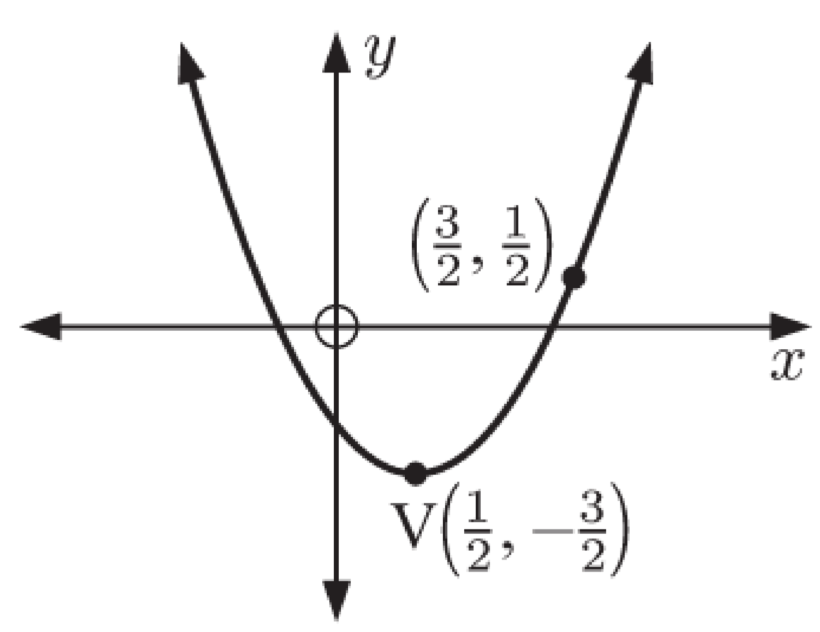


i



5 A quadratic has vertex $(2, -5)$, and passes through the point $(-1, 13)$. Find the value of the quadratic when $x = 4$.

INVESTIGATION 3 — FINDING QUADRATICS

For the quadratic $y = 2x^2 + 3x + 7$ we can construct a table of values for $x = 0, 1, 2, 3, 4, 5$.

x	0	1	2	3	4	5
y	7	12	21	34	51	72

We turn this table into a **difference table** by adding two further rows:

- the row Δ_1 gives the differences between successive y-values
- the row Δ_2 gives the differences between successive Δ_1-values.

x	0	1	2	3	4	5
y	7	12	21	34	51	72
Δ_1	5	9	13	17	21	
Δ_2	4	4	4	4		

$9 - 5$ $\quad$ $34 - 21$ $\quad$ $72 - 51$

What to do:

1 Construct difference tables for $x = 0, 1, 2, 3, 4, 5$ for each of the following quadratics:

a $y = x^2 + 4x + 3$ **b** $y = 3x^2 - 4x$

c $y = 5x - x^2$ **d** $y = 4x^2 - 5x + 2$

2 What do you notice about the Δ_2 row for each quadratic in **1**?

3 Consider the general quadratic $y = ax^2 + bx + c$, $a \neq 0$.

a Copy and complete the following difference table:

x	0	1	2	3	4	5
y	ⓒ	$a + b + c$	$4a + 2b + c$	......	......	......
Δ_1	◯	......	......	......	......	
Δ_2	◯	......	......	......		

b Comment on the Δ_2 row.

c What can the circled numbers be used for?

4 Use your observations in **3** to determine, if possible, the quadratics with the following tables of values:

a

x	0	1	2	3	4
y	6	5	8	15	26

b

x	0	1	2	3	4
y	8	10	18	32	52

c

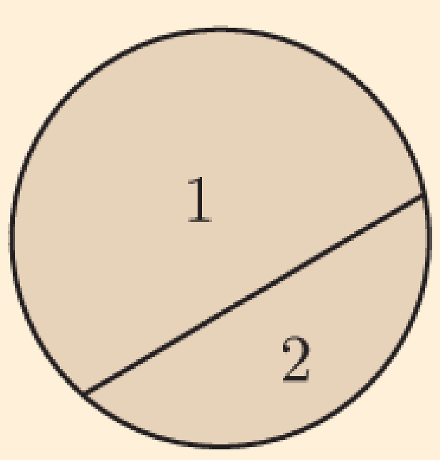
x	0	1	2	3	4
y	1	2	-1	-8	-19

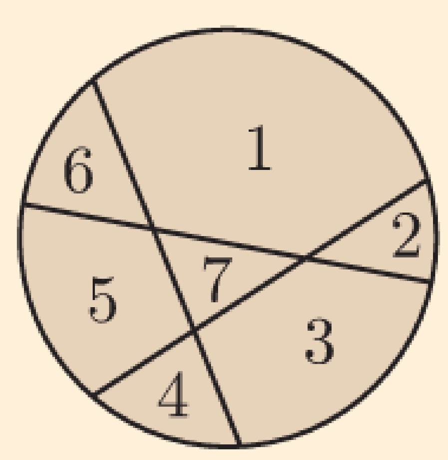
d

x	0	1	2	3	4
y	5	3	-1	-7	-15

5 We wish to determine the **maximum** number of pieces into which a pizza can be cut using n straight cuts across it. For example:

- for $n = 1$ we can make 2 pieces
- for $n = 3$ we can make 7 pieces.





a Copy and complete:

Number of cuts, n	0	1	2	3	4	5
Maximum number of pieces, P_n						

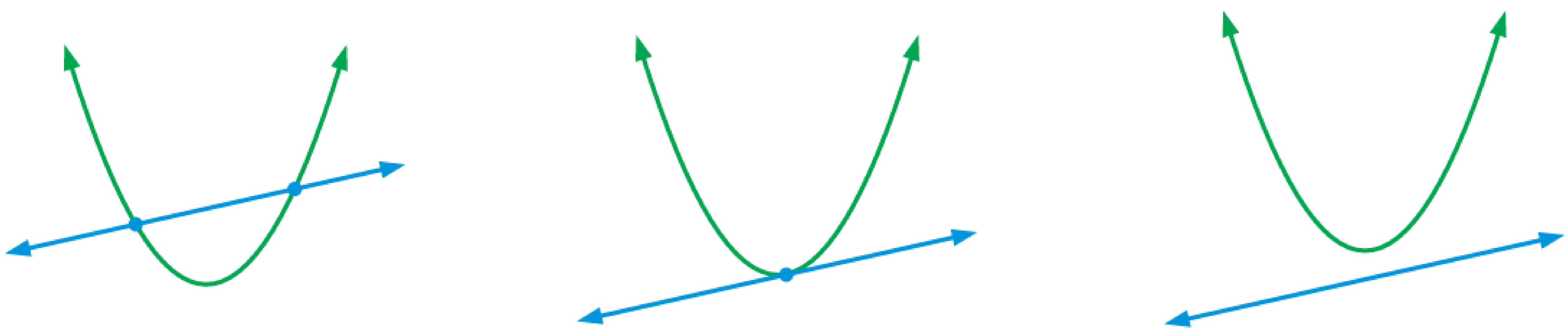
b Complete the Δ_1 and Δ_2 rows. Hence determine a quadratic formula for P_n.

c For a huge pizza with 12 cuts across it, find the maximum number of pieces which can result.

E THE INTERSECTION OF GRAPHS

Consider the graphs of a quadratic function and a linear function on the same set of axes.

There are three possible scenarios for intersection:



cutting (2 points of intersection) **touching** (1 point of intersection) **missing** (no points of intersection)

If the line *touches* the curve, we say that the line is a **tangent** to the curve.

The x-coordinates of any intersection points of the graphs can be found by solving the two equations **simultaneously**.

Example 14 Self Tutor

Find the coordinates of the point(s) of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$.

$y = x^2 - x - 18$ meets $y = x - 3$ where

$$x^2 - x - 18 = x - 3$$
$$\therefore \ x^2 - 2x - 15 = 0 \quad \{\text{RHS} = 0\}$$
$$\therefore \ (x-5)(x+3) = 0 \quad \{\text{factorising}\}$$
$$\therefore \ x = 5 \text{ or } -3$$

Substituting into $y = x - 3$, when $x = 5$, $y = 2$ and when $x = -3$, $y = -6$.

$\therefore$ the graphs meet at $(5, 2)$ and $(-3, -6)$.

Graphing each side of an inequality helps us to illustrate its solutions. Any points where the graphs intersect will lie at the endpoints of the interval(s) in the solution.

Example 15 **Self Tutor**

Consider the curves $y = x^2 + 5x + 6$ and $y = 2x^2 + 2x - 4$.

a Solve for x: $x^2 + 5x + 6 = 2x^2 + 2x - 4$.

b Graph the two curves on the same set of axes.

c Hence solve for x: $x^2 + 5x + 6 > 2x^2 + 2x - 4$.

a $x^2 + 5x + 6 = 2x^2 + 2x - 4$

$\therefore\ x^2 - 3x - 10 = 0$

$\therefore\ (x + 2)(x - 5) = 0$

$\therefore\ x = -2$ or 5

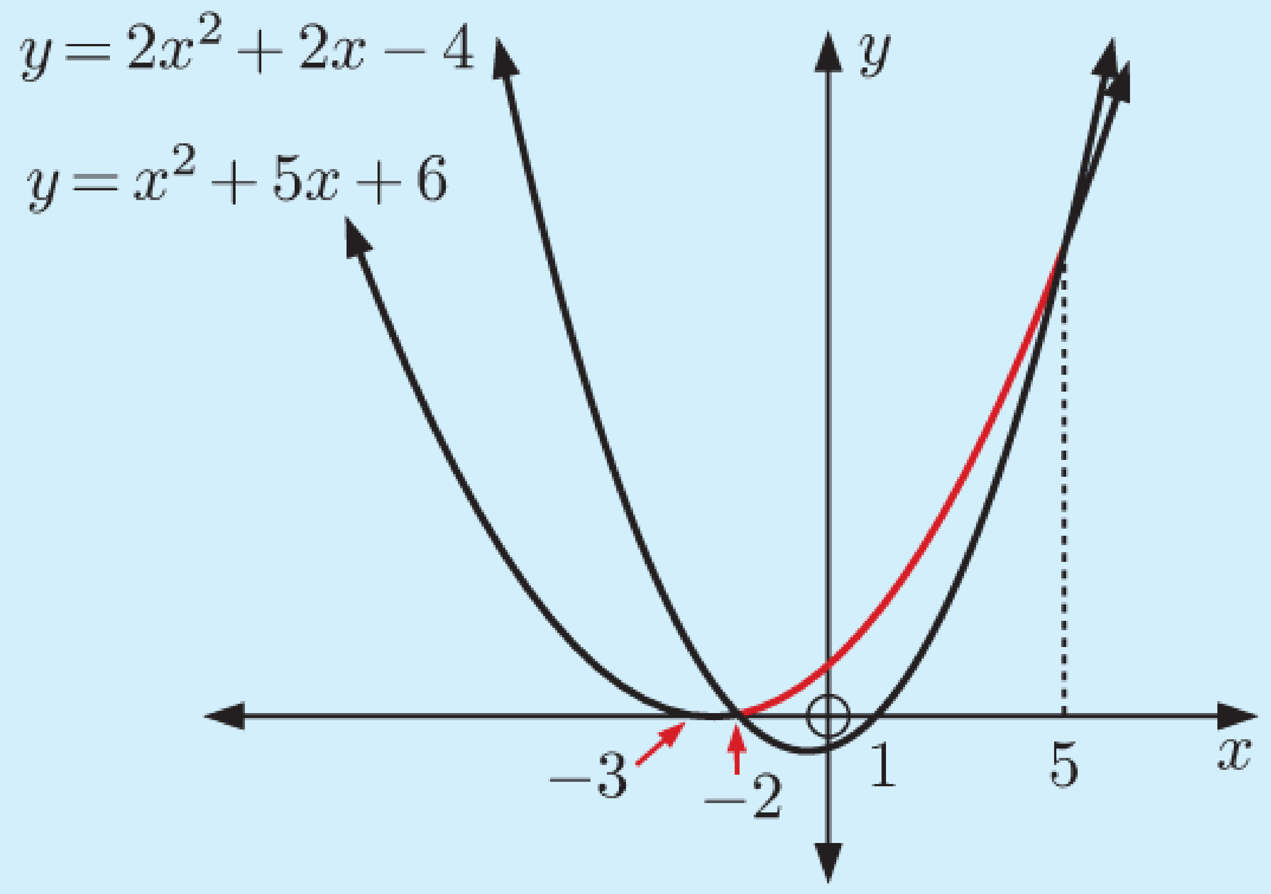
b $y = x^2 + 5x + 6$

$= (x + 2)(x + 3)$ has zeros -2 and -3.

$y = 2x^2 + 2x - 4$

$= 2(x^2 + x - 2)$

$= 2(x + 2)(x - 1)$ has zeros -2 and 1.



c If $x^2 + 5x + 6 > 2x^2 + 2x - 4$, the graph of $y = x^2 + 5x + 6$ is above the graph of $y = 2x^2 + 2x - 4$.

This occurs when $-2 < x < 5$.

EXERCISE 14E

1 Find the coordinates of the point(s) of intersection of:

a $y = x^2 - 2x + 8$ and $y = x + 6$

b $y = -x^2 + 3x + 9$ and $y = 2x - 3$

c $y = x^2 - 4x + 3$ and $y = 2x - 6$

d $y = -x^2 + 4x - 7$ and $y = 5x - 4$

2 Use technology to find the coordinates of the points of intersection of the graphs with equations:

GRAPHING PACKAGE

a $y = x^2 - 3x + 7$ and $y = x + 5$

b $y = x^2 - 5x + 2$ and $y = x - 7$

c $y = -x^2 - 2x + 4$ and $y = x + 8$

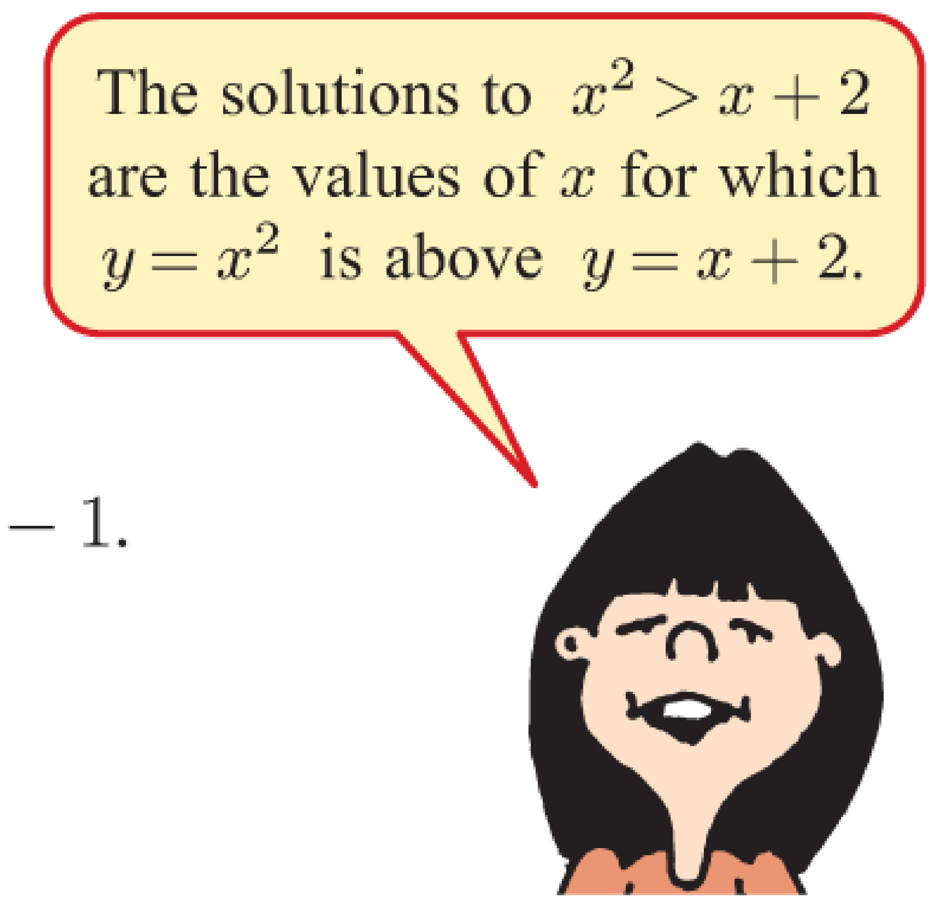
d $y = -x^2 + 4x - 2$ and $y = 5x - 6$.

3 Consider the graphs with equations $y = x^2$ and $y = x + 2$.

a Find the points where the graphs intersect.

b Plot the graphs on the same set of axes.

c Hence solve for x: $x^2 > x + 2$.



4 Consider the graphs with equations $y = x^2 + 2x - 3$ and $y = x - 1$.

a Find the points where the graphs intersect.

b Plot the graphs on the same set of axes.

c Hence solve for x: $x^2 + 2x - 3 > x - 1$.

5 Consider the curves $y = 2x^2 - x + 3$ and $y = 2 + x + x^2$.

  a Find the points where the curves intersect.

  b Plot the curves on the same set of axes.

  c Hence solve for x: $2x^2 - x + 3 > 2 + x + x^2$.

6 Consider the graphs with equations $y = \frac{4}{x}$ and $y = x + 3$.

  a Solve $\frac{4}{x} = x + 3$ using algebra.

  b Use technology to plot the graphs on the same set of axes.

  c Hence solve for x: $\frac{4}{x} > x + 3$.

Example 16 **Self Tutor**

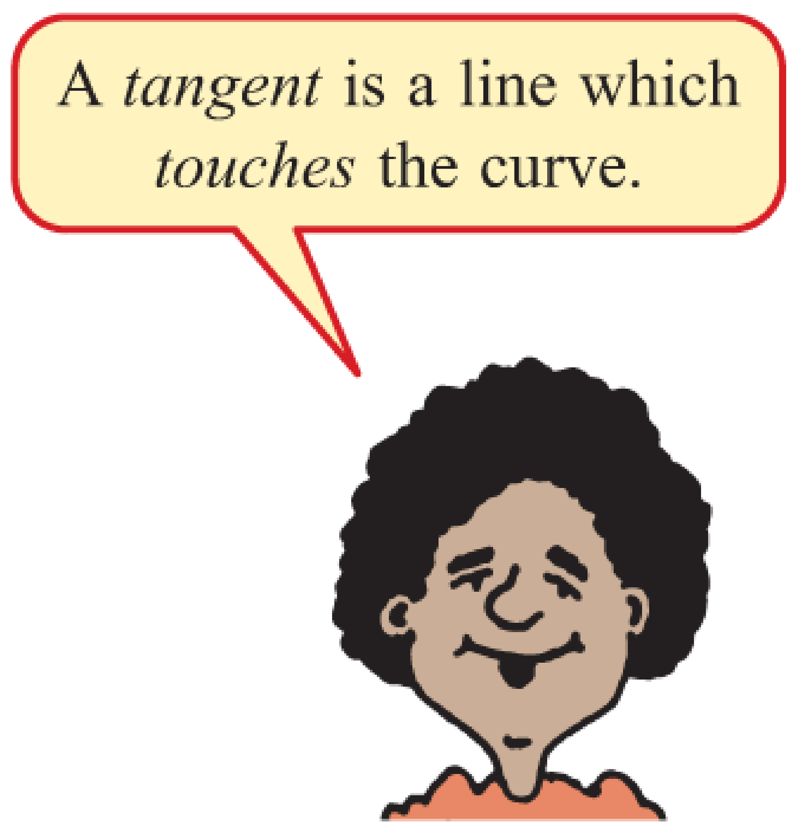
$y = 2x + k$ is a tangent to $y = 2x^2 - 3x + 4$. Find k.

$y = 2x + k$ meets $y = 2x^2 - 3x + 4$ where

$$2x^2 - 3x + 4 = 2x + k$$
$$\therefore\ 2x^2 - 5x + (4 - k) = 0$$

Since the graphs touch, this quadratic has $\Delta = 0$

$$\therefore\ (-5)^2 - 4(2)(4-k) = 0$$
$$\therefore\ 25 - 8(4-k) = 0$$
$$\therefore\ 25 - 32 + 8k = 0$$
$$\therefore\ 8k = 7$$
$$\therefore\ k = \tfrac{7}{8}$$



7 For what value of c is the line $y = 3x + c$ a tangent to the parabola with equation $y = x^2 - 5x + 7$?

8 Find the values of m for which the lines $y = mx - 2$ are tangents to the curve with equation $y = x^2 - 4x + 2$.

9 Find the gradients of the lines with y-intercept 1 that are tangents to the curve $y = 3x^2 + 5x + 4$.

10 **a** For what values of c do the lines $y = x + c$ never meet the parabola with equation $y = 2x^2 - 3x - 7$?

  b Choose one of the values of c found in part **a**. Illustrate with a sketch that these graphs never meet.

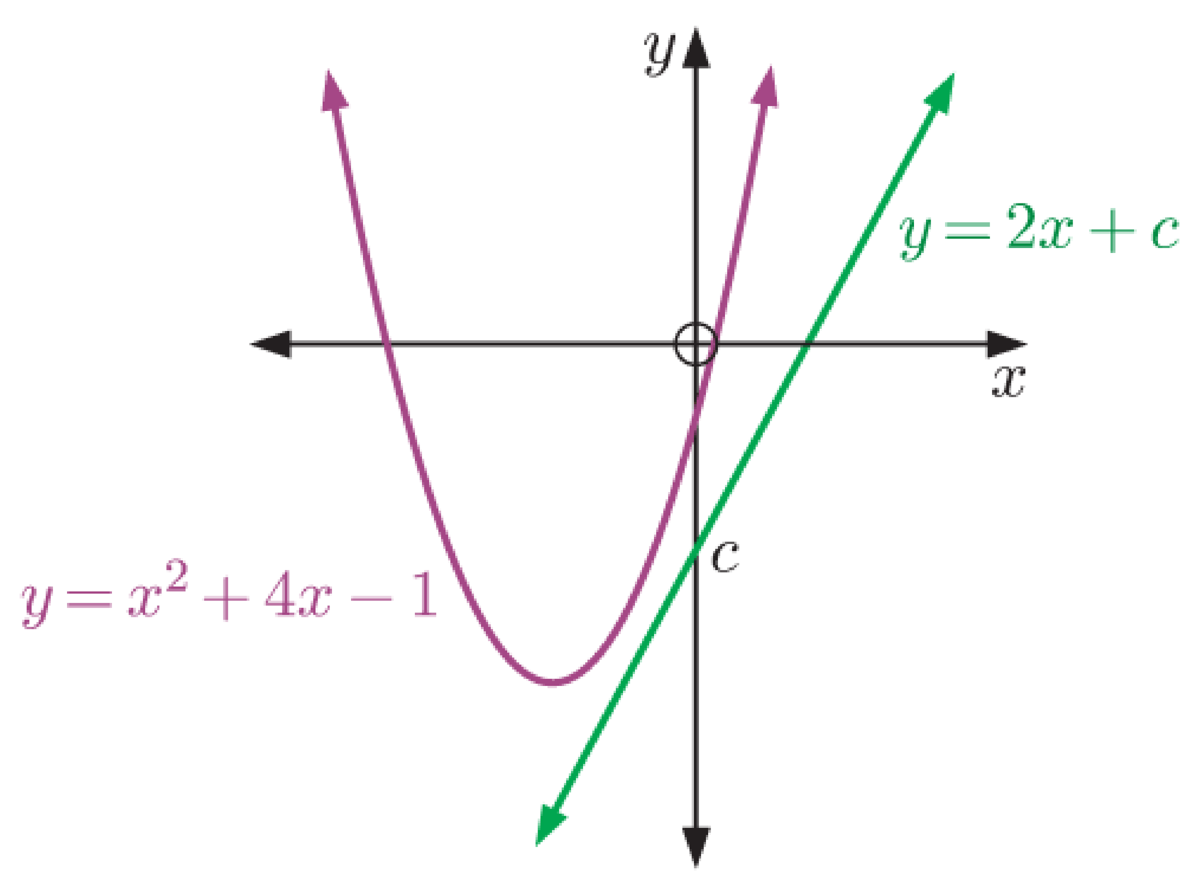
11 Prove that two quadratic functions can intersect at most twice.

12 Consider the curve $y = x^2 + 4x - 1$ and the line $y = 2x + c$. Find the values of c for which the line:

  a meets the curve twice

  b is a tangent to the curve

  c does not meet the curve.



DYNAMIC GEOMETRY PACKAGE

13 Show that any linear function passing through $(0, 3)$ will meet the curve $y = 2x^2 - x - 2$ twice.

14 The graphs of $y = (x-2)^2$ and $y = -x^2 + bx + c$ touch when $x = 3$.
Find the values of b and c.

15 Consider the quadratic function $y = x^2$. The point $P(a, a^2)$ lies on the function.

a The line $y = m(x-a) + c$ also passes through P. What, if anything, can you say about the values of m and c?

b Suppose $y = m(x-a) + a^2$ touches $y = x^2$ at P. Find the value of m.

F PROBLEM SOLVING WITH QUADRATICS

Some real-world problems can be solved using a quadratic equation.

Any answer we obtain must be checked to see if it is reasonable. For example:

- if we are finding a length then it must be positive and we reject any negative solutions
- if we are finding "how many people are present" then the answer must be a positive integer.

We employ the following general problem solving method:

Step 1: If the information is given in words, translate it into algebra using a variable such as x. Be sure to define what x represents, and include units if appropriate. Write down the resulting equation.

Step 2: Solve the equation by a suitable method.

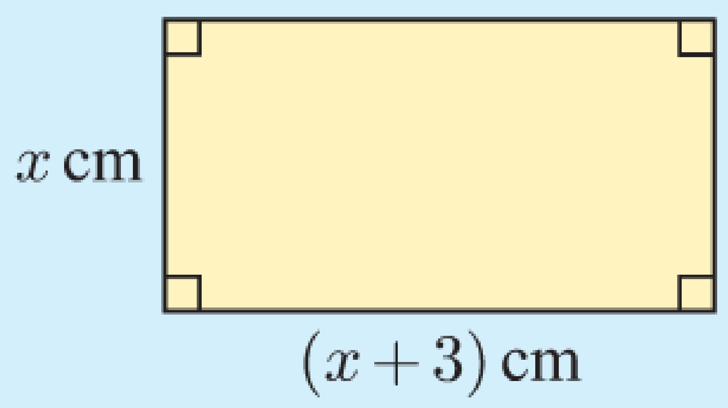
Step 3: Examine the solutions carefully to see if they are acceptable.

Step 4: Give your answer in a sentence, making sure you answer the question.

Example 17 **Self Tutor**

A rectangle has length 3 cm longer than its width, and its area is 42 cm^2. Find the width of the rectangle.

If the width is x cm then the length is $(x+3)$ cm.



$$\therefore \quad x(x+3) = 42 \quad \{\text{equating areas}\}$$
$$\therefore \quad x^2 + 3x - 42 = 0$$
$$\therefore \quad x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-42)}}{2}$$
$$\therefore \quad x = \frac{-3 \pm \sqrt{177}}{2}$$
$$\therefore \quad x \approx -8.15 \quad \text{or} \quad 5.15$$

We reject the negative solution as lengths are positive.

The width is about 5.15 cm.

EXERCISE 14F

1 Two integers differ by 12, and the sum of their squares is 74. Find the integers.

2 The sum of a number and its reciprocal is $\frac{26}{5}$. Find the number.

3 The sum of a natural number and its square is 210. Find the number.

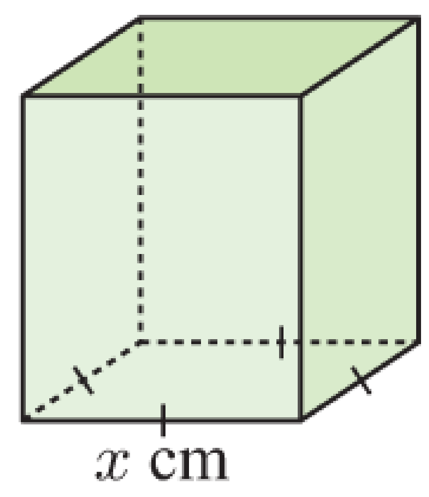
4 The product of two consecutive even numbers is 360. Find the numbers.

5 The product of two consecutive odd numbers is 255. Find the numbers.

6 The number of diagonals of an n-sided polygon is given by the formula $D = \frac{n}{2}(n-3)$.
A polygon has 90 diagonals. How many sides does it have?

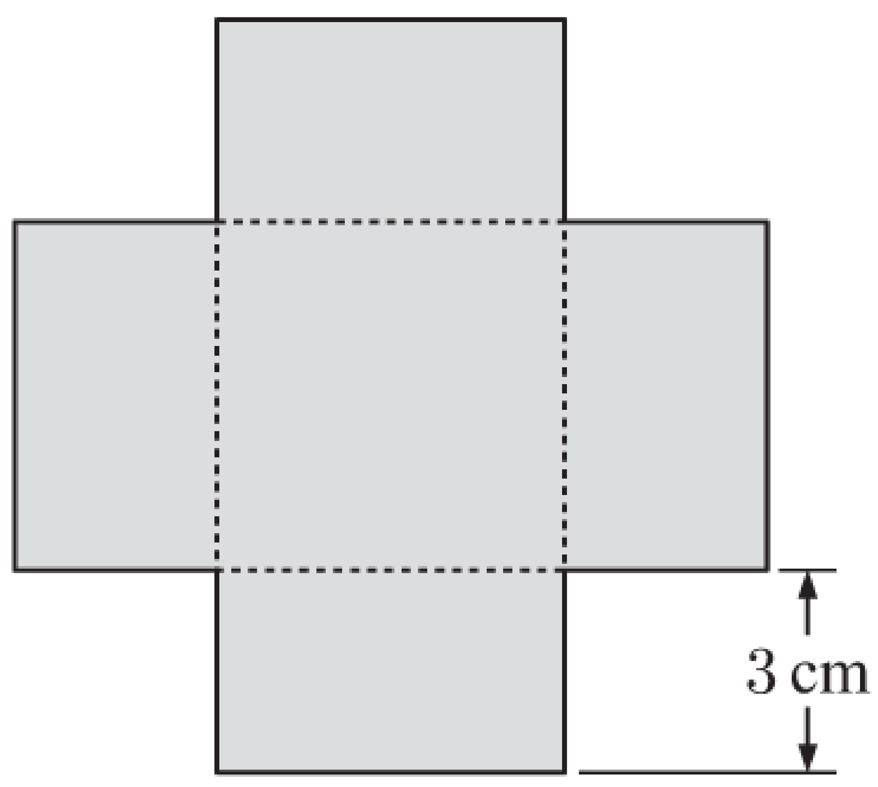
7 The length of a rectangle is 4 cm longer than its width. The rectangle has area 26 cm^2. Find its width.

8 A rectangular box has a square base. Its height is 1 cm longer than its base side length. The total surface area of the box is 240 cm^2.
Suppose the sides of the base are x cm long.



- **a** Show that the total surface area is given by $A = 6x^2 + 4x$ cm^2.
- **b** Find the dimensions of the box.

9



An open box can hold 80 cm^3. It is made from a square piece of tinplate with 3 cm squares cut from each of its 4 corners. Find the dimensions of the original piece of tinplate.

Example 18 — **Self Tutor**

Is it possible to bend a 12 cm length of wire to form the perpendicular sides of a right angled triangle with area 20 cm^2?

Suppose the wire is bent x cm from one end.

The area $A = \frac{1}{2}x(12-x)$

$\therefore\ \frac{1}{2}x(12-x) = 20$

$\therefore\ x(12-x) = 40$

$\therefore\ 12x - x^2 - 40 = 0$

$\therefore\ x^2 - 12x + 40 = 0$

Now $\Delta = (-12)^2 - 4(1)(40)$
$= -16$ which is < 0

There are no real solutions, indicating that this situation is **impossible**.

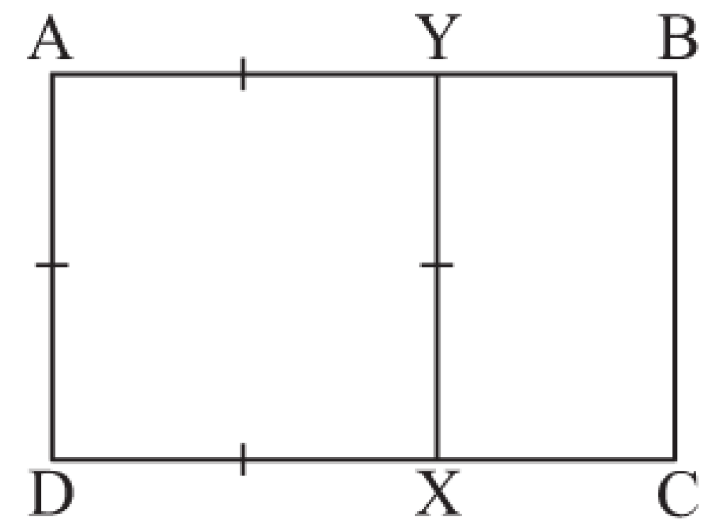
10 Is it possible to bend a 20 cm length of wire into a rectangle with area 30 cm^2?

11 The rectangle ABCD is divided into a square and a smaller rectangle by [XY] which is parallel to its shorter sides.
The smaller rectangle BCXY is *similar* to the original rectangle, so rectangle ABCD is a **golden rectangle**.

The ratio $\frac{AB}{AD}$ is called the **golden ratio**.

Show that the golden ratio is $\frac{1+\sqrt{5}}{2}$.

Hint: Let $AB = x$ units and $AD = 1$ unit.

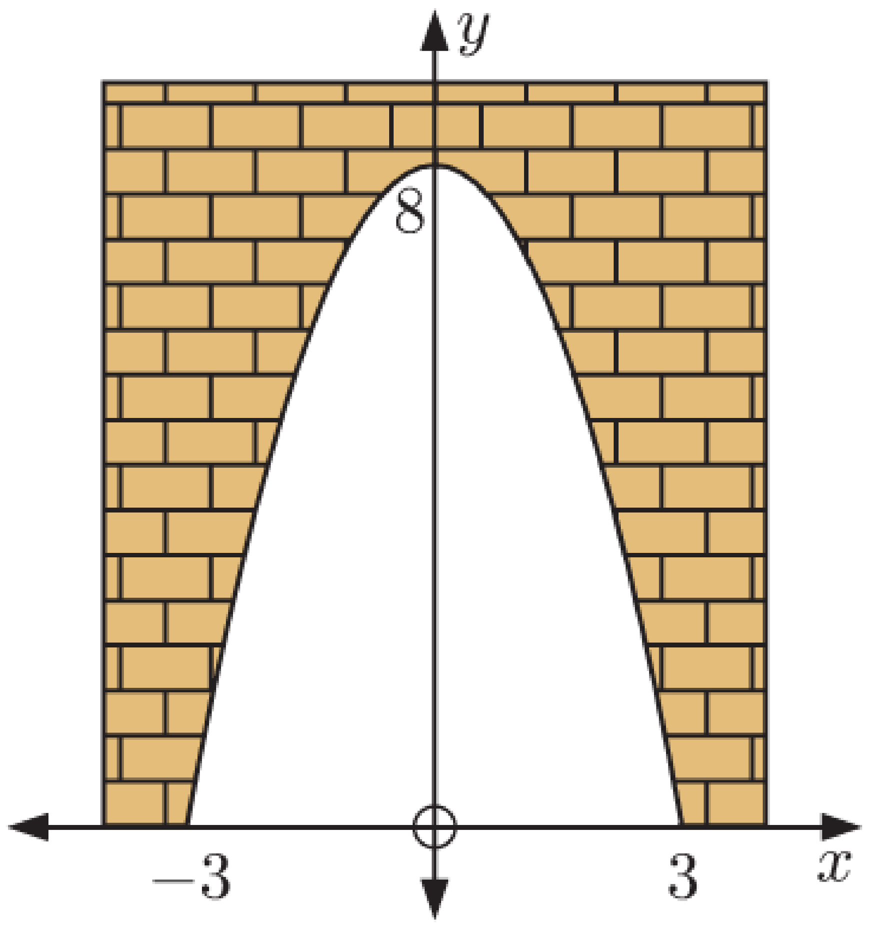


12 Two trains travel along a 160 km track each day. The express travels 10 km h^{-1} faster and takes 30 minutes less time than the normal train. Find the speed of the express.

13 A group of elderly citizens chartered a bus for \$160. Unfortunately, 8 of them fell ill and had to miss the trip. As a consequence, the other citizens had to pay an extra \$1 each. How many elderly citizens went on the trip?

14 A truck carrying a wide load needs to pass through the parabolic tunnel shown. The units are metres.
The truck is 5 m high and 4 m wide.

- **a** Find the quadratic function which describes the shape of the tunnel.
- **b** Determine whether the truck will fit.



15 A stone is thrown into the air from the top of a cliff 60 m above sea level. The stone reaches a maximum height of 80 m above sea level after 2 seconds.

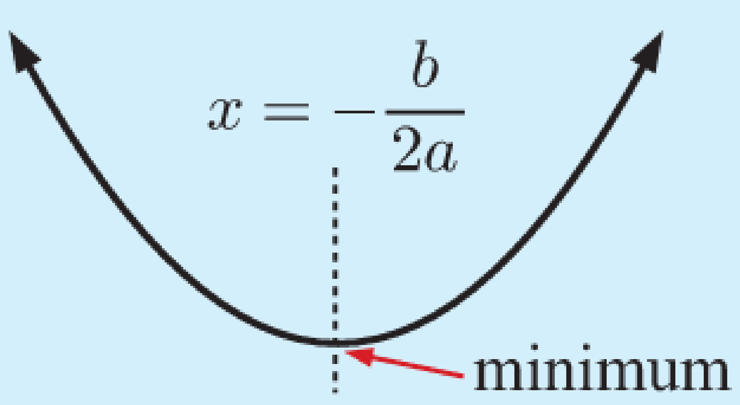
- **a** Find the quadratic function which describes the stone's height above sea level.
- **b** Find the stone's height above sea level after 3 seconds.
- **c** How long will it take for the stone to hit the water?

G OPTIMISATION WITH QUADRATICS

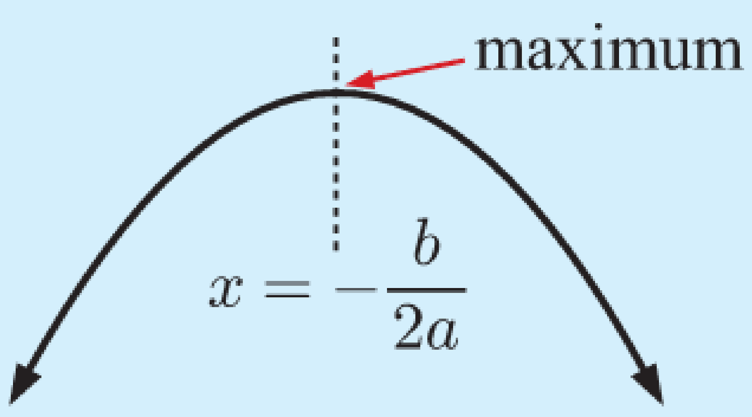
The process of finding a maximum or minimum value is called **optimisation**.

For the quadratic $y = ax^2 + bx + c$, we have seen that the vertex has x-coordinate $-\frac{b}{2a}$.

- If $a > 0$, the **minimum** value of y occurs at $x = -\frac{b}{2a}$.



- If $a < 0$, the **maximum** value of y occurs at $x = -\frac{b}{2a}$.



Example 19 Self Tutor

Find the maximum or minimum value of the quadratic, and the corresponding value of x:

a $y = x^2 + x - 3$

b $y = 3 + 3x - 2x^2$

a $y = x^2 + x - 3$ has $a = 1$, $b = 1$, and $c = -3$.

Since $a > 0$, the shape is 

The minimum value occurs when $x = \frac{-b}{2a} = -\frac{1}{2}$

and $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 3 = -3\frac{1}{4}$

So, the minimum value of y is $-3\frac{1}{4}$, occurring when $x = -\frac{1}{2}$.

b $y = -2x^2 + 3x + 3$ has $a = -2$, $b = 3$, and $c = 3$.

Since $a < 0$, the shape is 

The maximum value occurs when $x = \frac{-b}{2a} = \frac{-3}{-4} = \frac{3}{4}$

and $y = -2(\frac{3}{4})^2 + 3(\frac{3}{4}) + 3 = 4\frac{1}{8}$

So, the maximum value of y is $4\frac{1}{8}$, occurring when $x = \frac{3}{4}$.

EXERCISE 14G

1 Find the maximum or minimum value for each quadratic, and the corresponding value of x:

a $y = x^2 - 2x$ **b** $y = 7 - 2x - x^2$ **c** $y = 8 + 2x - 3x^2$

d $y = 2x^2 + x - 1$ **e** $y = 4x^2 - x + 5$ **f** $y = 7x - 2x^2$

2 The profit in manufacturing x refrigerators per day, is given by $P = -3x^2 + 240x - 800$ euros.

a How many refrigerators should be made each day to maximise the total profit?

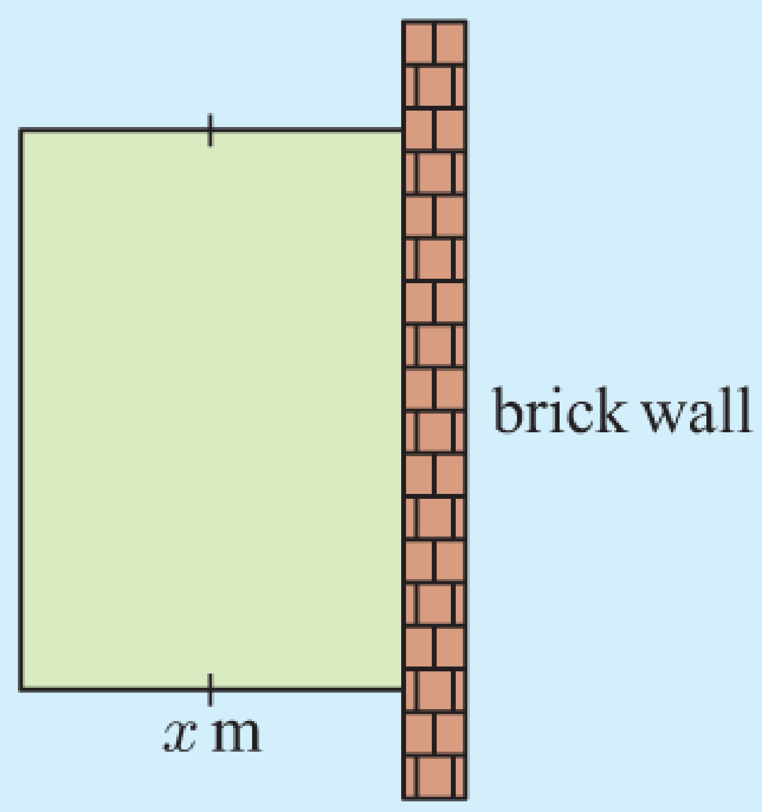
b What is the maximum profit?

Example 20 Self Tutor

A gardener has 40 m of fencing to enclose a rectangular garden plot, where one side is an existing brick wall. Suppose the two new equal sides are x m long.

a Show that the area enclosed is given by $A = x(40 - 2x)$ m^2.

b Find the dimensions of the garden of maximum area.



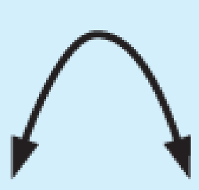
a Side [XY] has length $(40 - 2x)$ m.

Now, area = length × width

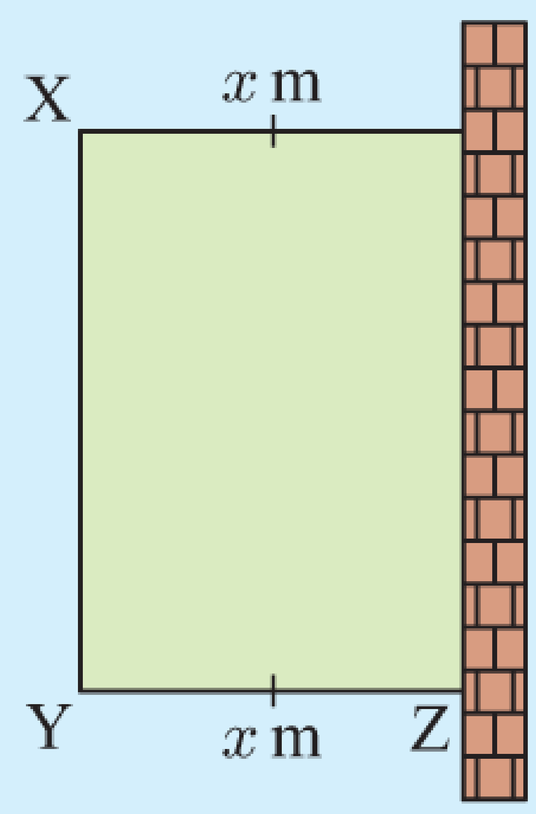
$\therefore$ $A = x(40 - 2x)$ m^2

b $A = 0$ when $x = 0$ or 20.

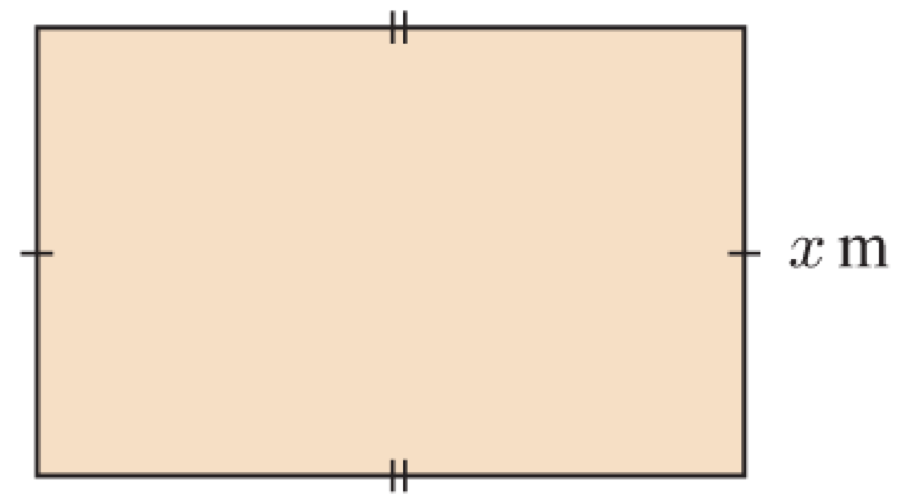
The vertex of the function lies midway between these values, so $x = 10$.

Since $a < 0$, the shape is 

$\therefore$ the area is maximised when YZ = 10 m and XY = 20 m.



3 A rectangular plot is enclosed by 200 m of fencing and has an area of A square metres. Show that:



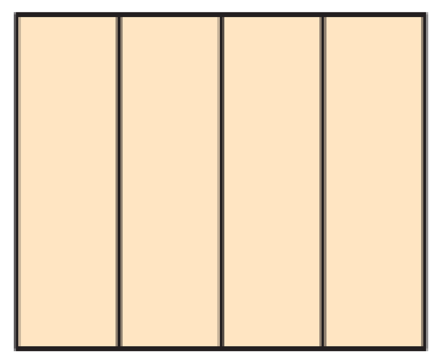
a $A = 100x - x^2$ where x m is the length of one of its sides

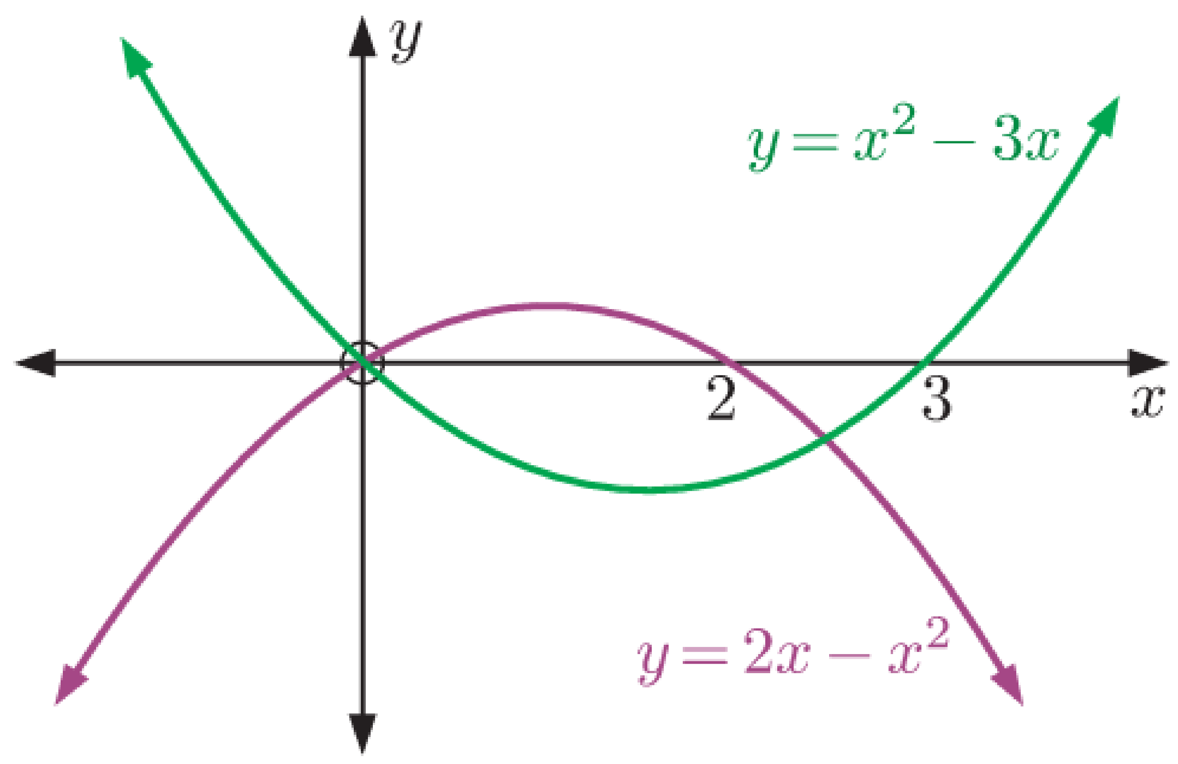
b the area is maximised if the rectangle is a square.

4 Three sides of a rectangular paddock are to be fenced, the fourth side being an existing straight water drain. If 1000 m of fencing is available, what dimensions should be used for the paddock to maximise its area?

5 500 m of fencing is available to make 4 rectangular pens of identical shape. Find the dimensions that maximise the area of each pen, if the plan is:

a 

b 

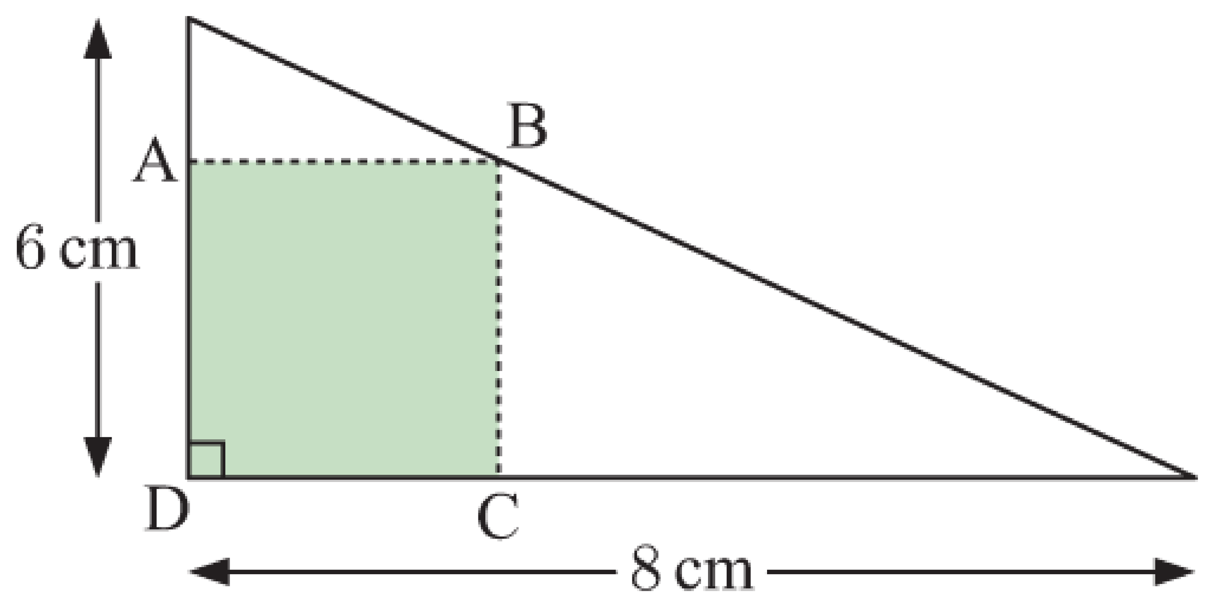
6 

The graphs of $y = x^2 - 3x$ and $y = 2x - x^2$ are illustrated.

a Show that the graphs meet where $x = 0$ and $x = 2\frac{1}{2}$.

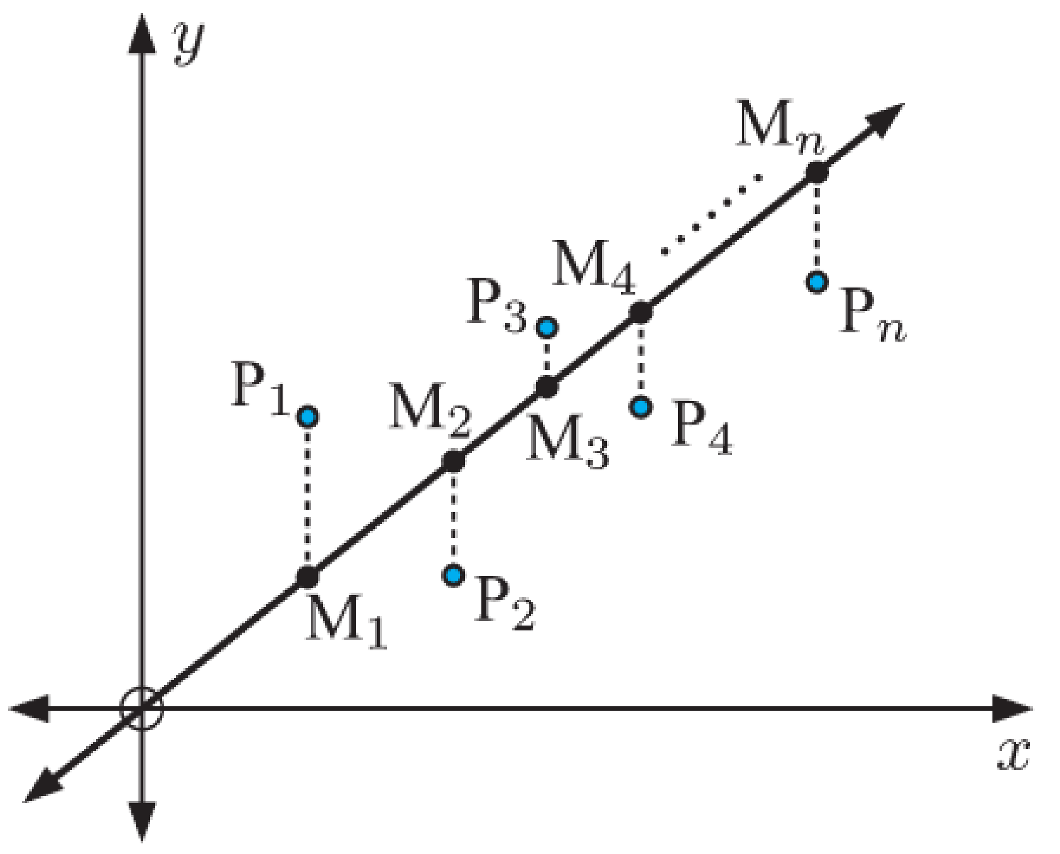
b Find the maximum vertical separation between the curves for $0 \leqslant x \leqslant 2\frac{1}{2}$.

7 Infinitely many rectangles may be inscribed within the right angled triangle shown alongside. One of them is illustrated.



a Let AB $= x$ cm and BC $= y$ cm.
Use similar triangles to find y in terms of x.

b Find the dimensions of rectangle ABCD of maximum area.

8 

$P_1(a_1, b_1)$, $P_2(a_2, b_2)$, $P_3(a_3, b_3)$, ..., $P_n(a_n, b_n)$ are a set of data points which are approximately linear through the origin O(0, 0).

To find the equation of the "line of best fit" through the origin, we minimise

$(P_1M_1)^2 + (P_2M_2)^2 + (P_3M_3)^2 + ... + (P_nM_n)^2$ where M_i lies on the line and has x-coordinate a_i.

Find the gradient m of the "line of best fit" in terms of a_i and b_i, $i = 1, 2, 3, 4, ..., n$.

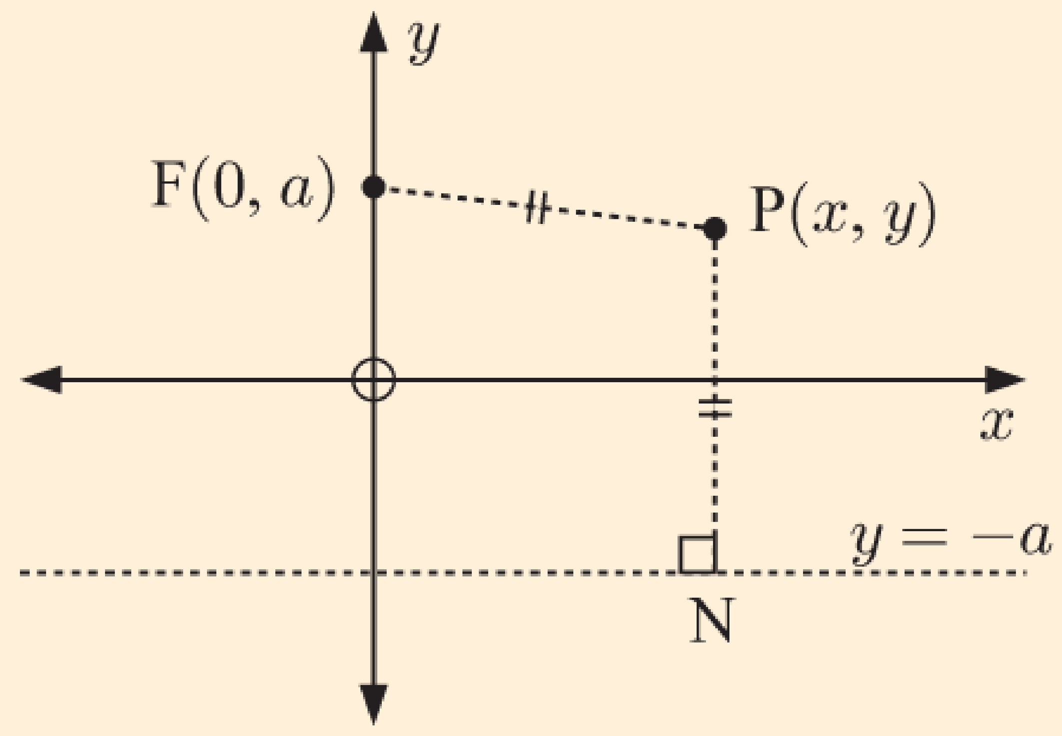
9 Assuming a and b are real constants, expand $y = (x-a-b)(x-a+b)(x+a-b)(x+a+b)$ and hence determine the least value of y.

10 By considering the function $y = (a_1x - b_1)^2 + (a_2x - b_2)^2$, use quadratic theory to prove the **Cauchy-Schwarz inequality** $|a_1b_1 + a_2b_2| \leqslant \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}$.

INVESTIGATION 4 THE GEOMETRIC DEFINITION OF A PARABOLA

A **parabola** is defined as the locus of all points which are equidistant from a fixed point called the **focus** and a fixed line called the **directrix**.

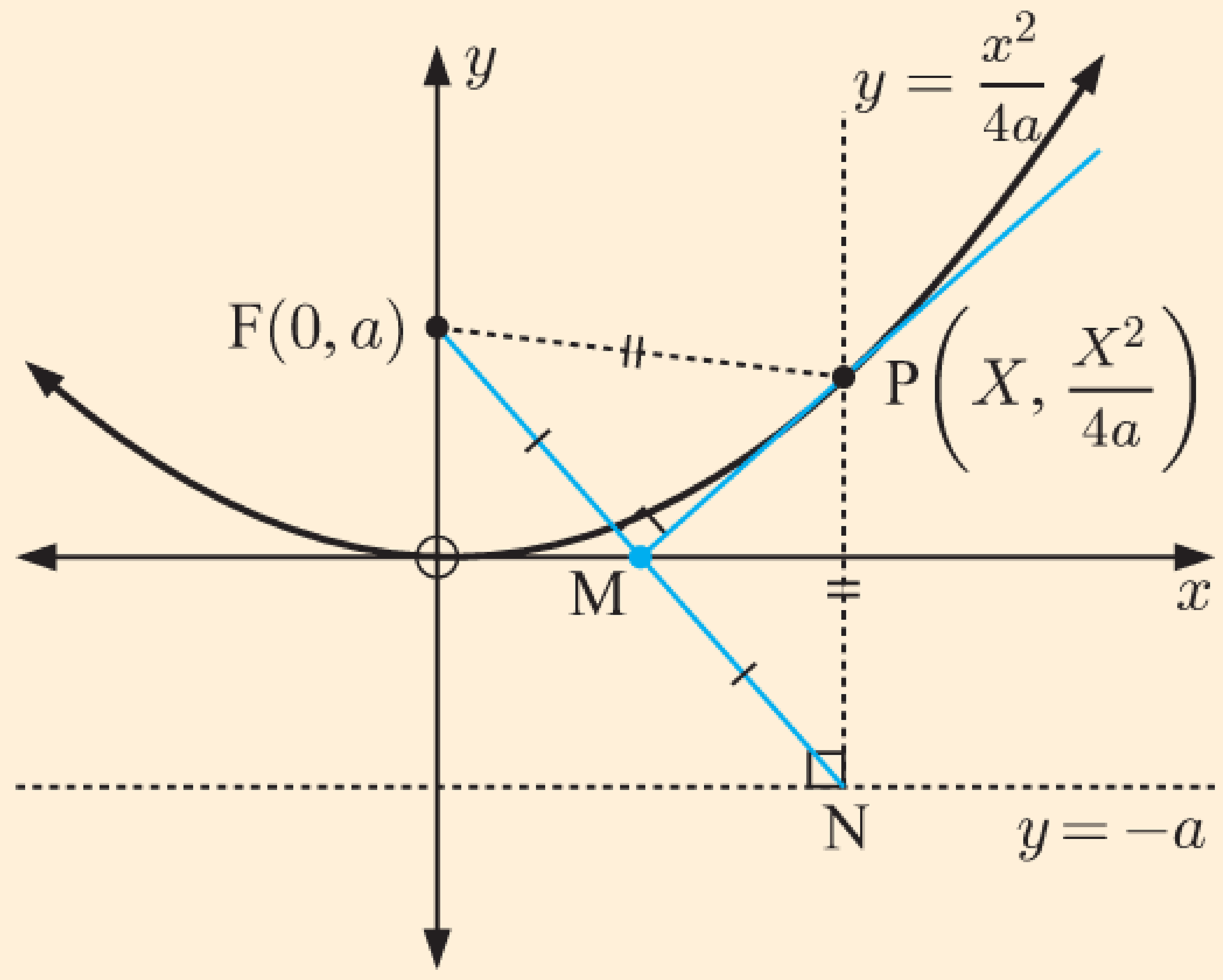
Suppose the focus is F$(0, a)$ and the directrix is the horizontal line $y = -a$. The parabola is the set of all points P such that FP = NP where N is the closest point on the directrix to P.



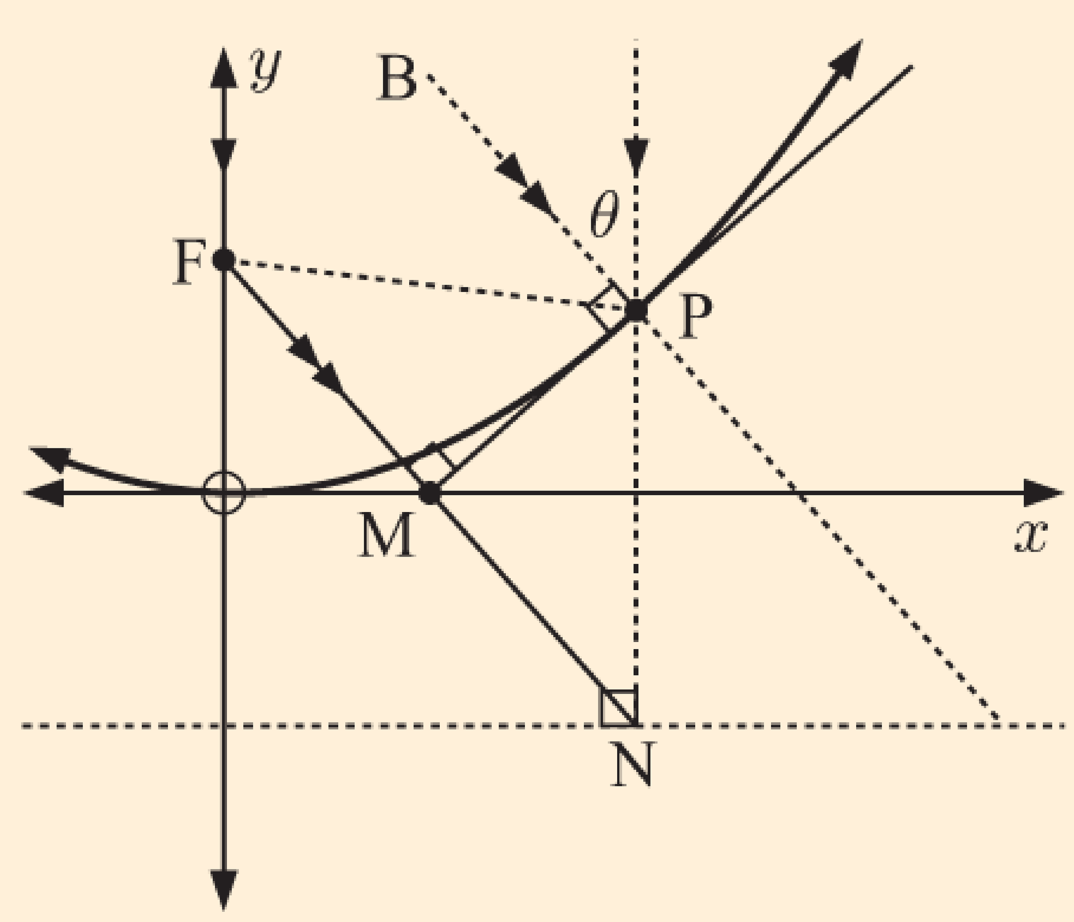
What to do:

PRINTABLE GRAPH PAPER

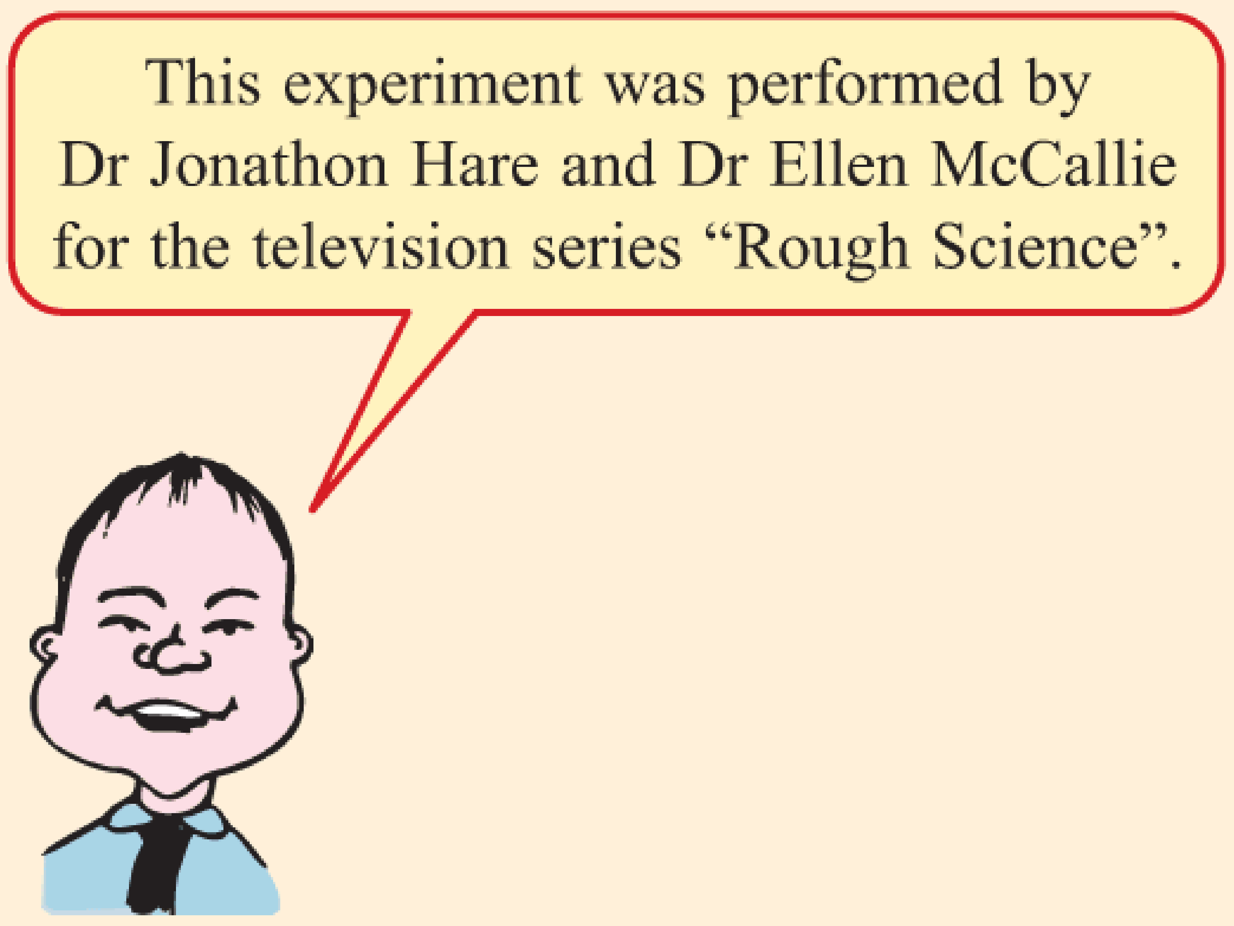
1. Suggest why it is convenient to let the focus be at $(0, a)$ and the directrix be the line $y = -a$.
2. Use the circular-linear graph paper provided to graph the parabola which has focus F$(0, 2)$ and directrix $y = -2$.
3. Using the definition above:
   a. Write down the coordinates of N.
   b. Write expressions for FP and NP.
   c. Show that the parabola has the equation $y = \dfrac{x^2}{4a}$.
4. Consider a point $\text{P}\left(X, \dfrac{X^2}{4a}\right)$ on the parabola $y = \dfrac{x^2}{4a}$ with focus F$(0, a)$ and directrix $y = -a$.
   a. Let N be the closest point on the directrix to P, and M be the midpoint of [FN].
      i. Show that [MP] has equation $y = \dfrac{X}{2a}\left(x - \dfrac{X}{2}\right)$.
      ii. Hence prove that (MP) is a tangent to the parabola.



   b. Let B lie on the normal to the parabola. Suppose a ray of light shines vertically down onto the parabola with angle of incidence θ as shown.
      i. Explain why $\widehat{\text{MNP}}$ must equal θ.
      ii. Hence explain why $\widehat{\text{MFP}}$ must equal θ.
      iii. Hence explain why $\widehat{\text{FPB}}$ must equal θ.
      iv. Hence explain why any vertical ray of light shining down onto a parabolic mirror will be reflected to the focus of the parabola F.



      v. Explain what shape Misha needs in the **Opening Problem** and where he needs to place his cup.



H QUADRATIC INEQUALITIES

A **quadratic inequality** can be written in either the form $ax^2 + bx + c \geqslant 0$ or $ax^2 + bx + c > 0$ where $a \neq 0$.

We have seen that the solutions to a quadratic equation are the x-intercepts of the corresponding quadratic function.

In a similar way, the solutions to a quadratic *inequality* are the values of x for which the corresponding function has a particular *sign*.

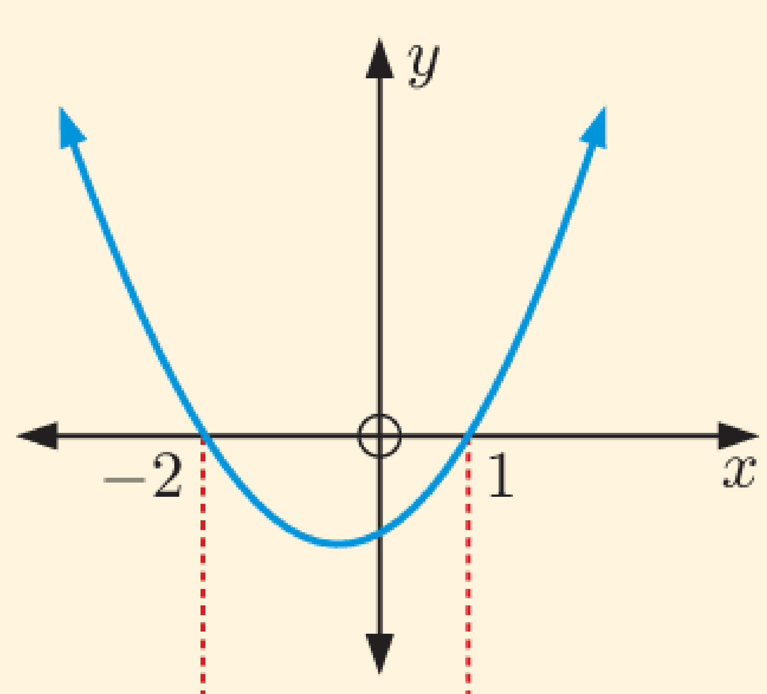
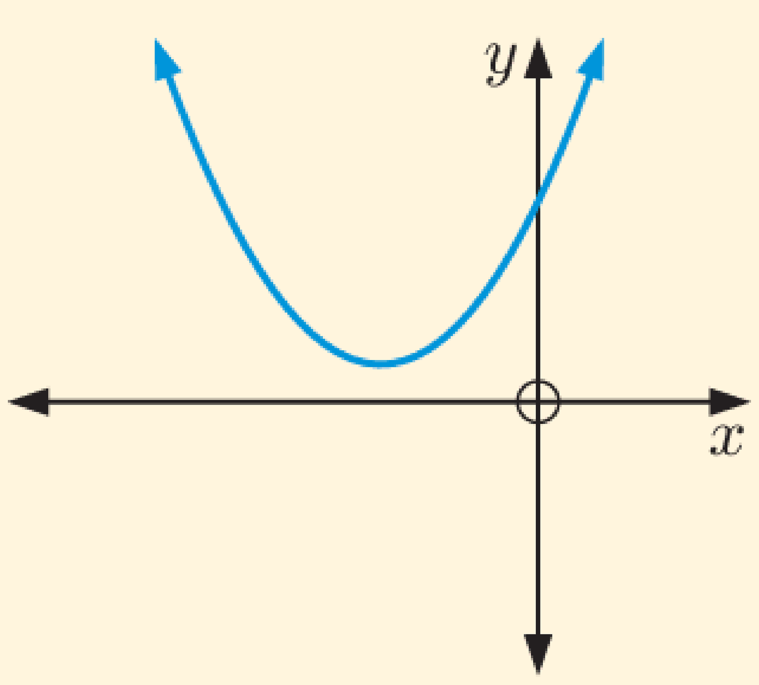
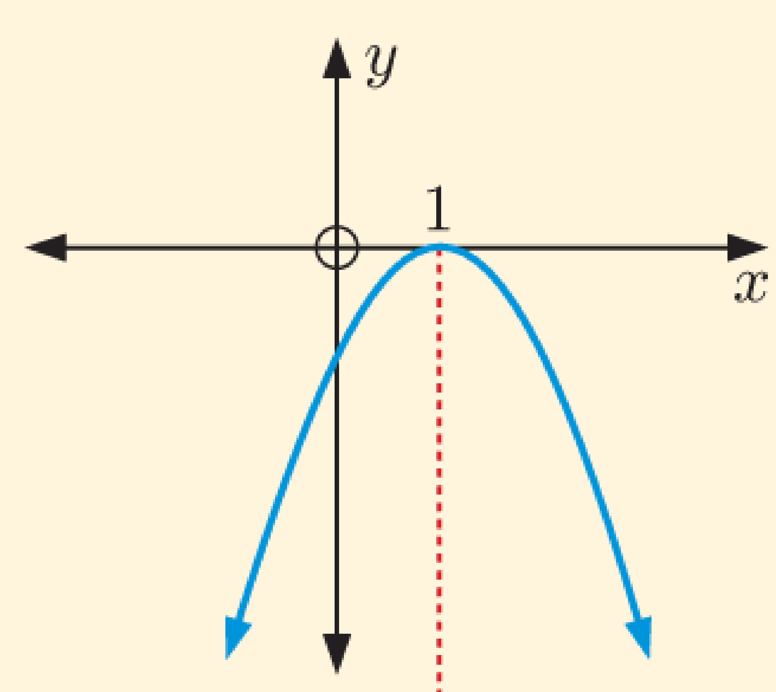
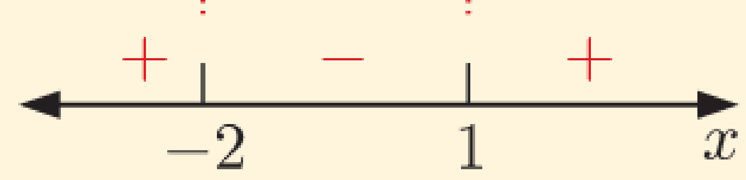
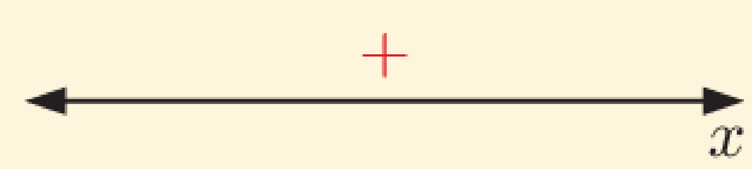
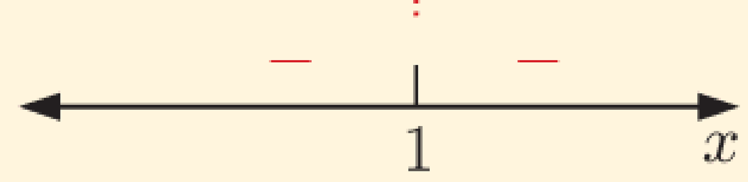
SIGN DIAGRAMS

A **sign diagram** is a number line which indicates the values of x for which a function is negative, zero, positive, or undefined.

A sign diagram consists of:

- a **horizontal line** which represents the x-axis
- **positive** (+) and **negative** (−) signs indicating where the graph is **above** and **below** the x-axis respectively
- the **zeros** of the function, which are the x-intercepts of its graph.

Consider the three functions below:

Function	$y = (x+2)(x-1)$	$y = (x+3)^2 + 2$	$y = -2(x-1)^2$
Graph			
Sign diagram			

You should notice that:

- A sign change occurs about a zero of the function for single linear factors such as $(x+2)$ and $(x-1)$. This indicates **cutting** of the x-axis.
- No sign change occurs about a zero of the function for squared linear factors such as $(x-1)^2$. This indicates **touching** of the x-axis.

DEMO

In general:

- when a linear factor has an **odd power** there is a change of sign about that zero
- when a linear factor has an **even power** there is no sign change about that zero.

Example 21 Self Tutor

Draw a sign diagram for:

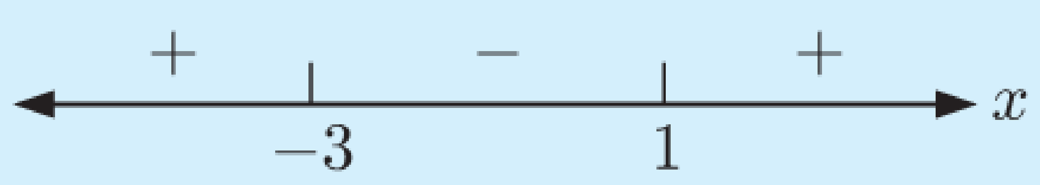
a x^2+2x-3

b $-4(x-3)^2$

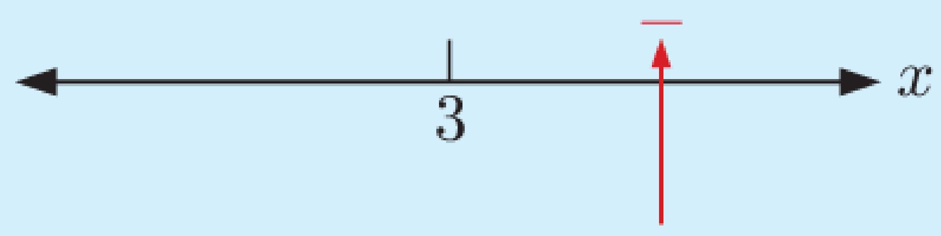
a $x^2+2x-3=(x+3)(x-1)$
which has zeros -3 and 1.

When $x=2$ we have $(5)(1)>0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

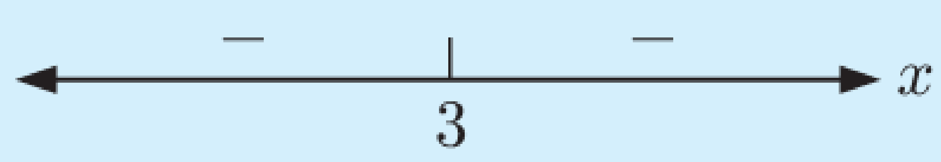


b $-4(x-3)^2$ has zero 3.



When $x=4$ we have $-4(1)^2<0$,
so we put a $-$ sign here.

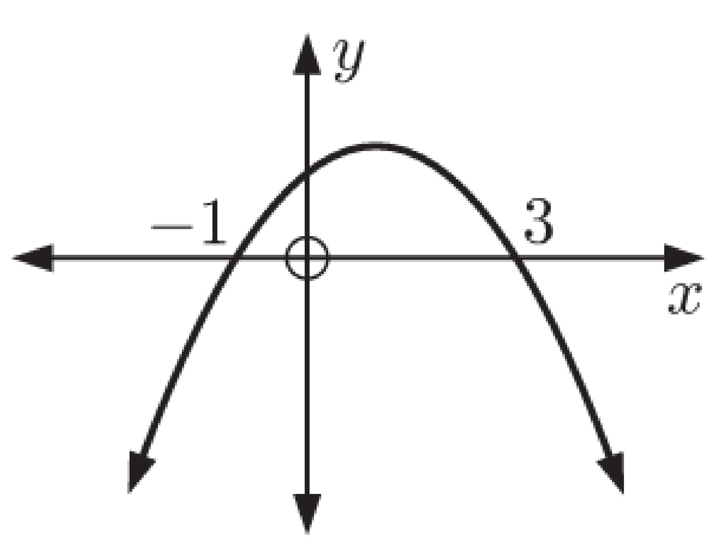
As the factor is squared, the signs do not change.



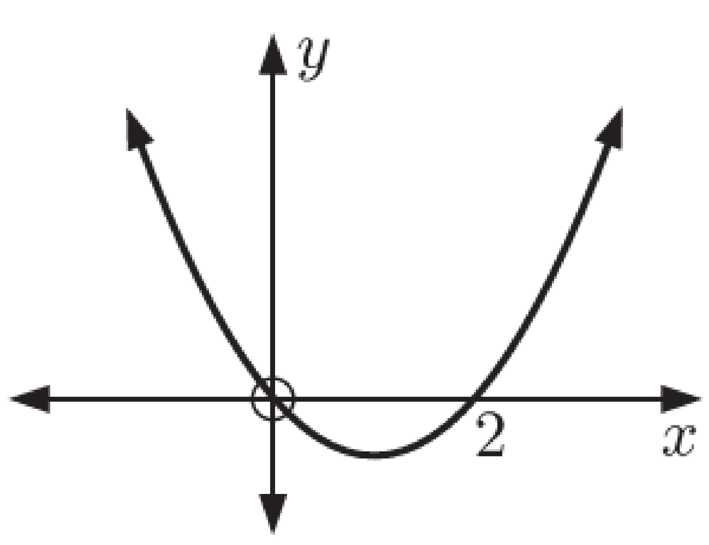
EXERCISE 14H.1

1 Draw a sign diagram for each graph:

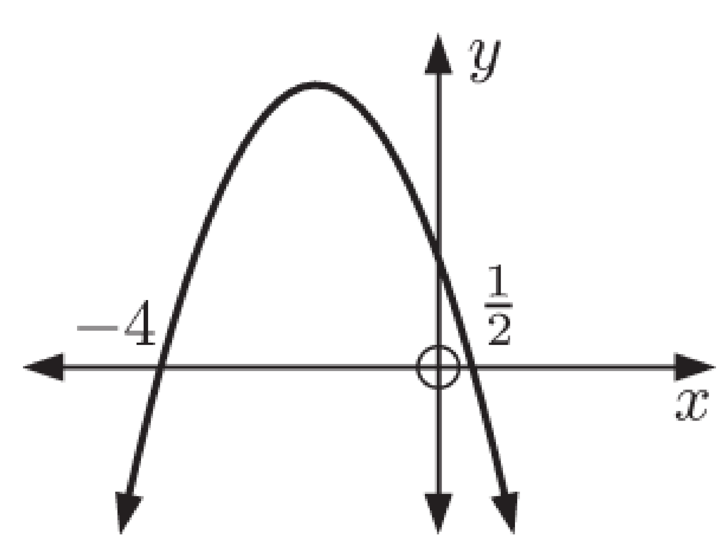
a



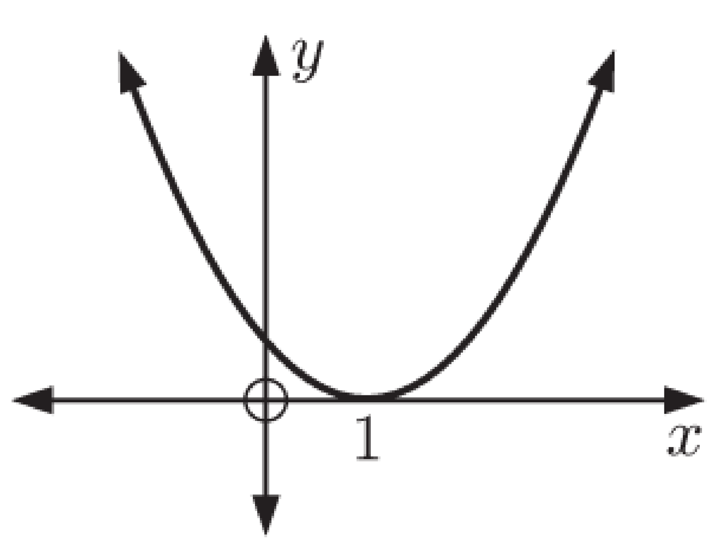
b



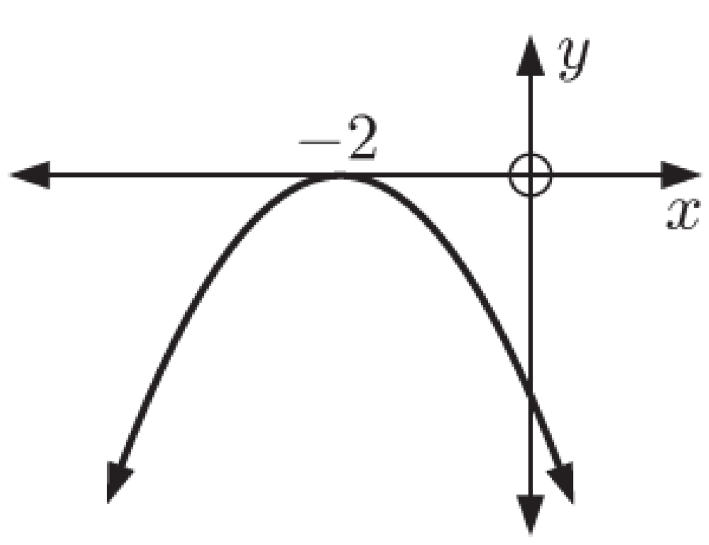
c



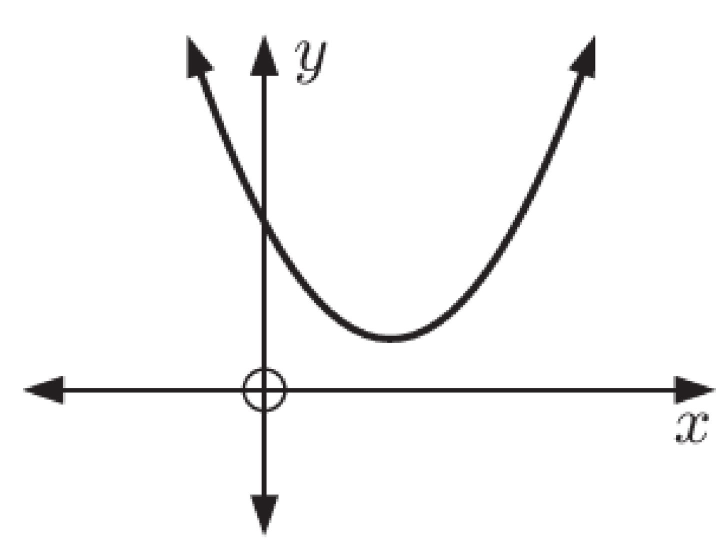
d



e



f



2 Draw a sign diagram for:

a $(x+4)(x-2)$

b $(x+1)(x-5)$

c $x(x-3)$

d $x(x+2)$

e $(2x+1)(x-4)$

f $-(x+1)(x-3)$

g $-(3x-2)(x+1)$

h $(2x-1)(3-x)$

i $(5-x)(1-2x)$

3 Draw a sign diagram for:

a $(x+2)^2$ **b** $(x-3)^2$ **c** $-(x-4)^2$

d $2(x+1)^2$ **e** $-3(x+4)^2$ **f** $-\frac{1}{2}(2x+5)^2$

4 Draw a sign diagram for:

a x^2-9 **b** $4-x^2$ **c** $5x-x^2$

d x^2-3x+2 **e** $2-8x^2$ **f** $6x^2+x-2$

g $6-16x-6x^2$ **h** $-2x^2+9x+5$ **i** $-15x^2-x+2$

5 Draw a sign diagram for:

a $x^2+10x+25$ **b** x^2-2x+1 **c** $-x^2+4x-4$

d $4x^2-4x+1$ **e** $-x^2-6x-9$ **f** $-4x^2+12x-9$

QUADRATIC INEQUALITIES

To solve quadratic inequalities we use the following procedure:

- Make the RHS zero by shifting all terms to the LHS.
- Fully factorise the LHS.
- Draw a sign diagram for the LHS.
- Determine the values required from the sign diagram.

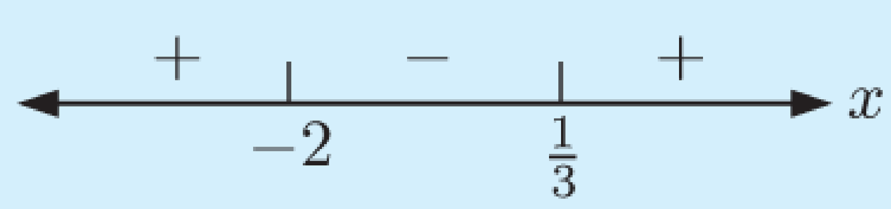
Example 22 Self Tutor

Solve for x:

a $3x^2+5x \geqslant 2$ **b** $x^2+9<6x$

a
$$3x^2+5x \geqslant 2$$
$$\therefore\ 3x^2+5x-2 \geqslant 0$$
$$\therefore\ (3x-1)(x+2) \geqslant 0$$

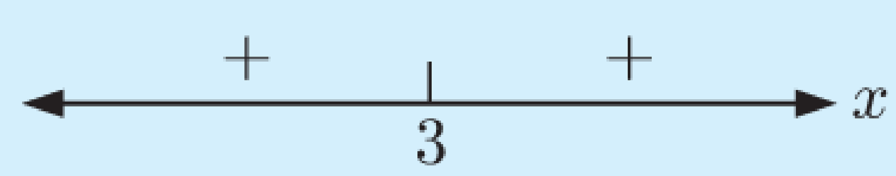
Sign diagram of LHS is



$x \leqslant -2$ or $x \geqslant \frac{1}{3}$

b
$$x^2+9<6x$$
$$\therefore\ x^2-6x+9<0$$
$$\therefore\ (x-3)^2<0$$

Sign diagram of LHS is



So, the inequality is not true for any real x.

EXERCISE 14H.2

1 Solve for x:

a $(x-2)(x+5) \leqslant 0$ **b** $(2-x)(x+3) \geqslant 0$ **c** $(x-1)^2<0$

d $(x+5)^2 \geqslant 0$ **e** $(2x+1)(3-x)>0$ **f** $(x-4)(2x+3)<0$

2 Solve for x:

a $x^2-x \geqslant 0$ **b** $3x^2+2x<0$ **c** $x^2+4x+4>0$

d $x^2+2x-15 \leqslant 0$ **e** $x^2-4x-12>0$ **f** $3x^2+9x-12<0$

3 Solve for x:

a $x^2 \geqslant 3x$ **b** $x^2 < 4$ **c** $2x^2 \geqslant 4$

d $x^2 - 21 \leqslant 4x$ **e** $x^2 + 30 > 11x$ **f** $x + 42 < x^2$

g $2x^2 \geqslant x + 3$ **h** $4x^2 - 4x + 1 < 0$ **i** $6x^2 + 7x < 3$

j $3x^2 > 8(x + 2)$ **k** $2x^2 - 4x + 2 > 0$ **l** $6x^2 + 1 \leqslant 5x$

m $1 + 5x < 6x^2$ **n** $12x^2 \geqslant 5x + 2$ **o** $2x^2 + 9 > 9x$

Example 23 **Self Tutor**

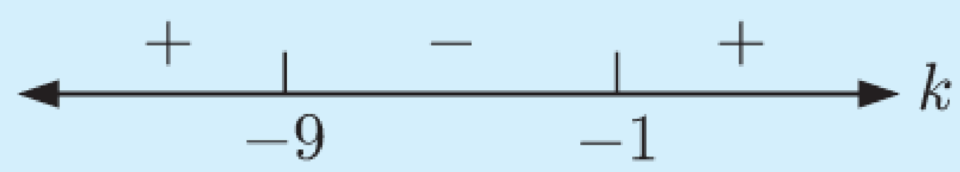
Find the value(s) of k for which the function $y = kx^2 + (k+3)x - 1$:

a cuts the x-axis twice **b** touches the x-axis **c** misses the x-axis.

$a = k, \quad b = k + 3, \quad c = -1$

$\therefore \quad \Delta = b^2 - 4ac$
$= (k+3)^2 - 4(k)(-1)$
$= k^2 + 6k + 9 + 4k$
$= k^2 + 10k + 9$
$= (k+9)(k+1)$

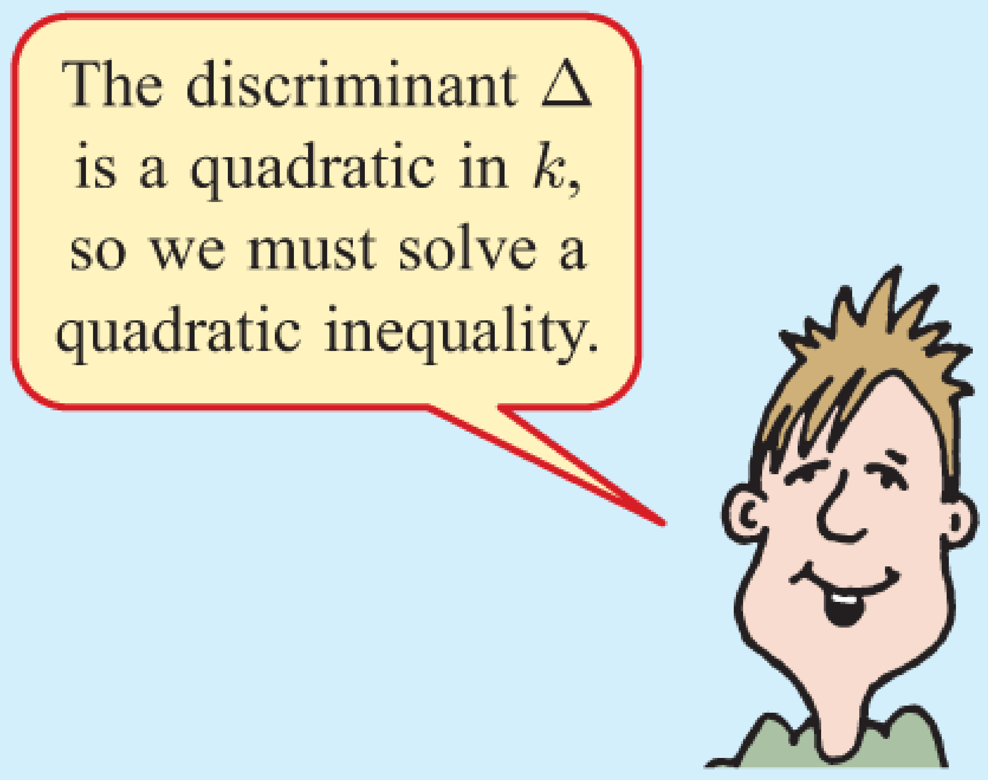
So, Δ has sign diagram:



a The graph cuts the x-axis twice if $\Delta > 0$
$\therefore \quad k < -9$ or $k > -1, \quad k \neq 0.$

b The graph touches the x-axis if $\Delta = 0$
$\therefore \quad k = -9$ or $k = -1.$

c The graph misses the x-axis if $\Delta < 0$
$\therefore \quad -9 < k < -1.$



4 For each quadratic function, find the values of k for which the function:

i cuts the x-axis twice **ii** touches the x-axis **iii** misses the x-axis.

a $y = 2x^2 + kx - k$ **b** $y = kx^2 - 2x + k$ **c** $y = x^2 + (k+2)x + 4$

5 For each quadratic equation, find the values of k for which the equation has:

i two real roots **ii** a repeated real root **iii** no real roots.

a $2x^2 + (k-2)x + 2 = 0$ **b** $x^2 + (3k-1)x + (2k+10) = 0$

c $(k+1)x^2 + kx + k = 0$

6 For what values of m is $y = (m-2)x^2 + 6x + 3m$:

a positive definite **b** negative definite?

7 Consider the curve $y = -x^2 + 3x - 6$ and the line $y = mx - 2$. Find the values of m for which the line:

a meets the curve twice **b** is a tangent to the curve

c does not meet the curve.

DYNAMIC GEOMETRY PACKAGE

8 For what values of a do the curves $y = ax^2 + 2x + 1$ and $y = -x^2 + ax - 1$:

a meet twice **b** touch **c** never meet?

REVIEW SET 14A

1 Use the vertex, axis of symmetry, and y-intercept to graph:

a $y = (x-2)^2 - 4$ **b** $y = -\frac{1}{2}(x+4)^2 + 6$

2 Find the points of intersection of $y = x^2 - 3x$ and $y = 3x^2 - 5x - 24$.

3 For what values of k does the graph of $y = -2x^2 + 5x + k$ *not* cut the x-axis?

4 Find the values of m for which $2x^2 - 3x + m = 0$ has:

a a repeated root **b** two distinct real roots **c** no real roots.

5 The sum of a number and its reciprocal is $2\frac{1}{30}$. Find the number.

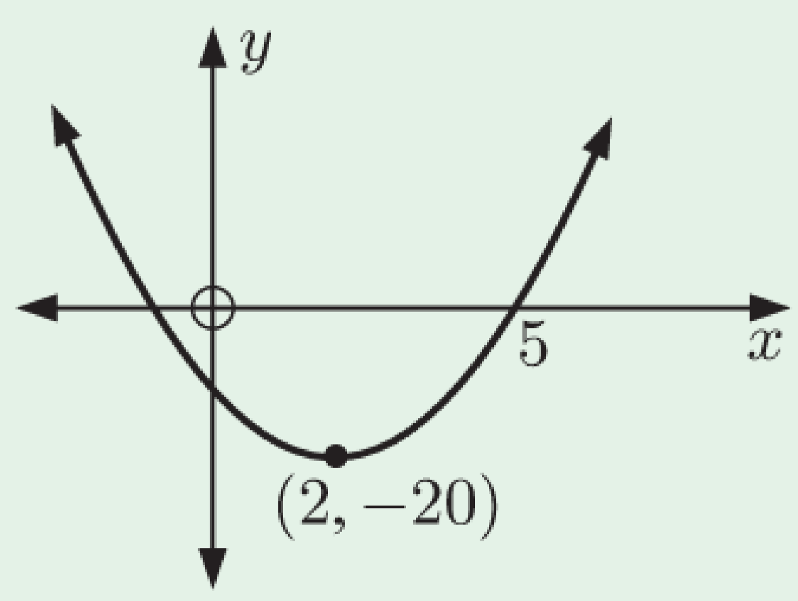
6 Show that no line with a y-intercept of 10 will ever be tangential to the curve with equation $y = 3x^2 + 7x - 2$.

7 **a** Write the quadratic $y = 2x^2 + 6x - 3$ in the form $y = a(x-h)^2 + k$.

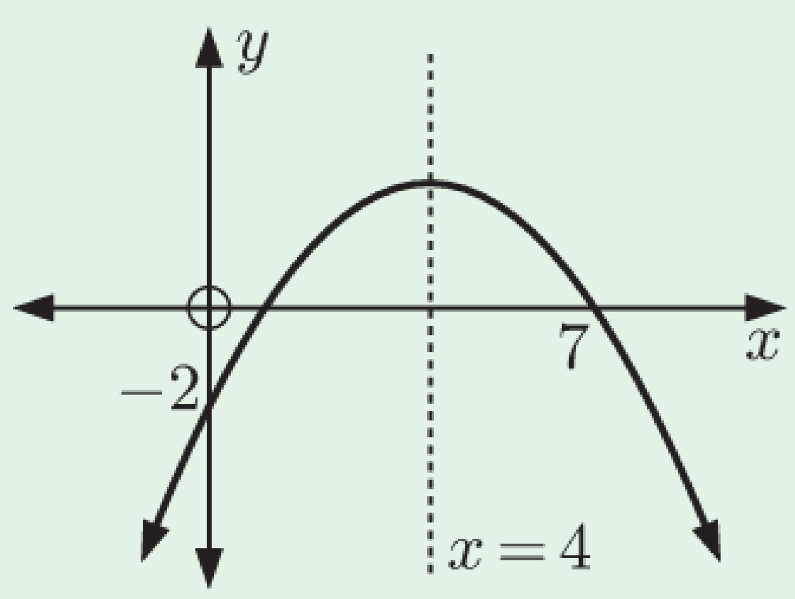
b Hence sketch the graph of the quadratic.

8 Find the equation of the quadratic with graph:

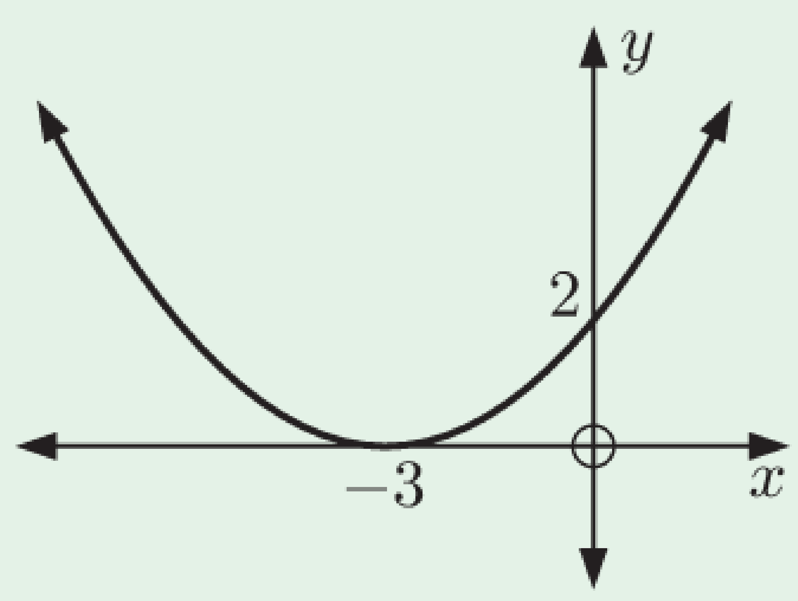
a



b



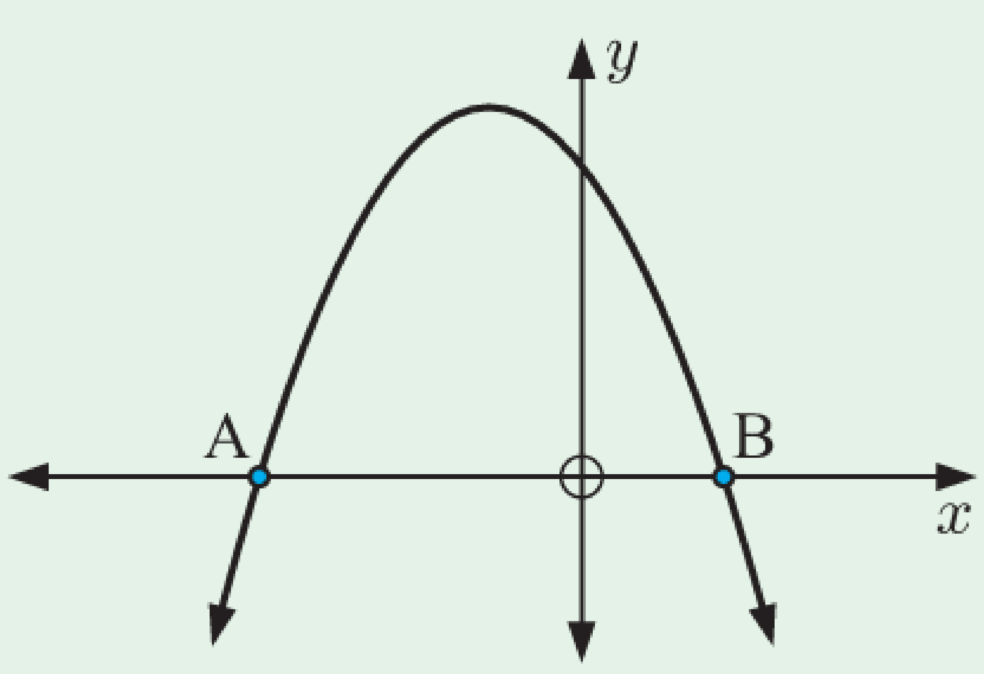
c



9 Draw the graph of $y = -x^2 + 2x$.

10 Find the y-intercept of the line with gradient -3 which is a tangent to the parabola $y = 2x^2 - 5x + 1$.

11 The graph shows the parabola $y = a(x+m)(x+n)$ where $m > n$.



a State the sign of:

i the discriminant Δ **ii** a.

b Find, in terms of m and n, the:

i coordinates of the x-intercepts A and B

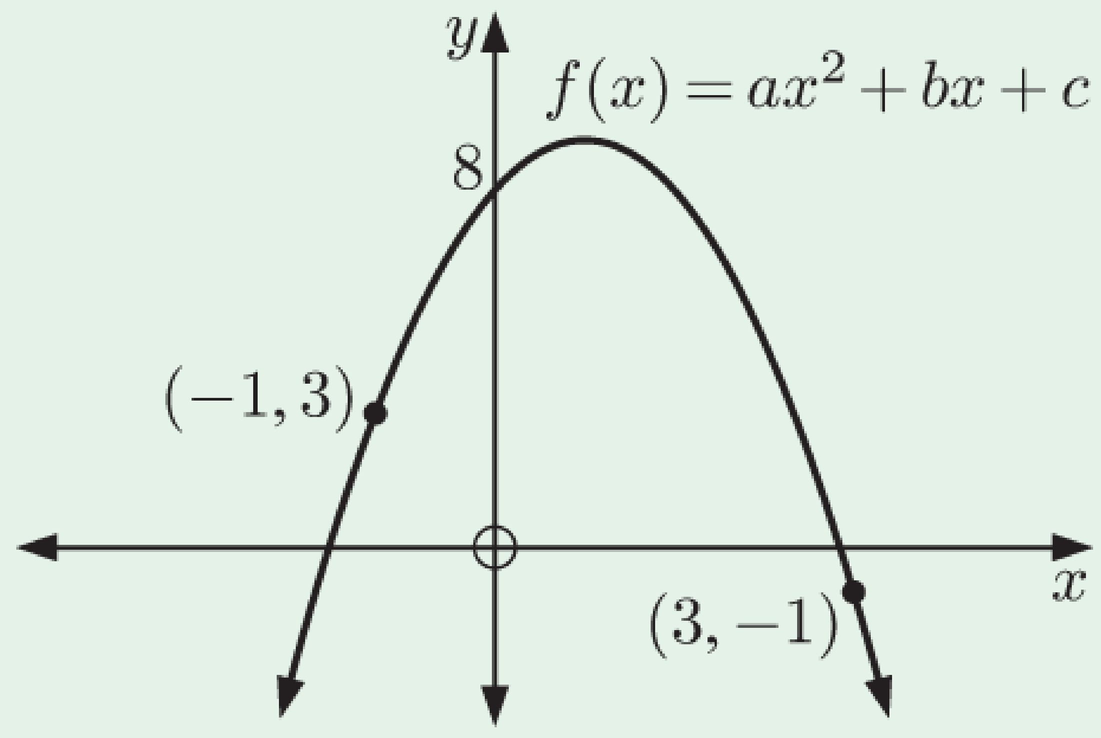
ii equation of the axis of symmetry.

12 For a quadratic function $y = ax^2 + bx + c$, suppose the constants a, b, and c are consecutive terms of a geometric sequence. Show that the function does not cut the x-axis.

13 Find the quadratic function which cuts the x-axis at 3 and -2 and which has y-intercept 24. Give your answer in the form $y = ax^2 + bx + c$.

14 Find the value of k for which the x-intercepts of $y = 3x^2 + 2kx + k - 1$ are closest together.

15 Consider the function $y = ax^2 + bx + c$ shown.



- **a** State the value of c.
- **b** Use the other information to write two equations involving a and b.
- **c** Find a and b, and hence state the equation of the quadratic.

16 For what values of m are the lines $y = mx - 10$ tangents to the parabola $y = 3x^2 + 7x + 2$?

17 When Annie hits a softball, the height of the ball above the ground after t seconds is given by $h = -4.9t^2 + 19.6t + 1.4$ metres. Find the maximum height reached by the ball.

18 Draw a sign diagram for:

- **a** $(3x + 2)(4 - x)$
- **b** $-x^2 + 3x + 18$

19 Solve for x:

- **a** $(3 - x)(x + 2) < 0$
- **b** $x^2 - 4x - 5 \leqslant 0$
- **c** $2x^2 + x > 10$

20 Find the values of k for which the function $f(x) = x^2 + kx + (3k - 4)$:

- **a** cuts the x-axis twice
- **b** touches the x-axis
- **c** misses the x-axis.

REVIEW SET 14B

1 Consider the quadratic $y = \frac{1}{2}(x - 2)^2 - 4$.

- **a** State the equation of the axis of symmetry.
- **b** Find the coordinates of the vertex.
- **c** Find the y-intercept.
- **d** Sketch the function.

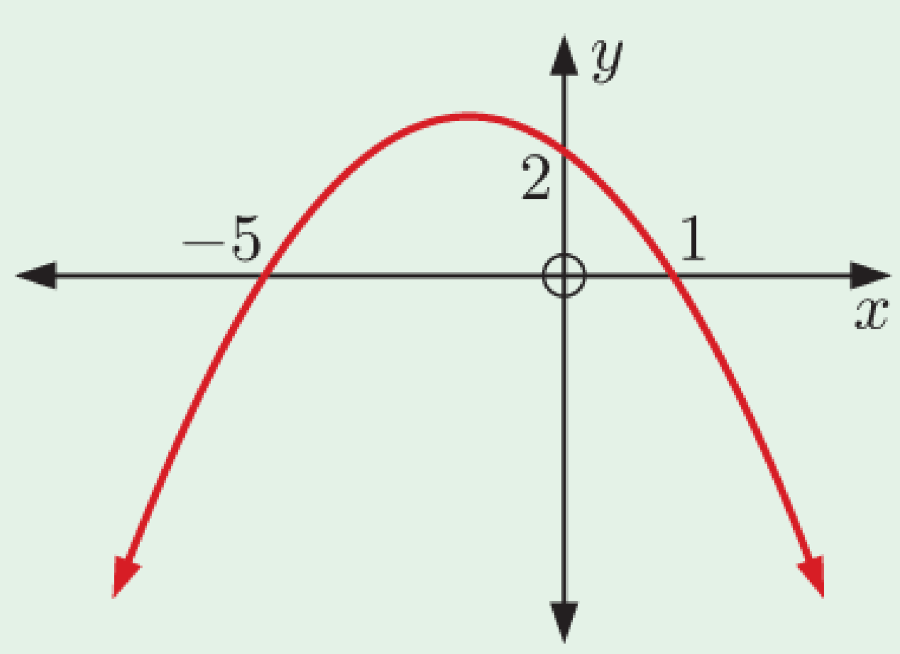
2 Consider the quadratic $y = -3x^2 + 8x + 7$. Find the equation of the axis of symmetry, and the coordinates of the vertex.

3 Use the discriminant only to find the relationship between the graph and the x-axis for:

- **a** $y = 2x^2 + 3x - 7$
- **b** $y = -3x^2 - 7x + 4$

4 Find the equation of the quadratic with vertex $(2, 25)$ and y-intercept 1.

5
- **a** Find the equation of the quadratic illustrated.
- **b** Hence find its vertex and axis of symmetry.



6 Consider the quadratic $y = 2x^2 + 4x - 1$.

a State the axis of symmetry.
b Find the coordinates of the vertex.
c Find the axes intercepts.
d Hence sketch the function.

7 Find, in the form $y = ax^2 + bx + c$, the quadratic function whose graph:

a touches the x-axis at 3 and passes through $(2, 2)$
b has x-intercepts 3 and -2, and y-intercept 3
c passes through $(-1, -9)$, $(1, 5)$, and $(2, 15)$
d has vertex $(3, 15)$ and passes through the point $(1, 7)$.

8 **a** For what values of c do the lines with equations $y = 3x + c$ intersect the parabola $y = x^2 + x - 5$ in two distinct points?
b Choose one such value of c and find the points of intersection in this case.

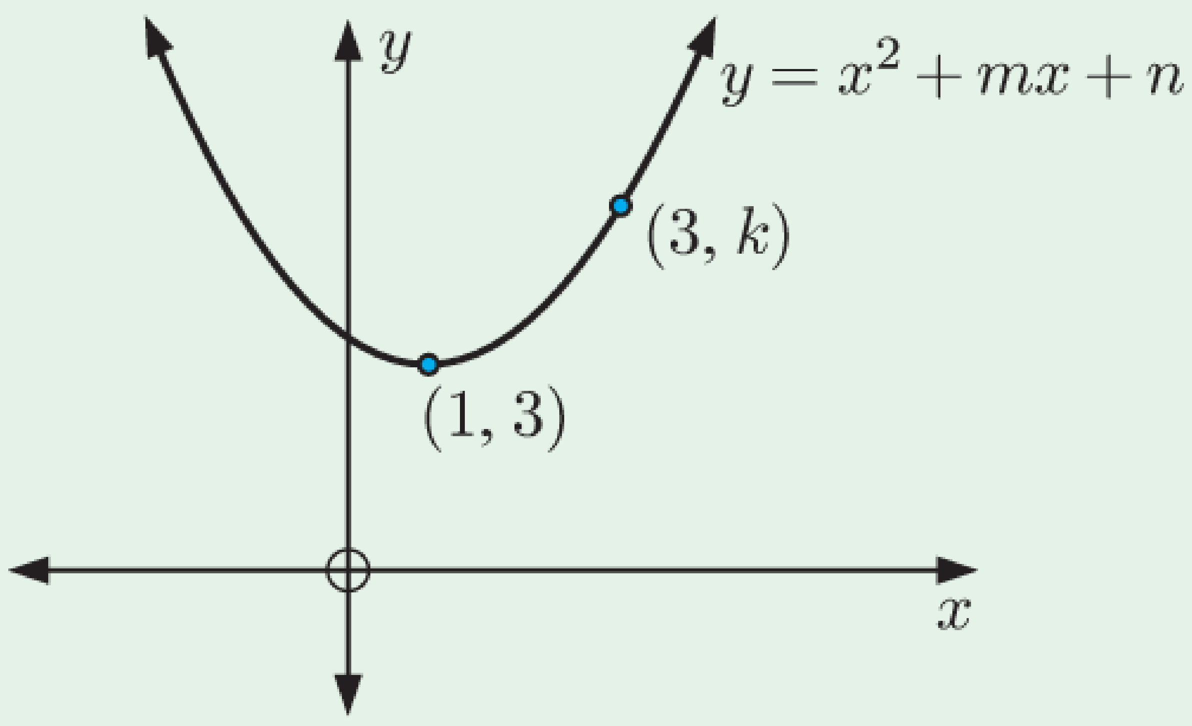
9 Find the maximum or minimum value of each quadratic, and the corresponding value of x:

a $y = 3x^2 + 4x + 7$
b $y = -2x^2 - 5x + 2$

10 The graph of a quadratic function cuts the x-axis at -2 and 3, and passes through $(-3, 18)$.

a Find the equation of the function in the form $y = ax^2 + bx + c$.
b Write down the y-intercept of the function.
c Find the coordinates of the vertex.

11



Consider the graph of $y = x^2 + mx + n$.

a Determine the values of m and n.
b Hence find the value of k.

12 An open square-based box has capacity 120 mL. It is made from a square piece of tinplate with 4 cm squares cut from each of its corners. Find the dimensions of the original piece of tinplate.

13 Consider $y = -x^2 - 3x + 4$ and $y = x^2 + 5x + 4$.

a Solve for x: $-x^2 - 3x + 4 = x^2 + 5x + 4$.
b Sketch the curves on the same set of axes.
c Hence solve for x: $x^2 + 5x + 4 > -x^2 - 3x + 4$.

14 For each of the following quadratics:

i Write the quadratic in completed square form.
ii Write the quadratic in factored form.
iii Sketch the graph of the quadratic, identifying its axes intercepts, vertex, and axis of symmetry.

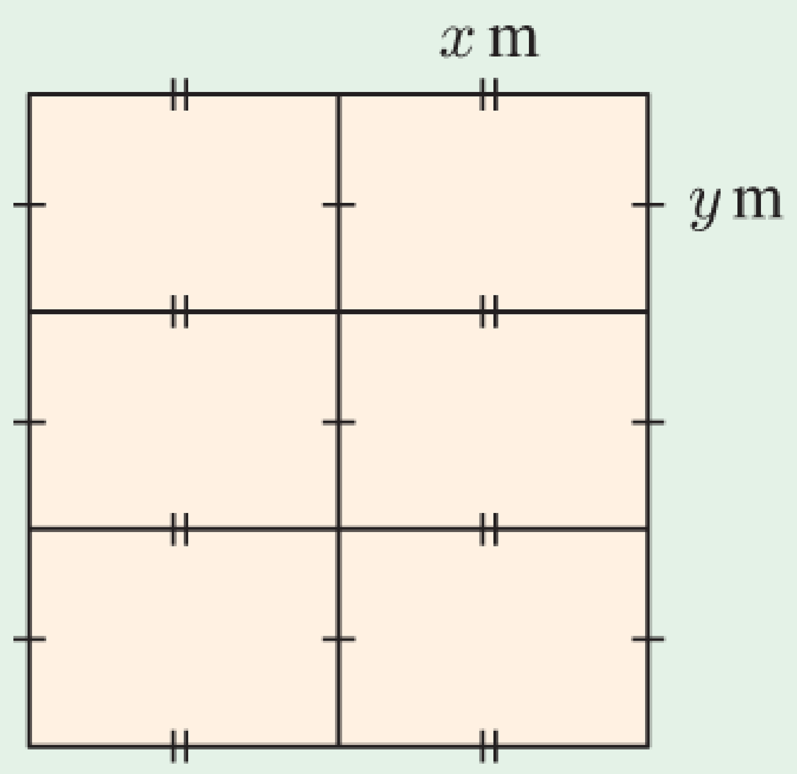
a $y = x^2 + 4x + 3$
b $y = x^2 + 2x - 3$
c $y = 2x^2 - 8x - 10$
d $y = -x^2 + 6x + 7$

15 Two different quadratic functions of the form $y = 9x^2 - kx + 4$ both *touch* the x-axis.

a Find the two values of k.

b Find the point of intersection of the two quadratic functions.

16 600 m of fencing is used to construct 6 rectangular animal pens as shown.



a Show that the area A of each pen is $A = x\left(\frac{600 - 8x}{9}\right)$ m^2.

b Find the dimensions of each pen so that it has the maximum possible area.

c What is the area of each pen in this case?

17 A retailer sells sunglasses for \$45, and has 50 customers per day. From market research, the retailer discovers that for every \$1.50 increase in the price of the sunglasses, he will lose a customer per day.

Let $\$x$ be the price increase of the sunglasses.

a Show that the revenue collected by the retailer each day is $R = (45 + x)\left(50 - \frac{x}{1.5}\right)$ dollars.

b Find the price the retailer should set for his sunglasses in order to maximise his daily revenue. How much revenue is made per day at this price?

18 Draw a sign diagram for:

a $x^2 - 3x - 10$ **b** $-(x + 3)^2$

19 Solve for x:

a $4x^2 - 3x < 0$ **b** $2x^2 - 3x - 5 \geqslant 0$ **c** $\frac{11}{3}x \leqslant 2x^2 + 1$

20 Find the values of m for which the function $y = mx^2 + 5x + (m + 12)$:

a cuts the x-axis twice **b** touches the x-axis **c** misses the x-axis.

2 **a** $x = 0$ or 5 **b** $x = 0$ or -3 **c** $x = -1$ or 3
d $x = 0$ or 7 **e** $x = -6$ or $\frac{3}{2}$ **f** $x = -\frac{1}{2}$ or $\frac{1}{2}$

3 **a** $x = 0$ or -5 **b** $x = 5$ **c** $x = \frac{1}{3}$ **d** $x = 5$ or $-\frac{2}{3}$
e $x = 0, -1,$ or 2 **f** $x = -2, -4,$ or $\frac{1}{2}$

4 **a** $a = 0, \ b \neq 0$ **b** $x = 0$ or $y = 0, \ z \neq 0$
c no solutions **d** $x = 0, \ y \neq 0$

EXERCISE 4C.1

1 **a** $x = 0$ or $-\frac{7}{4}$ **b** $x = 0$ or $-\frac{1}{3}$ **c** $x = 0$ or $\frac{7}{3}$
d $x = 0$ or $\frac{11}{2}$ **e** $x = 0$ or $\frac{8}{3}$ **f** $x = 0$ or $\frac{3}{2}$

2 **a** $x = 2$ or 3 **b** $x = 1$ **c** $x = -4$ or 2
d $x = -3$ or -4 **e** $x = -2$ or 4 **f** $x = 3$ or 7
g $x = 3$ **h** $x = -4$ or 3 **i** $x = -11$ or 3
j $x = -4$ or 1 **k** $x = -7$ or 5 **l** $x = -2$ or 5

3 **a** $x = \frac{2}{3}$ **b** $x = -\frac{1}{2}$ or 7 **c** $x = -\frac{2}{3}$ or 6
d $x = \frac{1}{3}$ or -2 **e** $x = \frac{3}{2}$ or 1 **f** $x = -\frac{2}{3}$ or -2
g $x = -\frac{2}{3}$ or 4 **h** $x = \frac{1}{2}$ or $-\frac{3}{2}$ **i** $x = -\frac{1}{4}$ or 3
j $x = -\frac{3}{4}$ or $\frac{5}{3}$ **k** $x = \frac{1}{7}$ or -1 **l** $x = -2$ or $\frac{28}{15}$

4 **a** $x = 2$ or 5 **b** $x = -3$ or 2 **c** $x = 0$ or $-\frac{3}{2}$
d $x = 1$ or 2 **e** $x = \frac{1}{2}$ or -1 **f** $x = 3$
g $x = 1$ or -2 **h** $x = 6$ or -4 **i** $x = 7$ or -5
j $x = 4$ or -2

EXERCISE 4C.2

1 **a** $x = 2 \pm \sqrt{3}$ **b** $x = -3 \pm \sqrt{7}$ **c** $x = 7 \pm \sqrt{3}$
d $x = 2 \pm \sqrt{7}$ **e** $x = -3 \pm \sqrt{2}$ **f** $x = 1 \pm \sqrt{7}$
g $x = -3 \pm \sqrt{11}$ **h** $x = 4 \pm \sqrt{6}$ **i** no real solutions

2 **a** $x = -1 \pm \frac{1}{\sqrt{2}}$ **b** $x = \frac{5}{2} \pm \frac{\sqrt{19}}{2}$ **c** $x = -2 \pm \sqrt{\frac{7}{3}}$
d $x = 1 \pm \sqrt{\frac{7}{3}}$ **e** $x = \frac{3}{2} \pm \sqrt{\frac{37}{20}}$ **f** $x = -\frac{1}{2} \pm \frac{\sqrt{6}}{2}$

3 **a** $x = \frac{2}{3} \pm \frac{\sqrt{10}}{3}$ **b** $x = -\frac{1}{10} \pm \frac{\sqrt{21}}{10}$ **c** $x = -\frac{5}{6} \pm \frac{\sqrt{13}}{6}$

4 $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

EXERCISE 4C.3

1 **a** $x = 2 \pm \sqrt{7}$ **b** $x = -3 \pm \sqrt{2}$ **c** $x = 2 \pm \sqrt{3}$
d $x = -2 \pm \sqrt{5}$ **e** $x = 2 \pm \sqrt{2}$ **f** $x = \dfrac{1 \pm \sqrt{7}}{2}$
g $x = \dfrac{5 \pm \sqrt{37}}{6}$ **h** $x = 2 \pm \sqrt{10}$ **i** $x = \dfrac{7 \pm \sqrt{33}}{4}$

2 **a** $x = -2 \pm 2\sqrt{2}$ **b** $x = \dfrac{-5 \pm \sqrt{57}}{8}$ **c** $x = \dfrac{5 \pm \sqrt{13}}{2}$
d $x = \dfrac{-4 \pm \sqrt{7}}{9}$ **e** $x = \dfrac{-7 \pm \sqrt{97}}{4}$ **f** $x = \dfrac{1 \pm \sqrt{145}}{8}$
g $x = \dfrac{1 \pm \sqrt{7}}{2}$ **h** $x = \dfrac{1 \pm \sqrt{5}}{2}$ **i** $x = \dfrac{3 \pm \sqrt{17}}{4}$

EXERCISE 4C.4

1 **a** $\Delta = 13$ **b** 2 distinct irrational roots **c** $x = \frac{7}{2} \pm \frac{\sqrt{13}}{2}$

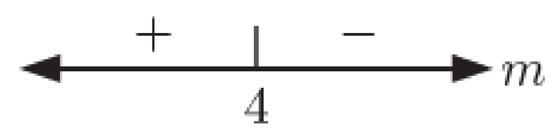
2 **a** $\Delta = 0$ **b** 1 root (repeated) **c** $x = \frac{1}{2}$

3 **a** $x^2 = -5$, $\therefore$ no real roots **b** $\Delta = -20$

4 **a** 2 distinct irrational roots **b** 2 distinct rational roots
c 2 distinct rational roots **d** 2 distinct irrational roots
e no real roots **f** a repeated root

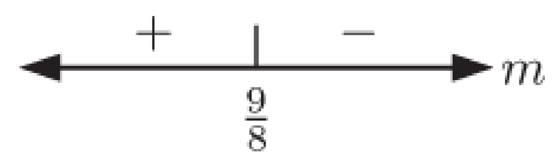
5 **a**, **c**, **d**, and **f**

6 **a** $\Delta = 16 - 4m$



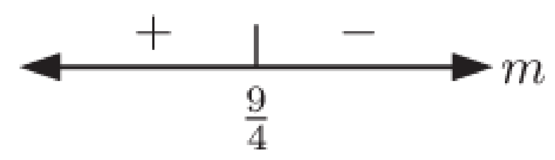
i $m = 4$ **ii** $m < 4$ **iii** $m > 4$

b $\Delta = 9 - 8m$



i $m = \frac{9}{8}$ **ii** $m < \frac{9}{8}, \ m \neq 0$ **iii** $m > \frac{9}{8}$

c $\Delta = 9 - 4m$



i $m = \frac{9}{4}$ **ii** $m < \frac{9}{4}, \ m \neq 0$ **iii** $m > \frac{9}{4}$

7 For $k = -8 + 4\sqrt{7}$, repeated root is $x = 1 - \frac{\sqrt{7}}{2}$.
For $k = -8 - 4\sqrt{7}$, repeated root is $x = 1 + \frac{\sqrt{7}}{2}$.

EXERCISE 4C.5

1 **a** **i** sum $= -4$, product $= -21$ **ii** roots are -7 and 3
b **i** sum $= 5$, product $= 5$ **ii** roots are $\frac{5}{2} \pm \frac{\sqrt{5}}{2}$
c **i** sum $= 3$, product $= \frac{5}{4}$ **ii** roots are $\frac{1}{2}$ and $\frac{5}{2}$
d **i** sum $= \frac{4}{3}$, product $= -\frac{2}{3}$ **ii** roots are $\frac{2}{3} \pm \frac{\sqrt{10}}{3}$

2 $k = -\frac{3}{5}$, roots are -1 and $\frac{1}{3}$

3 **a** $3\alpha = \dfrac{6}{a}$, $2\alpha^2 = \dfrac{a-2}{a}$
b $a = 4$, roots are $\frac{1}{2}$ and 1; or $a = -2$, roots are -1 and -2

4 $k = 4$, roots are $-\frac{1}{2}$ and $\frac{3}{2}$; or $k = 16$, roots are $-\frac{5}{4}$ and $\frac{3}{4}$

5 **a** $x^2 + 2x - 15 = 0$ **b** $x^2 - 4x + 1 = 0$

6 **a** $3x^2 - 2x - 4 = 0$ **b** $3x^2 + 4x - 16 = 0$

7 **a** $2x^2 - 13x + 17 = 0$
b $k(8x^2 - 10x - 1) = 0, \ k \in \mathbb{R}, \ k \neq 0$

8 $k(x^2 + 7x - 44) = 0, \ k \in \mathbb{R}, \ k \neq 0$

9 $k(4x^2 - 61x + 81), \ k \in \mathbb{R}, \ k \neq 0$

10 $7x^2 - 48x + 64 = 0$

11 $k(8x^2 - 70x + 147) = 0, \ k \in \mathbb{R}, \ k \neq 0$

EXERCISE 4D

1 **a** $x = -2$ or -7 **b** $x = 4$
c $x \approx 1.29$ or -1.54 **d** $x = 1.5$ or -2.5
e $x \approx 1.18$ or 2.82 **f** no real solutions
g $x \approx 0.360$ or 1.39 **h** $x \approx -5.99$ or 4.18

2 **a** $x = -3$ or -4 **b** $x \approx 1.85$ or -4.85
c $x \approx 0.847$ or -1.18 **d** no real solutions

3 **a** $x = -3, 0,$ or 3 **b** $x \approx -1.13$ **c** $x = 3, 2,$ or -4
d $x = 1$ **e** $x = 0.5, \approx 0.618,$ or ≈ -1.62
f $x \approx 4.36, 0.406,$ or -2.26

4 **a** no real solutions **b** $x \approx 1.34$ or -3.17
c $x = 1$ or -1 **d** no real solutions **e** $x \approx -2.27$ or 2.43
f $x \approx -3.36, -1.65, 0.192,$ or 2.82

5 **a** $x = 0, \approx 1.73,$ or -1.73 **b** $x \approx -0.811$

6 **a** $x^3 - 6x^2 + 2x - 6 = 0$ **b** $x \approx 5.83$

EXERCISE 4E

1 **a** $x = -2$ or 3 **b** $x = -2$ or 3

2 **a** $x \approx 3.21$ **b** $x \approx 0.387$ or -1.72 **c** $x \approx 2.46$
d $x \approx 5.17$ **e** $x \approx 1.52$ or 2.83 **f** $x \approx 3.56$ or -1.30

11 **a** ≈ 88 students **b** $m \approx 24$

c

Time (t min)	*Frequency*
$5 \leqslant t < 10$	20
$10 \leqslant t < 15$	40
$15 \leqslant t < 20$	48
$20 \leqslant t < 25$	42
$25 \leqslant t < 30$	28
$30 \leqslant t < 35$	17
$35 \leqslant t < 40$	5

12 **a** $\sigma^2 \approx 63.0,\ \sigma \approx 7.94$ **b** $\sigma^2 \approx 0.969,\ \sigma \approx 0.984$

13 **a** $\overline{x} \approx 49.6$ matches, $\sigma \approx 1.60$ matches, $s \approx 1.60$ matches
b The claim is not justified, but a larger sample is needed.

14 **a** $\overline{x} \approx 33.6$ L **b** $\sigma \approx 7.63$ L, $s \approx 7.66$ L

15 **a** No, extreme values have less effect on the standard deviation of a larger population.
b **i** mean **ii** standard deviation
c A low standard deviation means that the weight of biscuits in each packet is, on average, close to 250 g.

REVIEW SET 13B

1 **a**

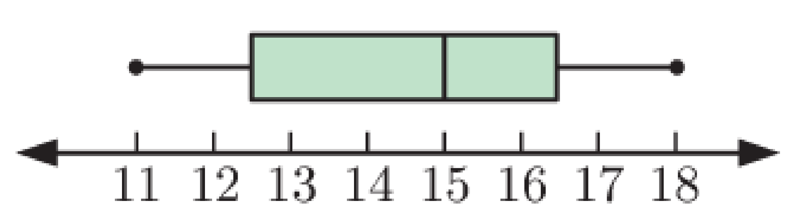
	mean (seconds)	median (seconds)
Week 1	≈ 16.0	16.3
Week 2	≈ 15.1	15.1
Week 3	≈ 14.4	14.3
Week 4	14.0	14.0

b Yes, Heike's mean and median times have gradually decreased each week which indicates that her speed has improved over the 4 week period.

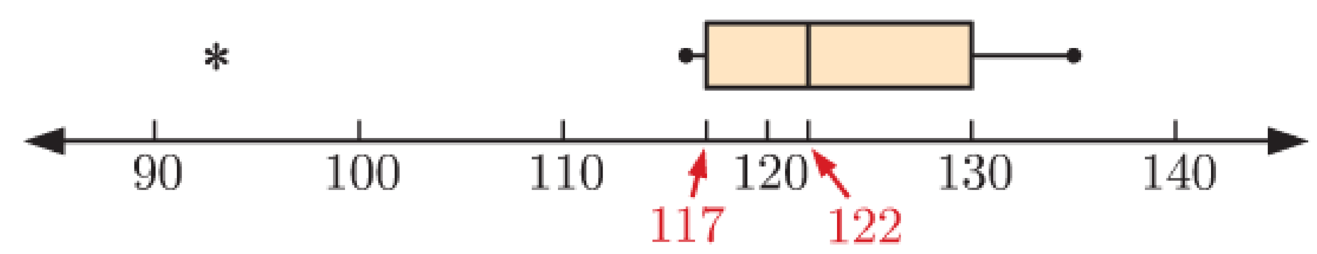
2 **a** 5 **b** 3.52 **c** 3.5 **3** **a** $x = 7$ **b** 6

4 $p = 7,\ q = 9$ (or $p = 9,\ q = 7$) **5** ≈ 414 patrons

6



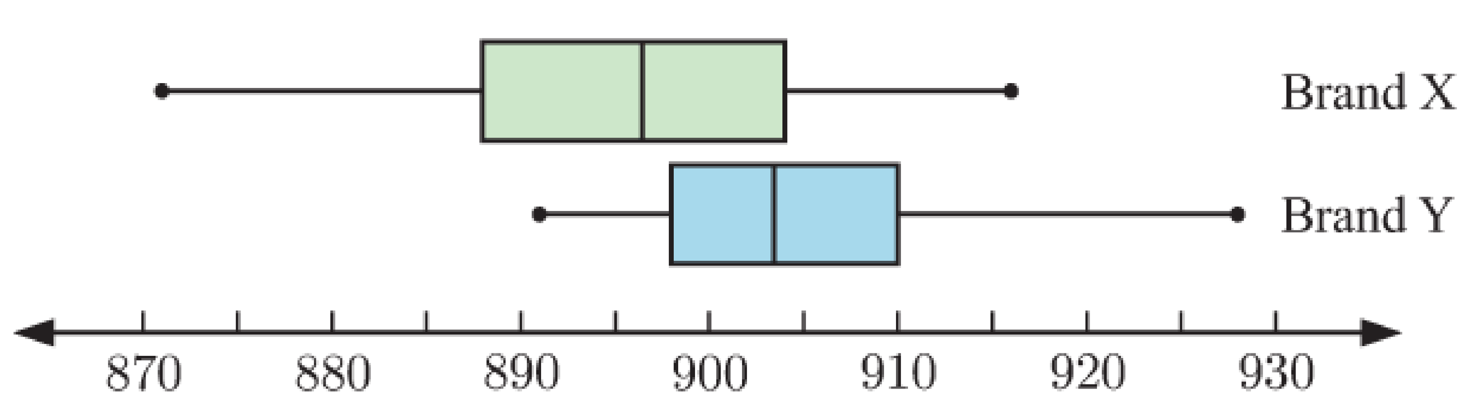
7 **a** $\sigma \approx 11.7,\ s \approx 12.4$ **b** $Q_1 = 117,\ Q_3 = 130$
c yes, 93
d



8 **a**

	Brand X	*Brand Y*
min	871	891
Q_1	888	898
median	896.5	903.5
Q_3	904	910
max	916	928
IQR	16	12

b



c **i** Brand Y, as the median is higher.
ii Brand Y, as the IQR is lower, so less variation.

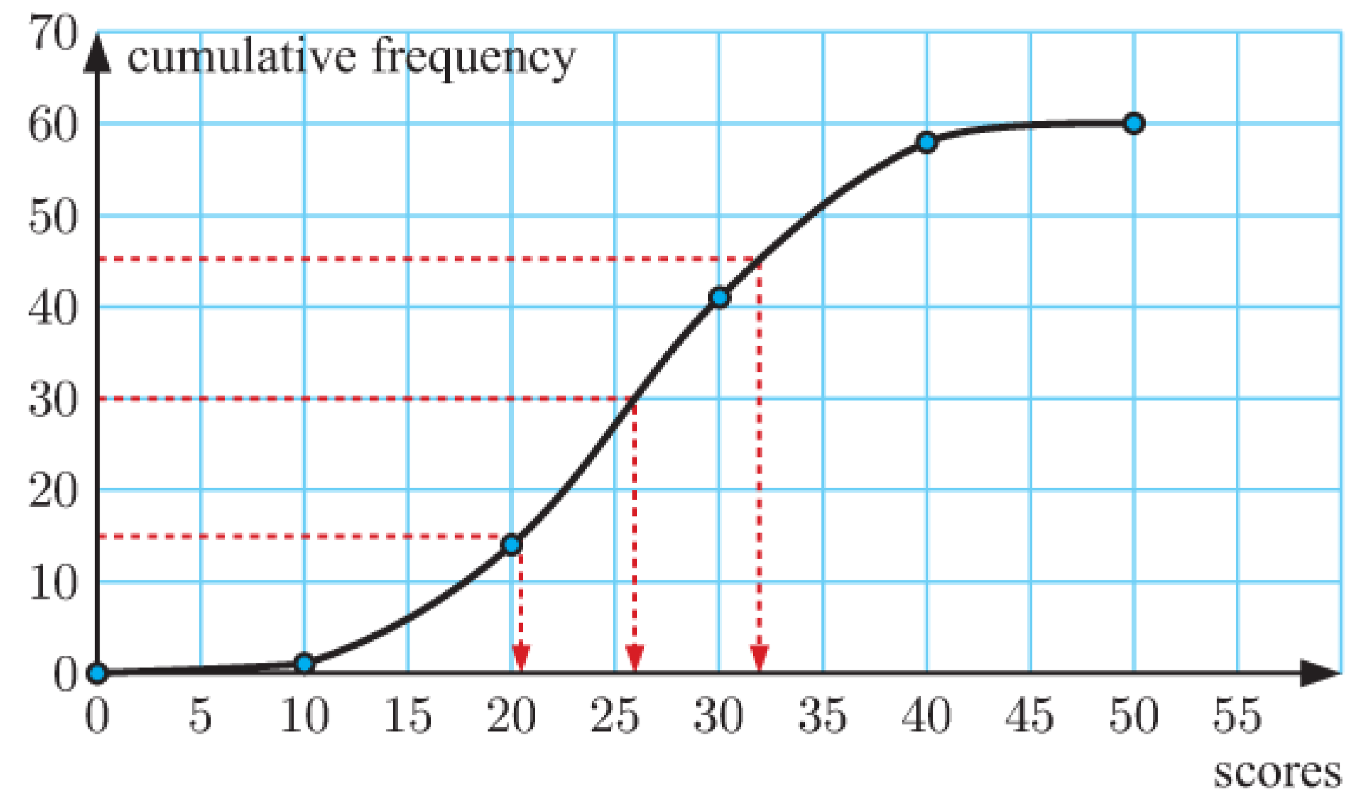
9 **a** $p = 12,\ m = 6$ **b**

Measure	*Value*
mode	9
median	9
range	4

c $\overline{x} = \frac{254}{30} = \frac{127}{15}$

10 **a** ≈ 77 days **b** ≈ 12 days

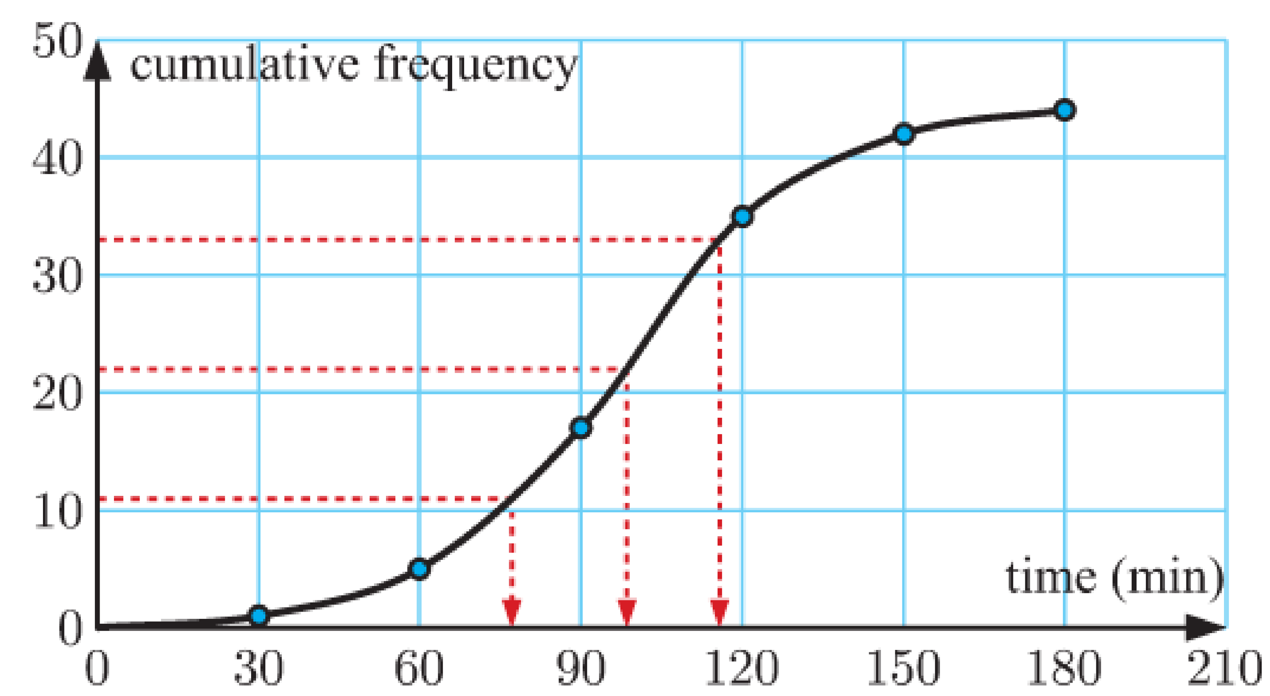
11 **a**



b **i** median ≈ 26 **ii** IQR ≈ 11.5
iii $\overline{x} \approx 26.0$ **iv** $\sigma \approx 8.31$

12 **a** 44 players **b** $90 \leqslant t < 120$ min

c



d **i** ≈ 98.6 min **ii** ≈ 96.8 min **iii** no
e ".... between 77.2 and 115.7 minutes."

13 **a** $\overline{x} \approx$ €207.02 **b** $\sigma =$ €38.80, $s \approx$ €38.89

14 **a** Kevin: $\overline{x} = 41.2$ min; Felicity: $\overline{x} = 39.5$ min
b Kevin: $\sigma \approx 7.61$ min, $s \approx 7.81$ min;
Felicity: $\sigma \approx 9.22$ min, $s \approx 9.46$ min
c Felicity **d** Kevin

15 10 data values

EXERCISE 14A

1 **a** $y = x^2 - 3x + 1$

x	-2	-1	0	1	2
y	11	5	1	-1	-1

b $y = x^2 + 2x - 5$

x	-2	-1	0	1	2
y	-5	-6	-5	-2	3

c $y = 2x^2 - x + 3$

x	-4	-2	0	2	4
y	39	13	3	9	31

d $y = -3x^2 + 2x + 4$

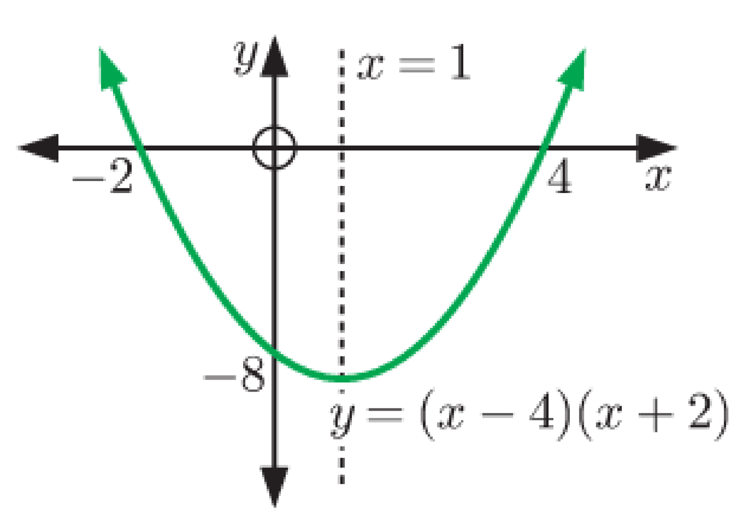
x	-4	-2	0	2	4
y	-52	-12	4	-4	-36

2 **a** no **b** yes **c** yes **d** yes **e** no **f** yes

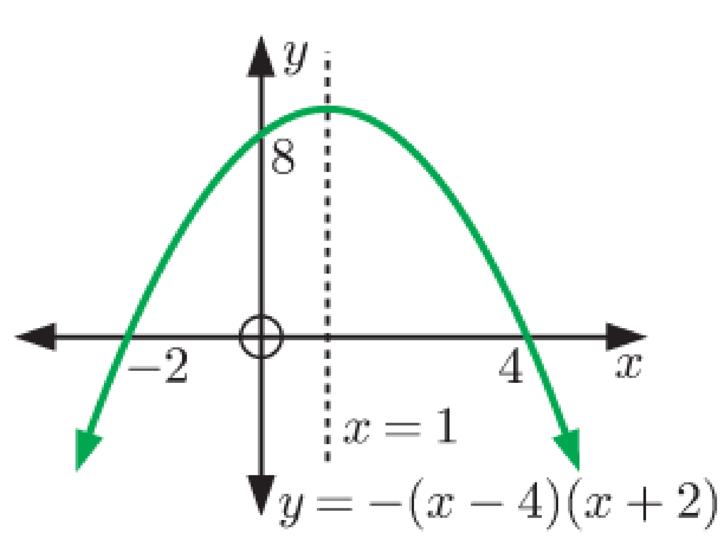
3 **a** $x = -1$ or -2 **b** $x = 2$ **c** $x = 1$ or 5
d $x = -3$ or $\frac{1}{2}$ **e** $x = -6$ or 1 **f** no real solutions

EXERCISE 14B.1

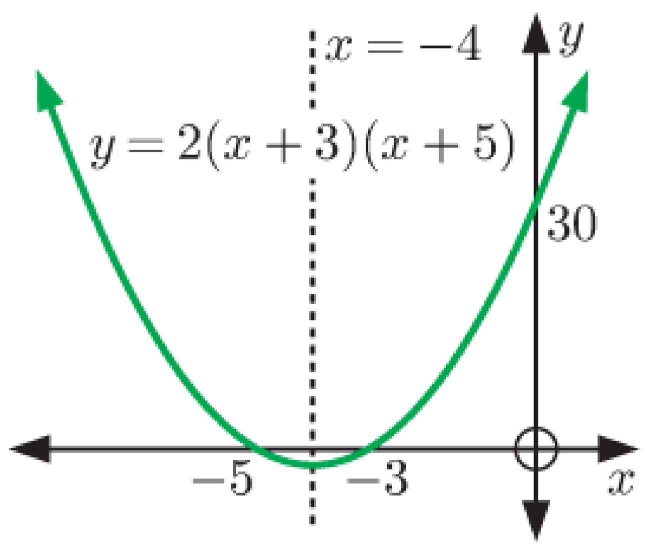
1 **a**



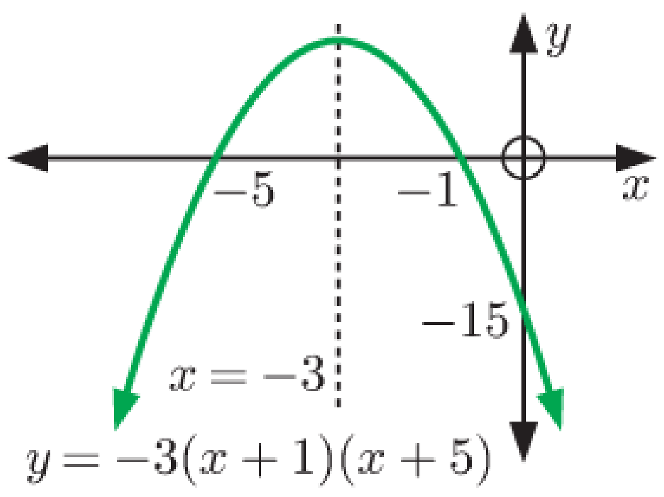
b



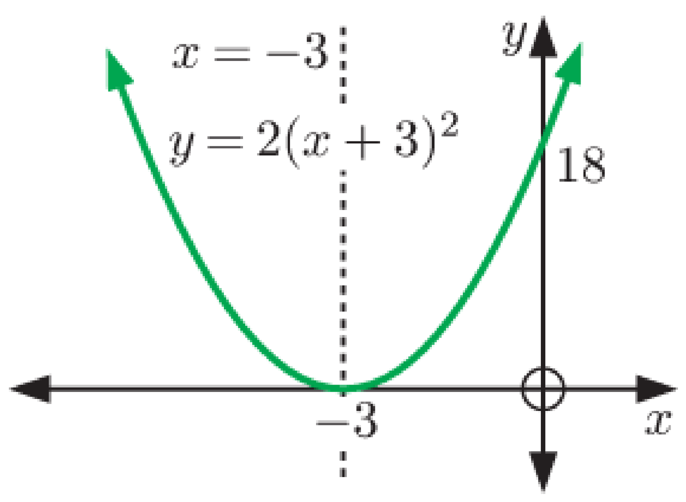
c



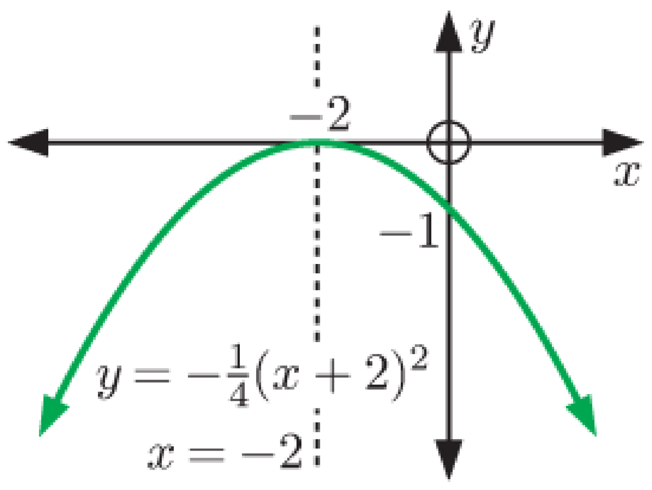
d



e

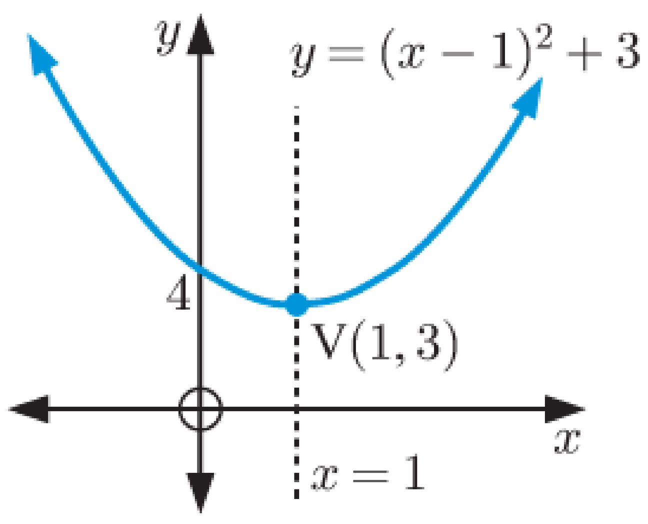


f

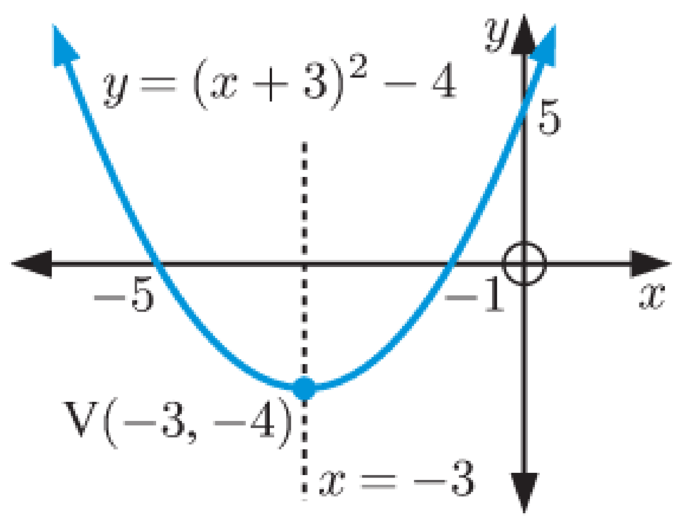


2 **a** C **b** E **c** B **d** F **e** G **f** H
g I **h** A **i** D

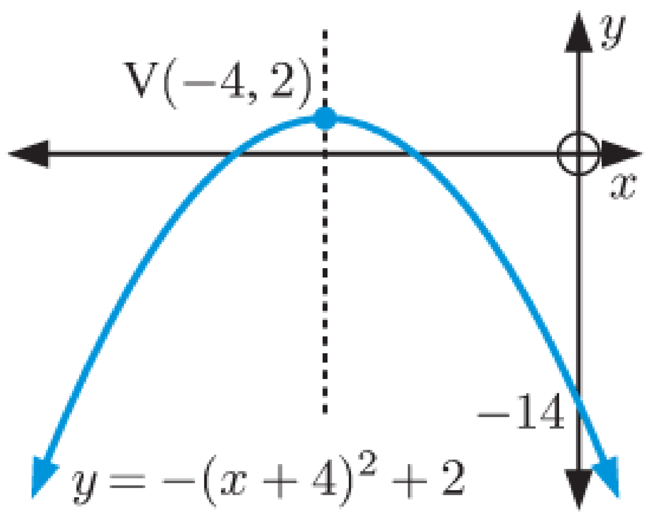
3 **a**



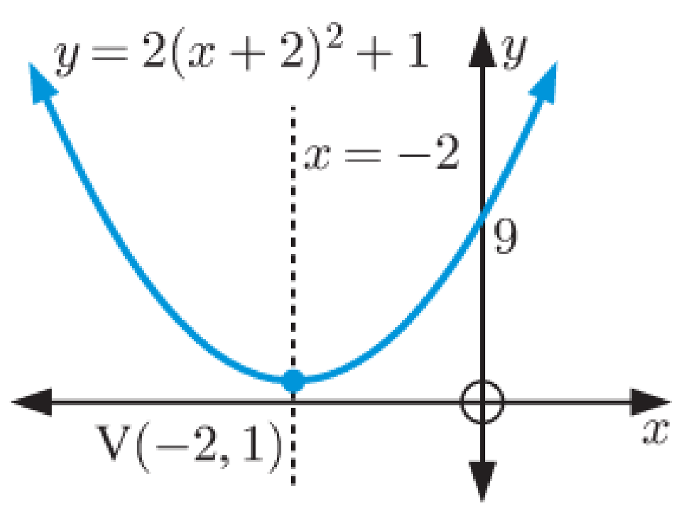
b



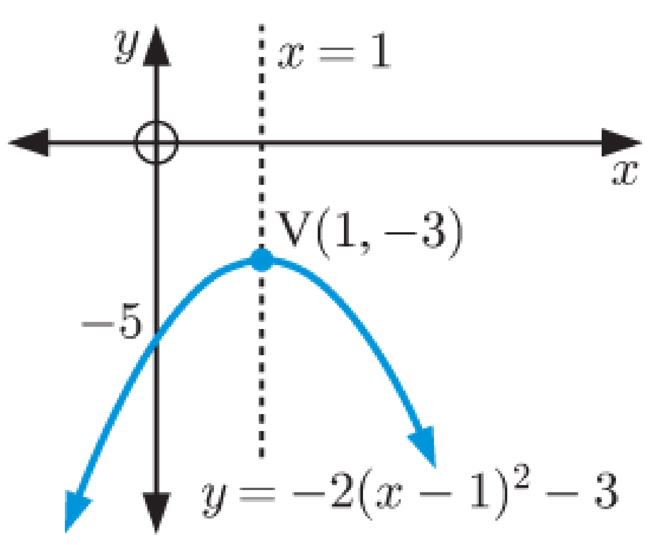
c



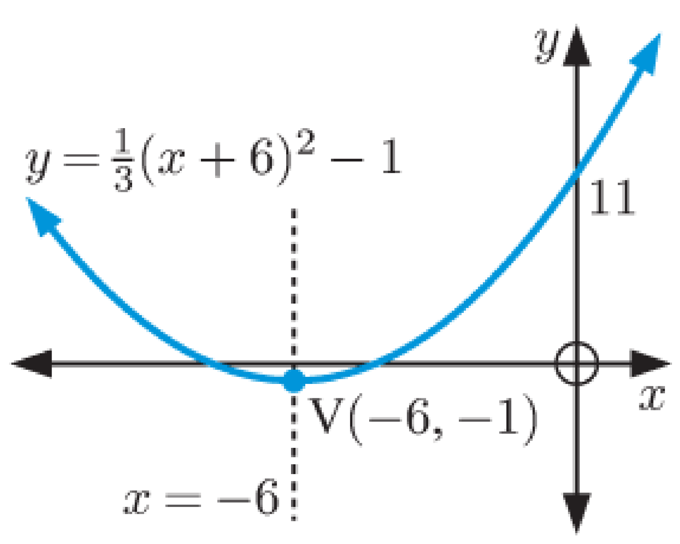
d



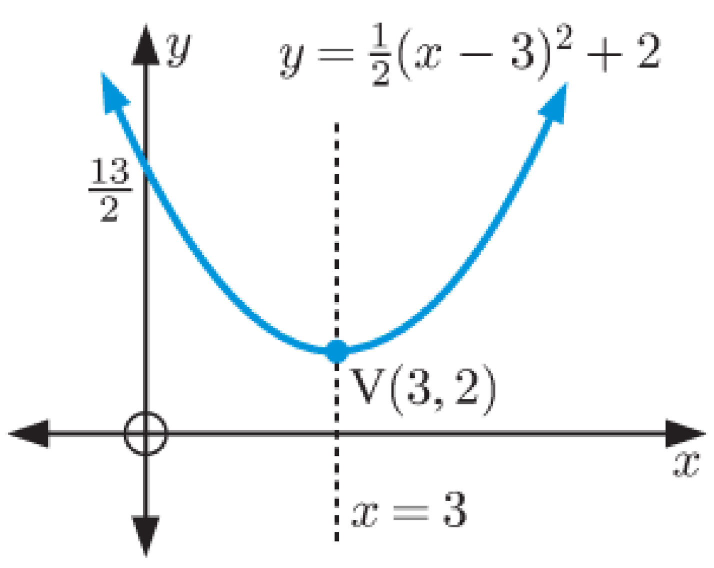
e



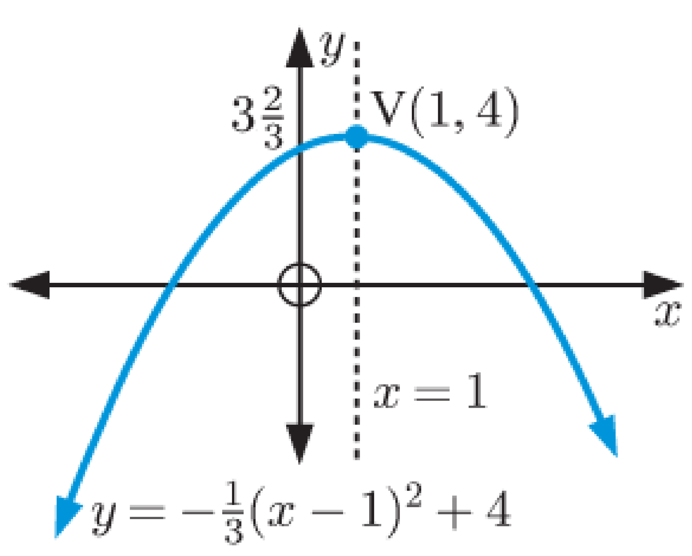
f



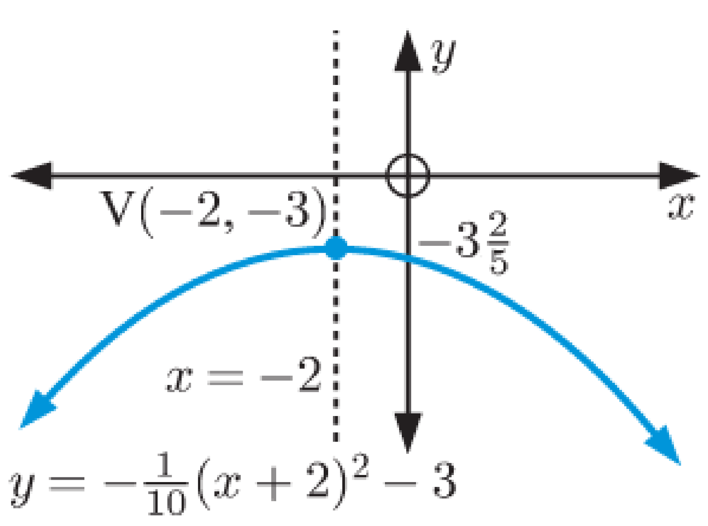
g



h

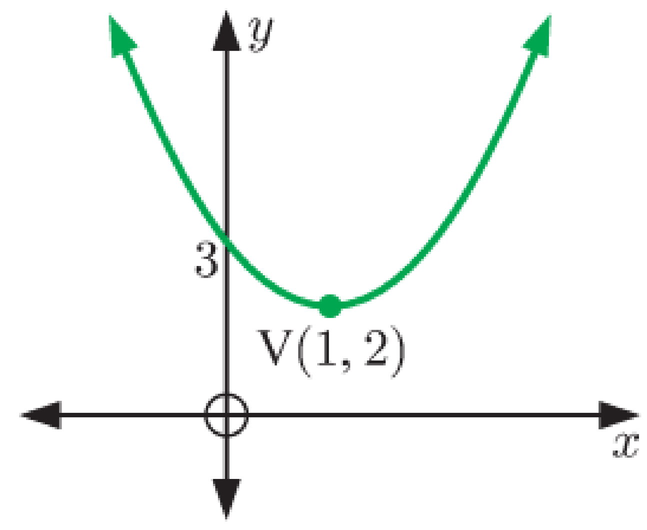


i

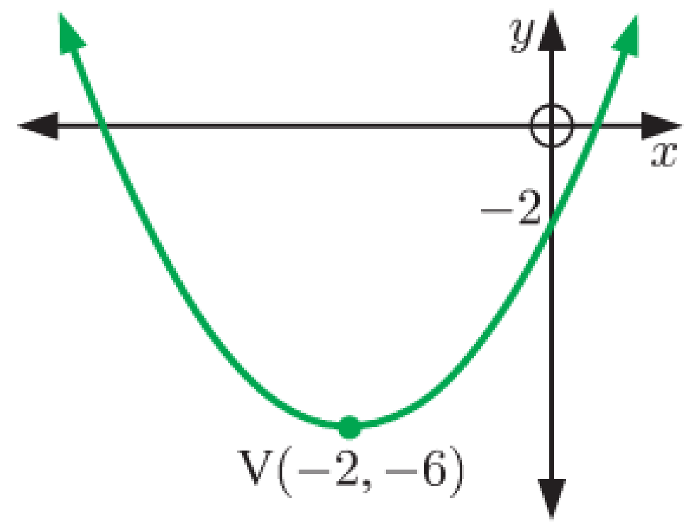


EXERCISE 14B.2

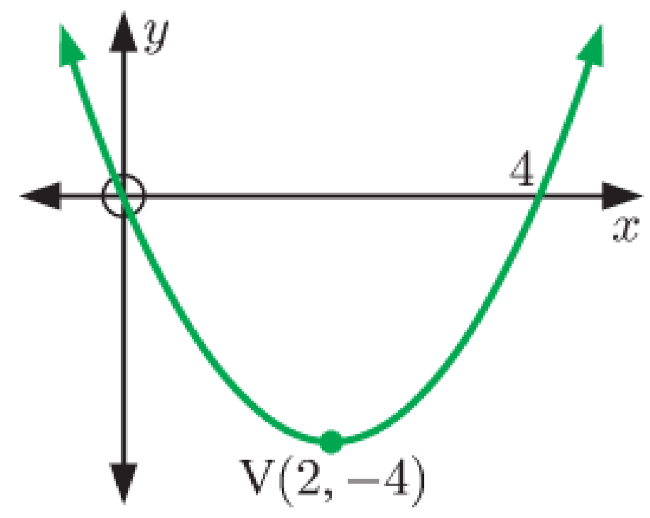
1 **a** $y = (x-1)^2 + 2$



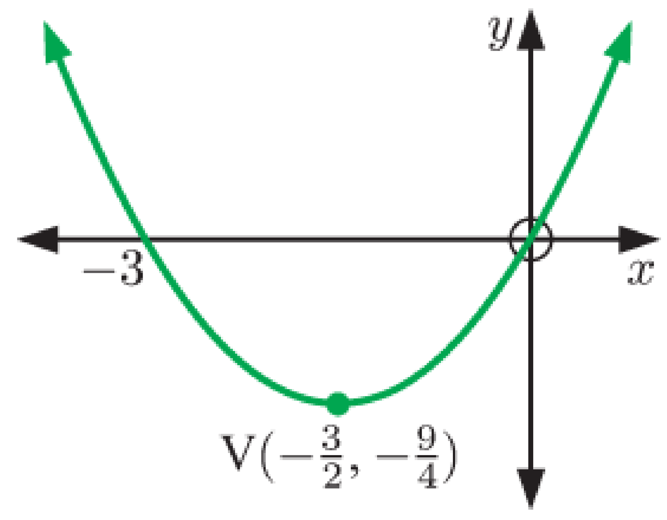
b $y = (x+2)^2 - 6$



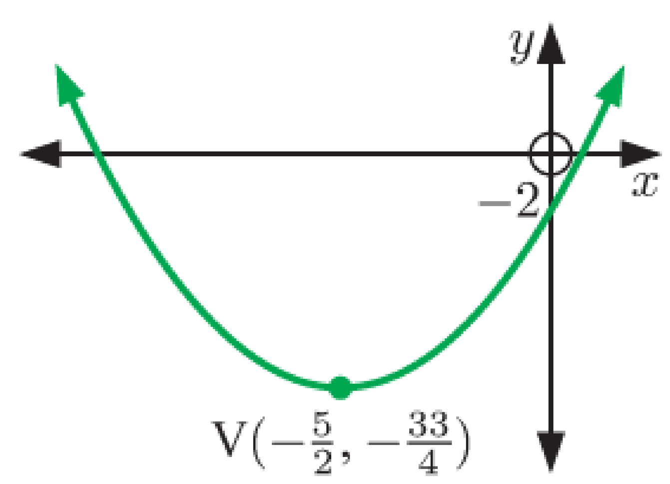
c $y = (x-2)^2 - 4$



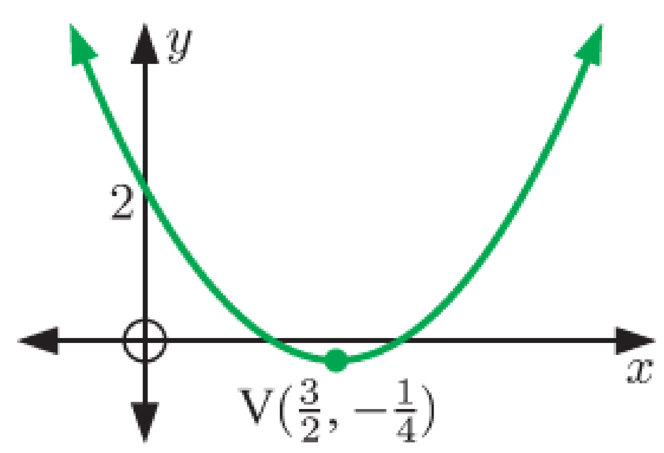
d $y = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$



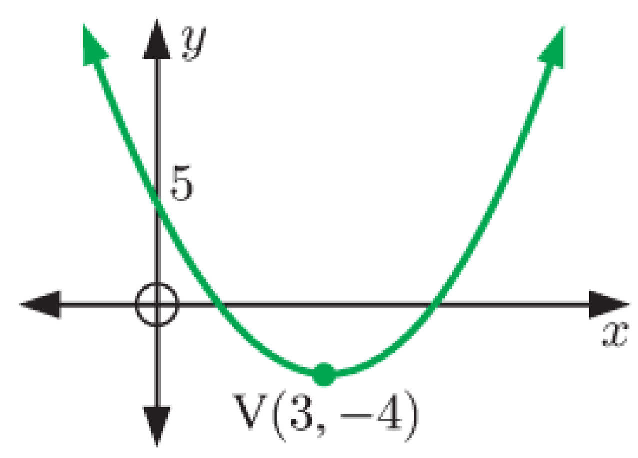
e $y = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$



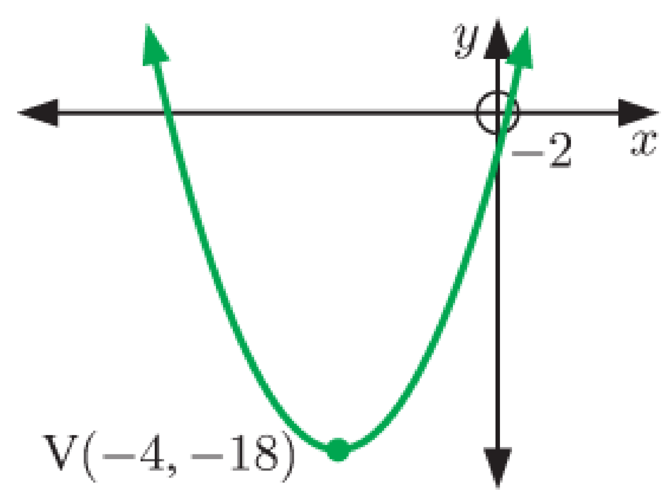
f $y = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$



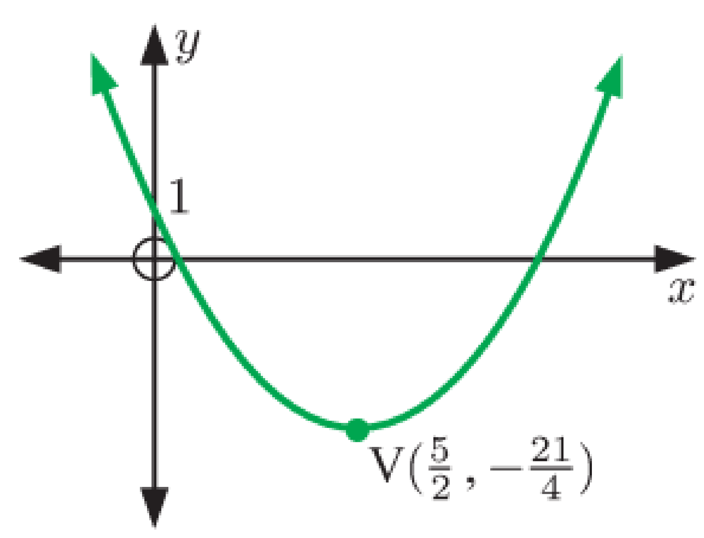
g $y = (x-3)^2 - 4$



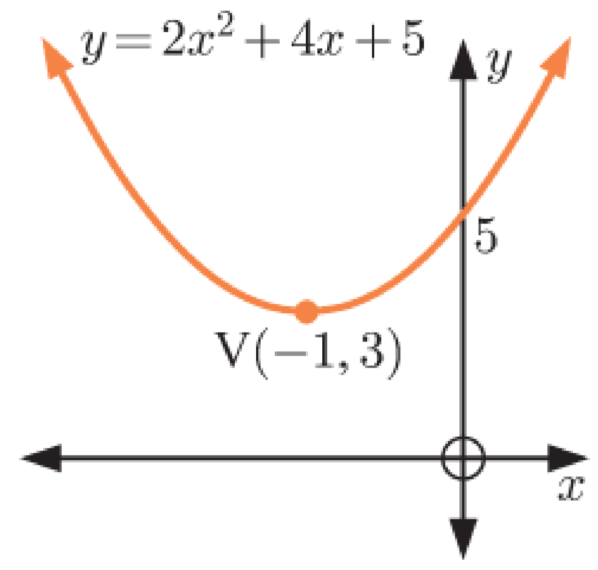
h $y = (x+4)^2 - 18$



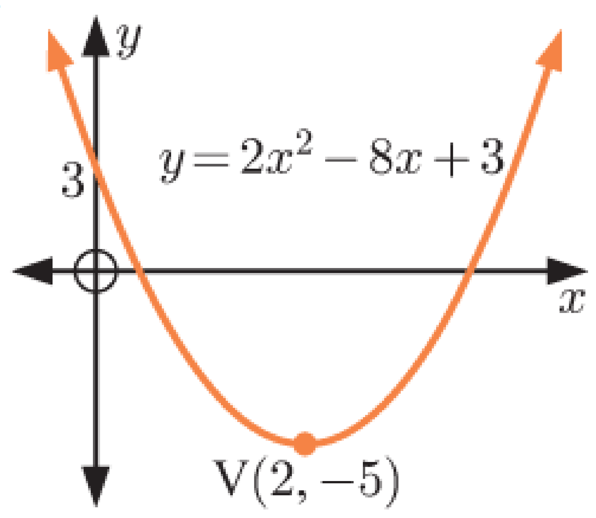
i $y = \left(x - \frac{5}{2}\right)^2 - \frac{21}{4}$



2 **a** **i** $y = 2(x+1)^2 + 3$
ii $(-1, 3)$ **iii** 5
iv

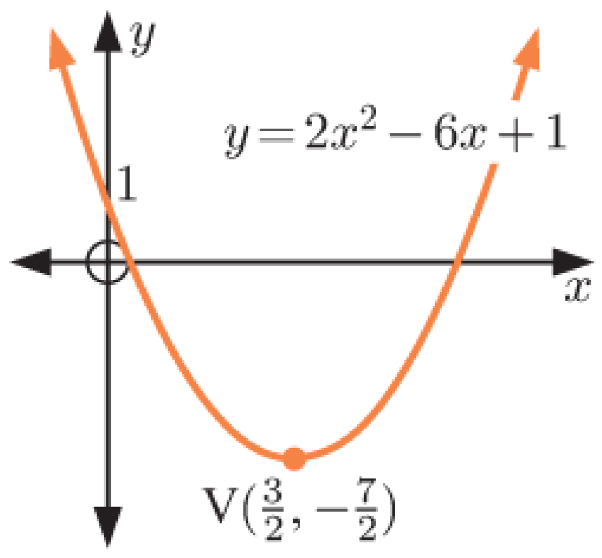


b **i** $y = 2(x-2)^2 - 5$
ii $(2, -5)$ **iii** 3
iv



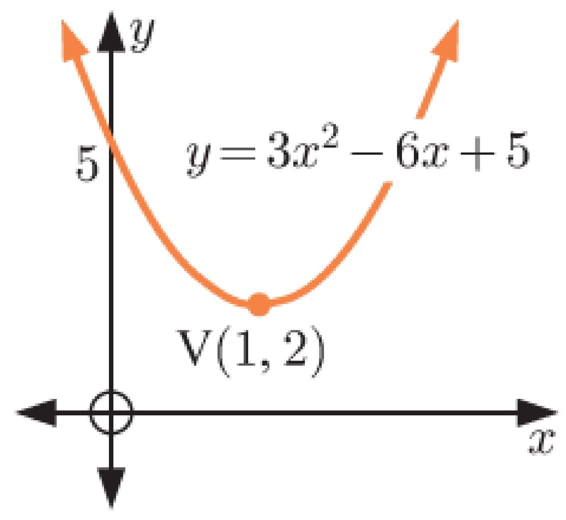
c **i** $y = 2(x - \frac{3}{2})^2 - \frac{7}{2}$ **ii** $(\frac{3}{2}, -\frac{7}{2})$ **iii** 1

iv



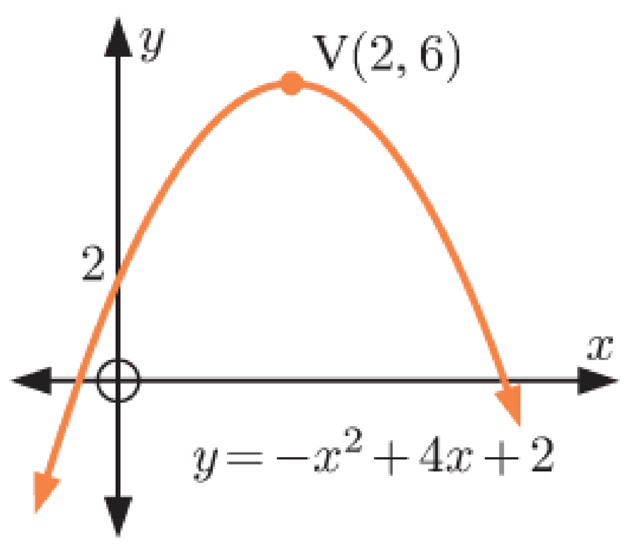
d **i** $y = 3(x - 1)^2 + 2$ **ii** $(1, 2)$ **iii** 5

iv



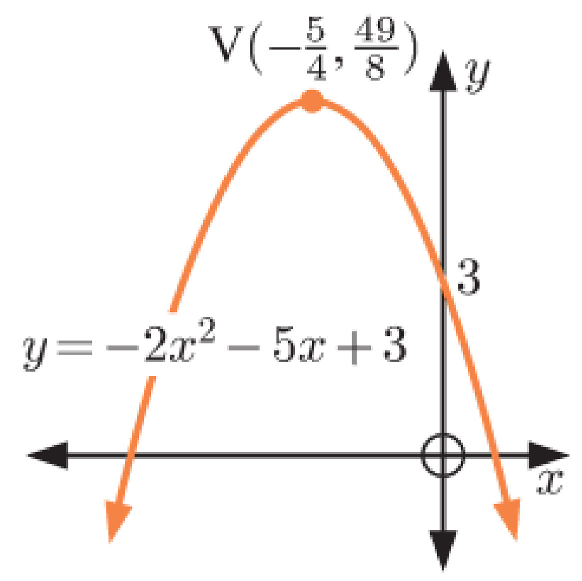
e **i** $y = -(x - 2)^2 + 6$ **ii** $(2, 6)$ **iii** 2

iv



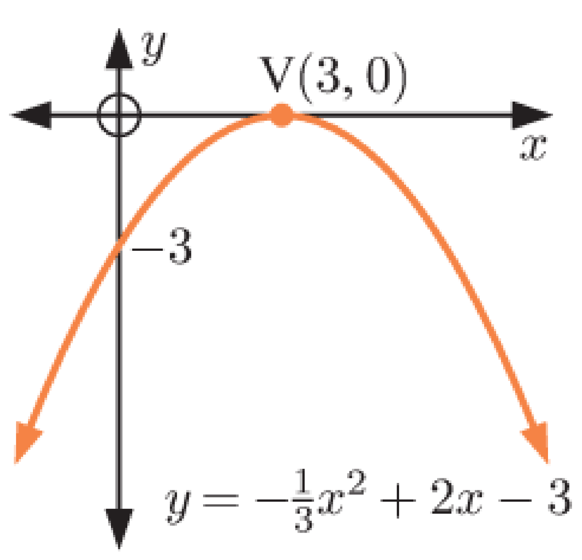
f **i** $y = -2(x + \frac{5}{4})^2 + \frac{49}{8}$ **ii** $(-\frac{5}{4}, \frac{49}{8})$ **iii** 3

iv



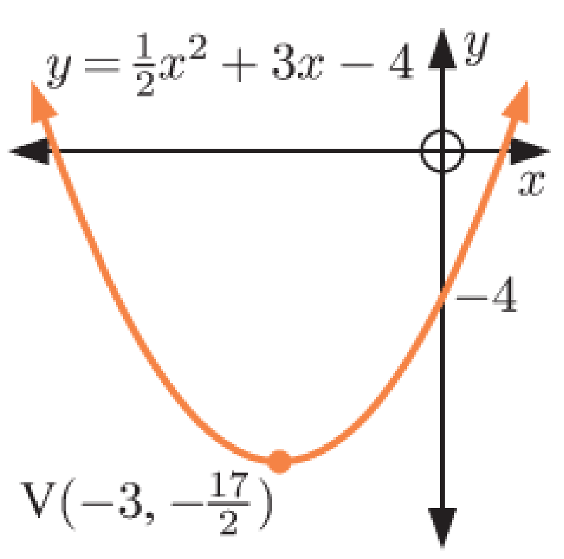
g **i** $y = -\frac{1}{3}(x - 3)^2$ **ii** $(3, 0)$ **iii** -3

iv



h **i** $y = \frac{1}{2}(x + 3)^2 - \frac{17}{2}$ **ii** $(-3, -\frac{17}{2})$ **iii** -4

iv

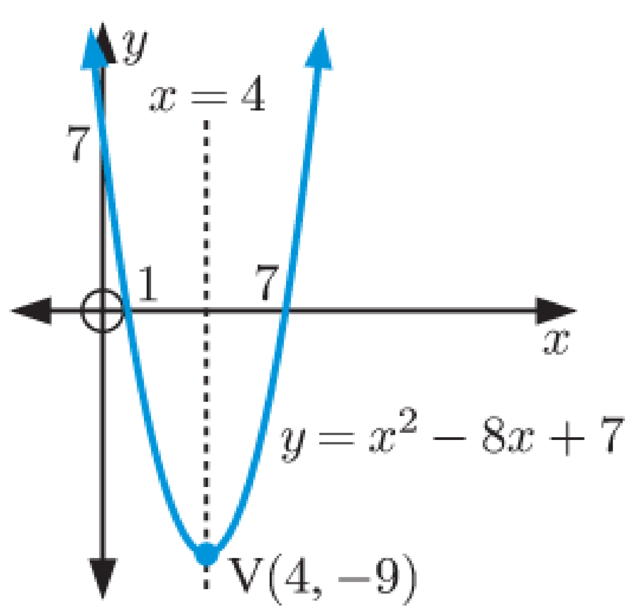


EXERCISE 14B.3

1 **a** **i** $(2, -2)$ **ii** minimum turning point
b **i** $(-1, -4)$ **ii** minimum turning point
c **i** $(0, 4)$ **ii** minimum turning point
d **i** $(0, 1)$ **ii** maximum turning point
e **i** $(-2, -15)$ **ii** minimum turning point
f **i** $(-2, -5)$ **ii** maximum turning point
g **i** $(-\frac{3}{2}, -\frac{11}{2})$ **ii** minimum turning point
h **i** $(\frac{5}{2}, -\frac{19}{2})$ **ii** minimum turning point
i **i** $(1, -\frac{9}{2})$ **ii** maximum turning point
j **i** $(14, -43)$ **ii** minimum turning point

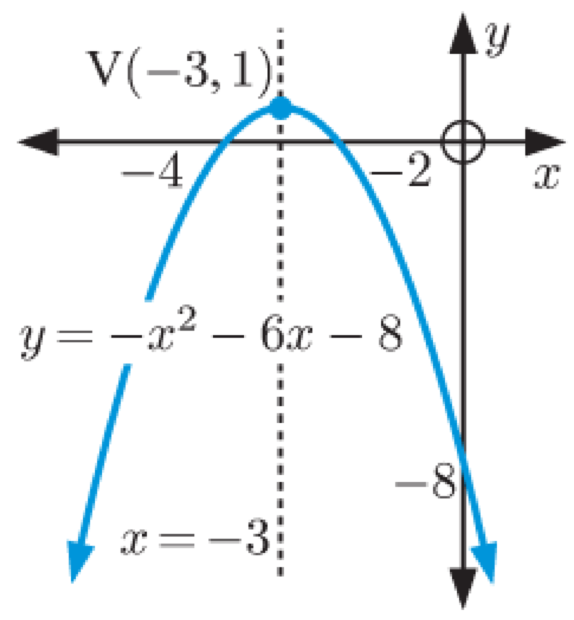
2 **a** **i** $x = 4$ **ii** $(4, -9)$ **iii** x-intercepts 1, 7, y-intercept 7

iv



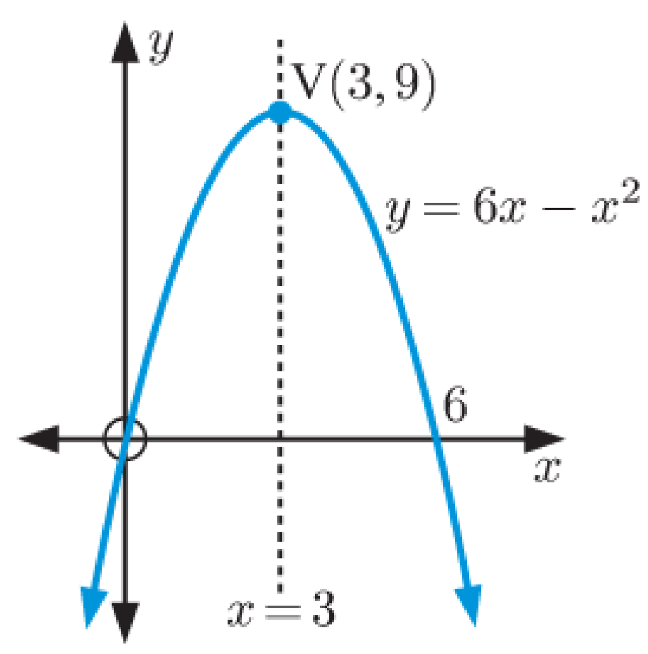
b **i** $x = -3$ **ii** $(-3, 1)$ **iii** x-int. $-2, -4$, y-intercept -8

iv



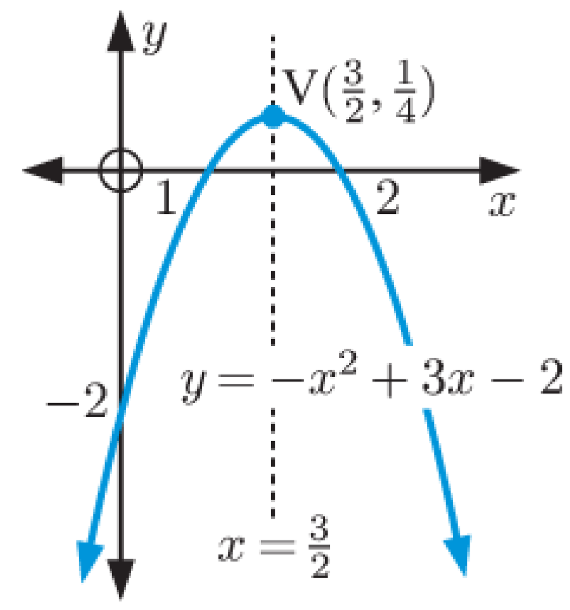
c **i** $x = 3$ **ii** $(3, 9)$ **iii** x-intercepts 0, 6, y-intercept 0

iv



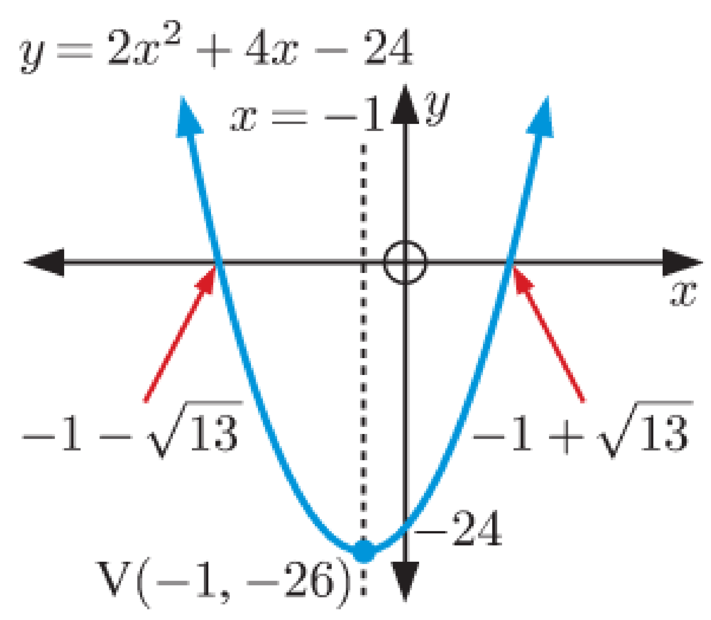
d **i** $x = \frac{3}{2}$ **ii** $(\frac{3}{2}, \frac{1}{4})$ **iii** x-intercepts 1, 2, y-intercept -2

iv



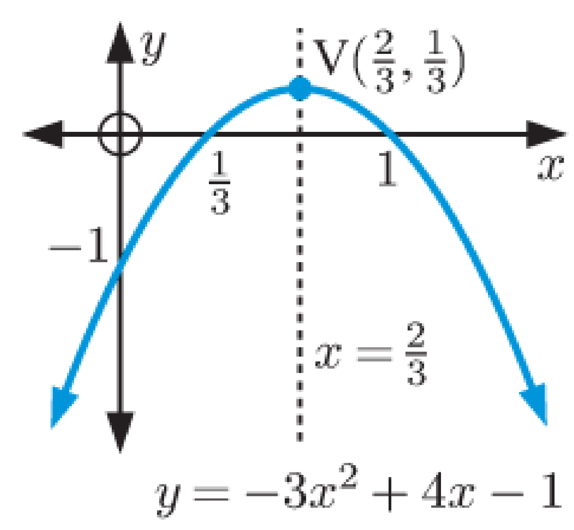
e **i** $x = -1$ **ii** $(-1, -26)$ **iii** x-int. $-1 \pm \sqrt{13}$, y-intercept -24

iv



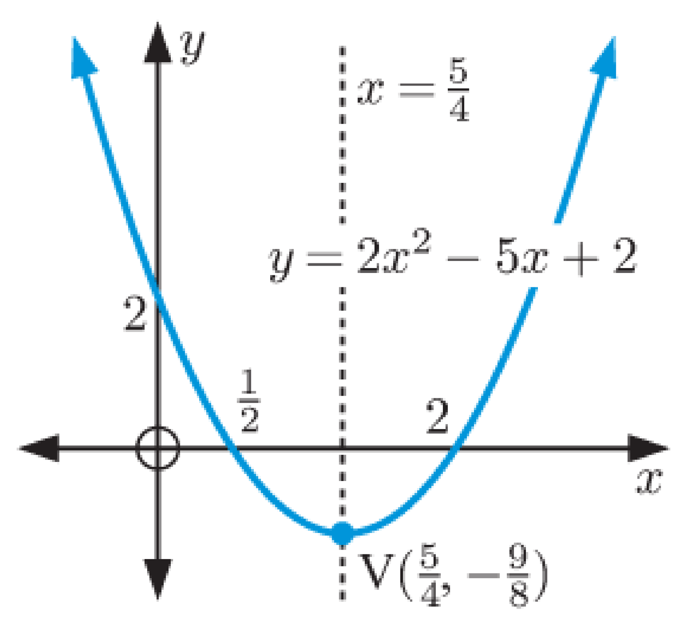
f **i** $x = \frac{2}{3}$ **ii** $(\frac{2}{3}, \frac{1}{3})$ **iii** x-intercepts $\frac{1}{3}$, 1, y-intercept -1

iv



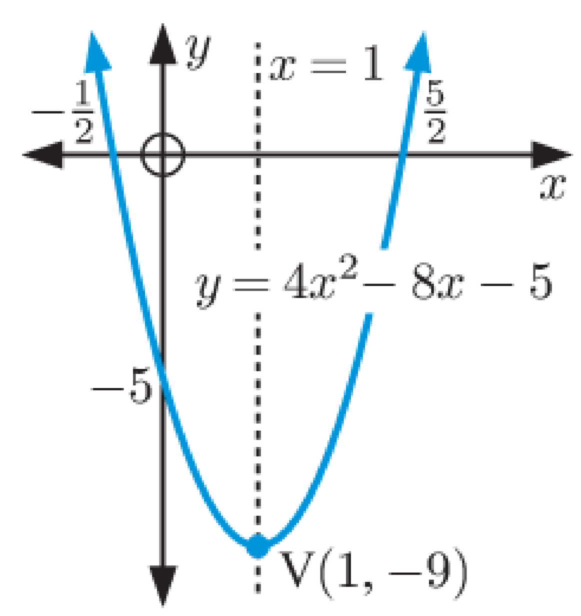
g **i** $x = \frac{5}{4}$ **ii** $(\frac{5}{4}, -\frac{9}{8})$ **iii** x-intercepts $\frac{1}{2}$, 2, y-intercept 2

iv



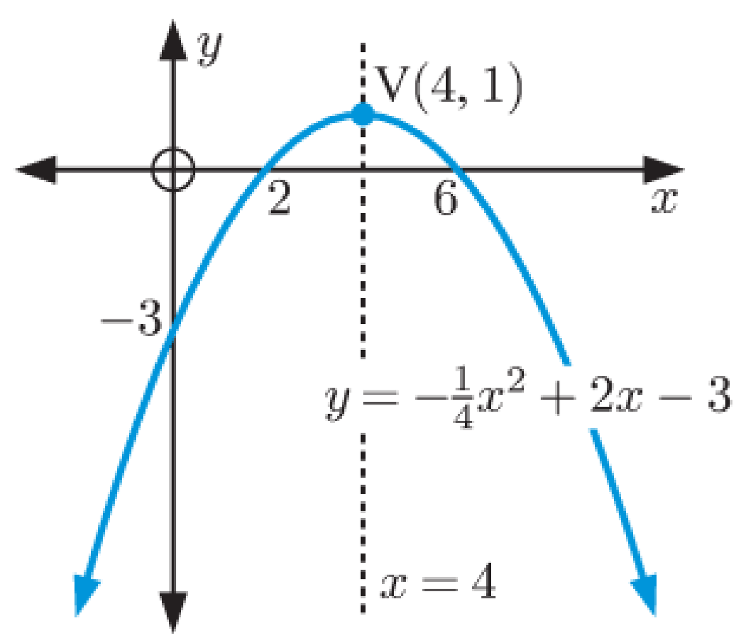
h **i** $x = 1$ **ii** $(1, -9)$ **iii** x-intercepts $-\frac{1}{2}, \frac{5}{2}$, y-intercept -5

iv



i **i** $x = 4$ **ii** $(4, 1)$ **iii** x-intercepts 2, 6, y-intercept -3

iv



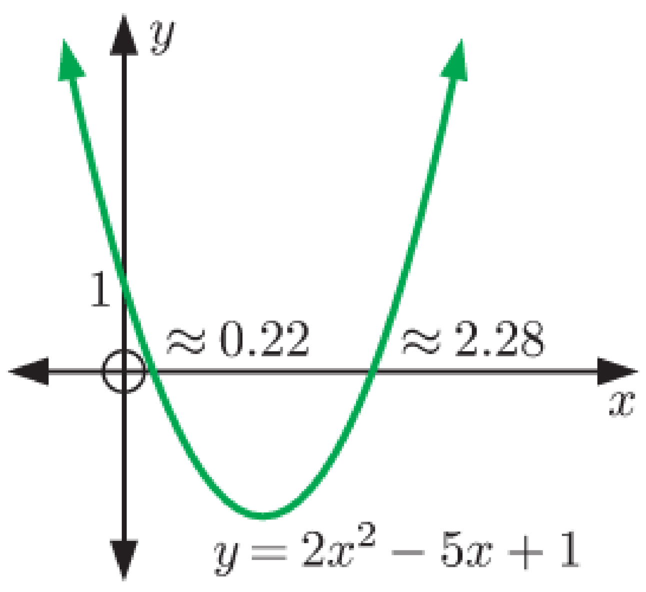
3 **Hint:** $y = ax^2 + bx + c$ has vertex with x-coordinate $-\dfrac{b}{2a}$ and y-coordinate $a\left(-\dfrac{b}{2a}\right)^2 + b\left(-\dfrac{b}{2a}\right) + c$.

EXERCISE 14C

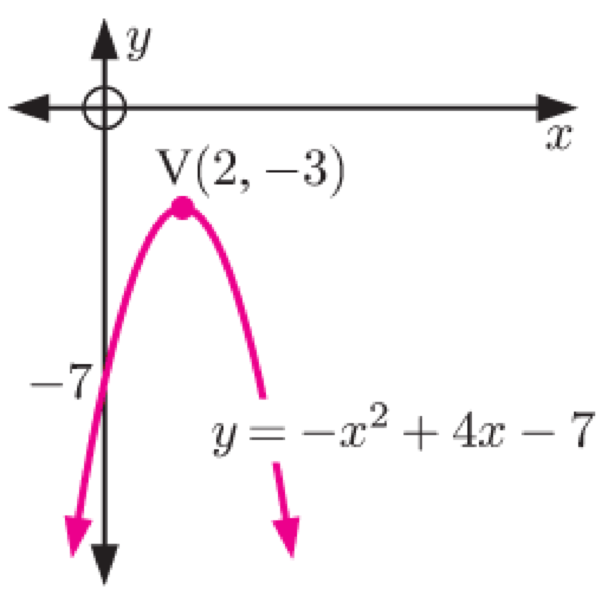
1 **a** $\Delta = 9$ which is > 0, graph cuts x-axis twice; is concave up.
b $\Delta = 12$ which is > 0, graph cuts x-axis twice; is concave up.
c $\Delta = -12$ which is < 0, graph lies entirely below the x-axis; is concave down, negative definite.
d $\Delta = 57$ which is > 0, graph cuts x-axis twice; is concave up.

e $\Delta = 0$, graph touches x-axis; is concave up.
f $\Delta = 17$ which is > 0, graph cuts x-axis twice; is concave down.
g $\Delta = 121$ which is > 0, graph cuts x-axis twice; is concave up.
h $\Delta = 25$ which is > 0, graph cuts x-axis twice; is concave down.
i $\Delta = 0$, graph touches x-axis; is concave up.

2 **a** concave up
b $\Delta = 17$ which is > 0
$\therefore$ cuts x-axis twice
c x-intercepts
≈ 0.22 and 2.28
d y-intercept is 1
e



3 **a** $\Delta = -12$ which is < 0
$\therefore$ does not cut x-axis
b negative definite, since
$a < 0$ and $\Delta < 0$
c vertex is $(2, -3)$,
y-intercept is -7
d



4 **a** $a = 2$ which is > 0 and $\Delta = -40$ which is < 0
$\therefore$ positive definite.
b $a = -2$ which is < 0 and $\Delta = -23$ which is < 0
$\therefore$ negative definite.
c $a = 1$ which is > 0 and $\Delta = -15$ which is < 0
$\therefore$ positive definite so $x^2 - 3x + 6 > 0$ for all x.
d $a = -1$ which is < 0 and $\Delta = -8$ which is < 0
$\therefore$ negative definite so $4x - x^2 - 6 < 0$ for all x.

5

Constant	a	b	c	d	e	f	Δ_1	Δ_2
Sign	+	−	+	−	+	0	−	+

6 **a** **i** $k < \frac{9}{4}$ **ii** $k = \frac{9}{4}$ **iii** $k > \frac{9}{4}$
b **i** $k < 4$ **ii** $k = 4$ **iii** $k > 4$
c **i** $k > -\frac{4}{3}$ **ii** $k = -\frac{4}{3}$ **iii** $k < -\frac{4}{3}$

7 $a = 3$ which is > 0 and $\Delta = k^2 + 12$ which is always > 0
$\{$as $k^2 \geqslant 0$ for all $k\}$ $\therefore$ cannot be positive definite.

8 $k = -2$, the graph touches the x-axis in this case.

EXERCISE 14D

1 **a** $y = 2(x-1)(x-2)$ **b** $y = 3(x-2)^2$
c $y = (x-1)(x-3)$ **d** $y = -(x-3)(x+1)$
e $y = -3(x-1)^2$ **f** $y = -2(x+2)(x-3)$

2 **a** $y = \frac{3}{2}(x-2)(x-4)$ **b** $y = -\frac{1}{2}(x+4)(x-2)$
c $y = -\frac{4}{3}(x+3)^2$

3 **a** $y = 3x^2 - 18x + 15$ **b** $y = -4x^2 + 6x + 4$
c $y = -x^2 + 6x - 9$ **d** $y = 4x^2 + 16x + 16$
e $y = \frac{3}{2}x^2 - 6x + \frac{9}{2}$ **f** $y = -\frac{1}{3}x^2 + \frac{2}{3}x + 5$

4 **a** $y = -(x-2)^2 + 4$ **b** $y = 2(x-2)^2 - 1$
c $y = \frac{1}{3}(x+3)^2 - 4$ **d** $y = -2(x-3)^2 + 8$
e $y = \frac{2}{3}(x-4)^2 - 6$ **f** $y = -\frac{5}{9}(x+2)^2 + 5$
g $y = -2(x-2)^2 + 3$ **h** $y = \frac{3}{2}(x+4)^2 + 3$
i $y = 2(x - \frac{1}{2})^2 - \frac{3}{2}$

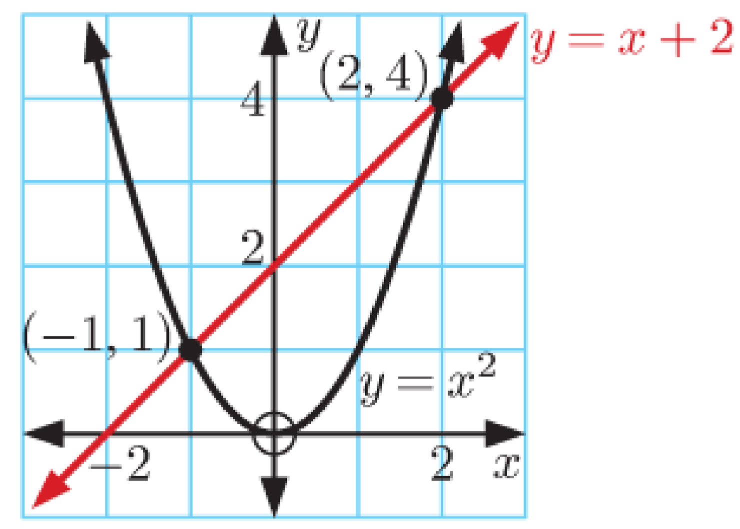
5 $y = 3$

EXERCISE 14E

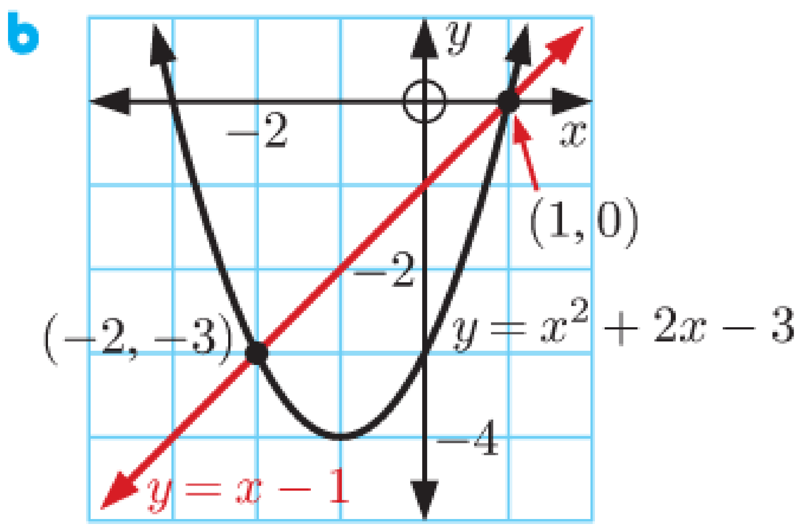
1 **a** $(1, 7)$ and $(2, 8)$ **b** $(4, 5)$ and $(-3, -9)$
c $(3, 0)$ (touching) **d** graphs do not meet

2 **a** $(0.586, 5.59)$ and $(3.41, 8.41)$
b $(3, -4)$ (touching) **c** graphs do not meet
d $(-2.56, -18.8)$ and $(1.56, 1.81)$

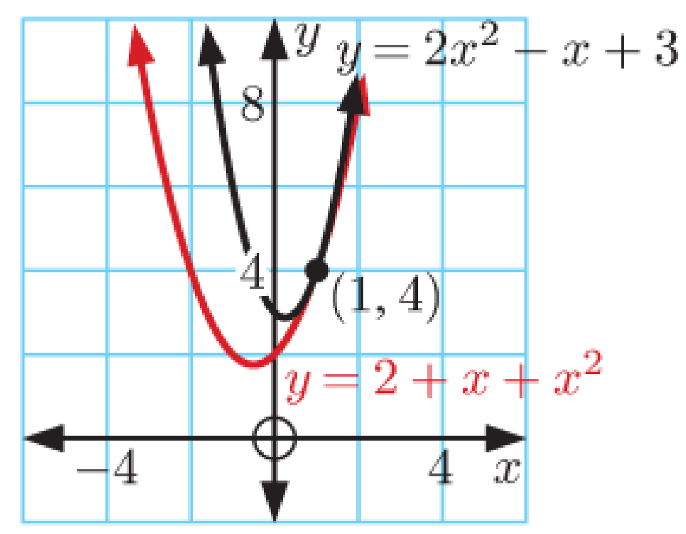
3 **a** $(-1, 1)$ and $(2, 4)$
c $x < -1$ or $x > 2$
b



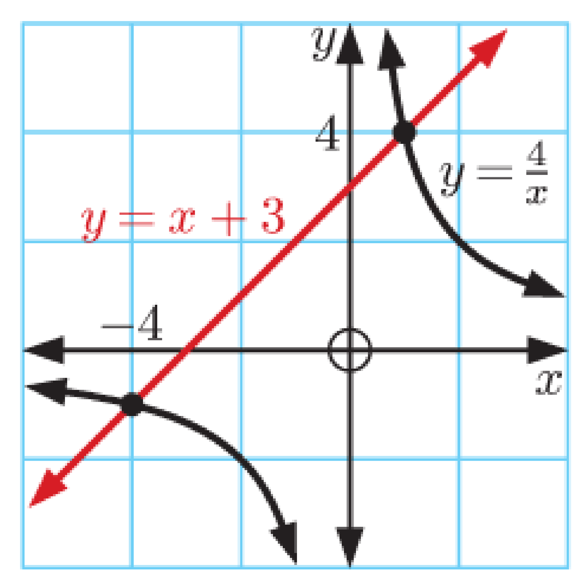
4 **a** $(-2, -3)$ and $(1, 0)$
c $x < -2$ or $x > 1$
b



5 **a** $(1, 4)$
c $x \in \mathbb{R}$, $x \neq 1$
b

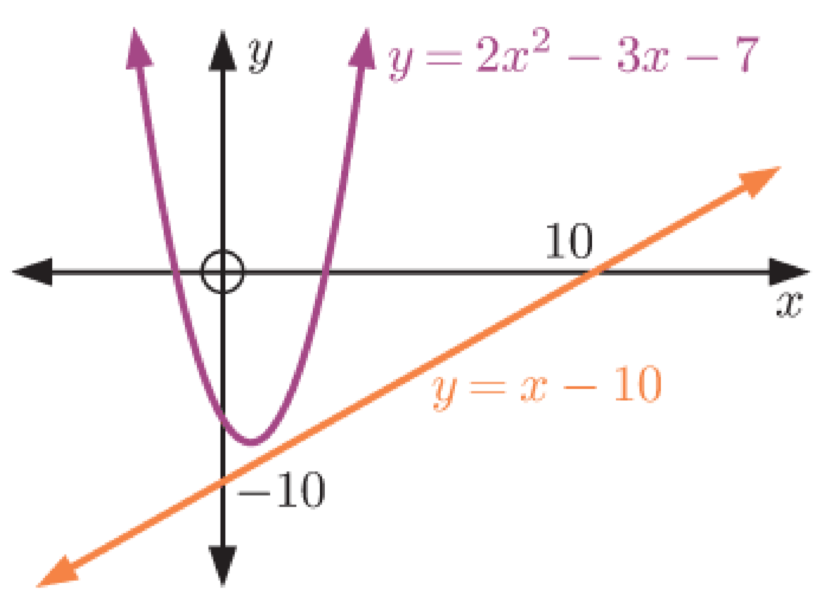


6 **a** $x = -4$ or 1
c $x < -4$ or
$0 < x < 1$
b



7 $c = -9$ **8** $m = 0$ or -8 **9** -1 or 11

10 **a** $c < -9$
b example: $c = -10$



12 **a** $c > -2$ **b** $c = -2$ **c** $c < -2$

13 **Hint:** A straight line through $(0, 3)$ will have an equation of the form $y = mx + 3$.

14 $b = 8$, $c = -14$ **15** **a** $c = a^2$, $m \in \mathbb{R}$ **b** $m = 2a$

EXERCISE 14F

1 7 and −5 or −7 and 5 **2** 5 or $\frac{1}{5}$ **3** 14
4 18 and 20 or −18 and −20
5 15 and 17 or −15 and −17 **6** 15 sides **7** ≈ 3.48 cm
8 **b** 6 cm by 6 cm by 7 cm **9** ≈ 11.2 cm square
10 no **12** ≈ 61.8 km h^{-1} **13** 32 elderly citizens

14 **a** $y = -\frac{8}{9}x^2 + 8$
b No, as the tunnel is only 4.44 m high when it is the same width as the truck.

15 **a** $h = -5(t-2)^2 + 80$ **b** 75 m **c** 6 seconds

EXERCISE 14G

1 **a** min. -1, when $x = 1$ **b** max. 8, when $x = -1$
c max. $8\frac{1}{3}$, when $x = \frac{1}{3}$ **d** min. $-1\frac{1}{8}$, when $x = -\frac{1}{4}$
e min. $4\frac{15}{16}$, when $x = \frac{1}{8}$ **f** max. $6\frac{1}{8}$, when $x = \frac{7}{4}$

2 **a** 40 refrigerators **b** €4000

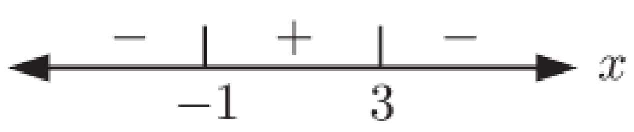
4 500 m by 250 m

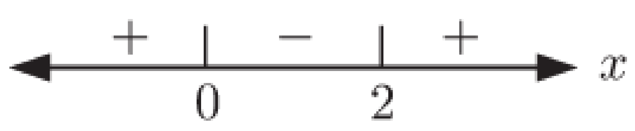
5 **a** $41\frac{2}{3}$ m by $41\frac{2}{3}$ m **b** 50 m by $31\frac{1}{4}$ m

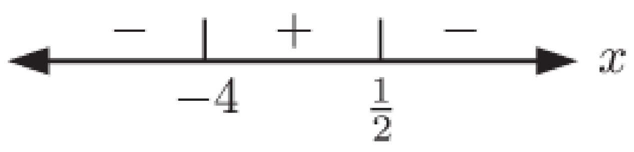
6 **b** $3\frac{1}{8}$ units **7** **a** $y = 6 - \frac{3}{4}x$ **b** 3 cm by 4 cm

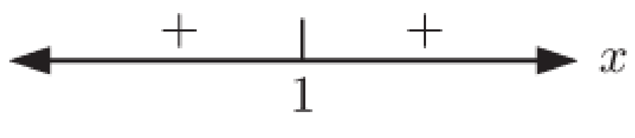
8 $m = \dfrac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} {a_i}^2}$ **9** $y = x^4 - 2(a^2+b^2)x^2 + (a^2-b^2)^2$
least value $= -4a^2b^2$

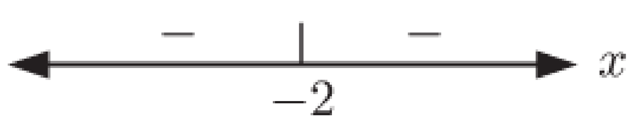
EXERCISE 14H.1

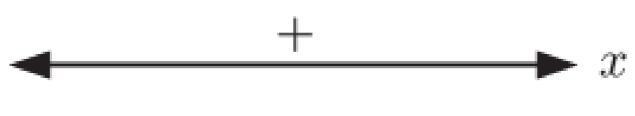
1 **a** 

b 

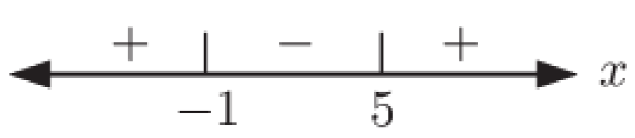
c 

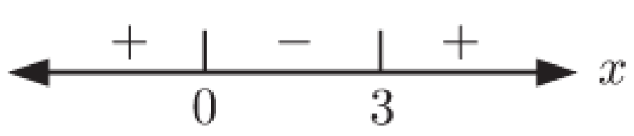
d 

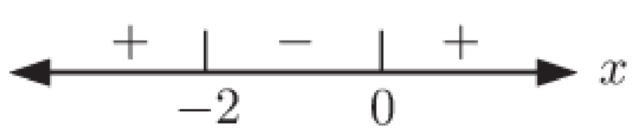
e 

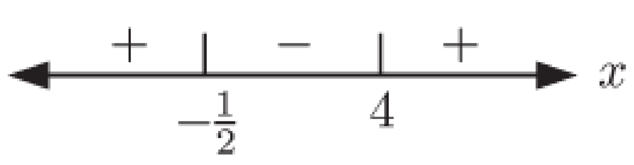
f 

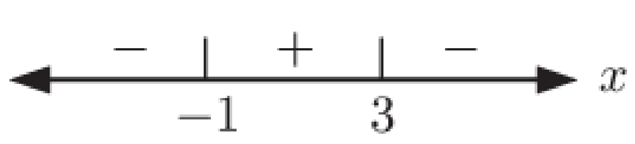
2 **a** 

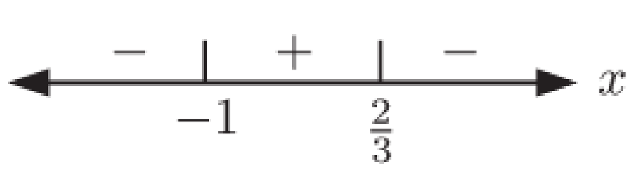
b 

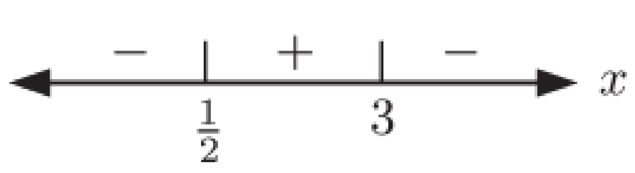
c 

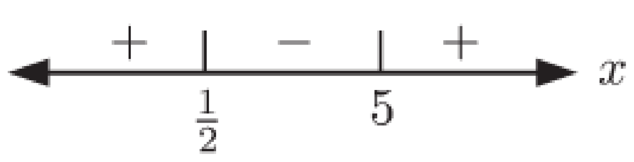
d 

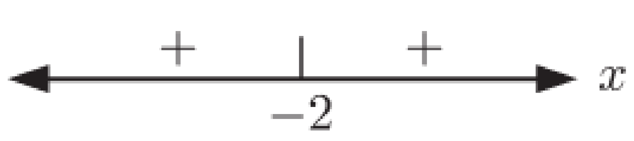
e 

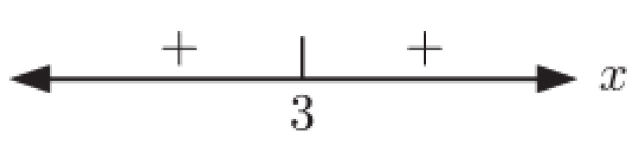
f 

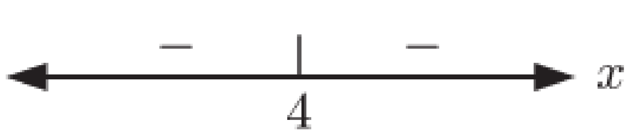
g 

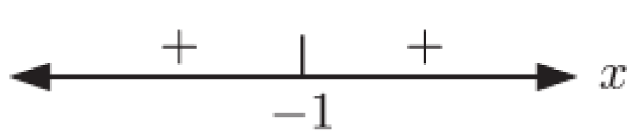
h 

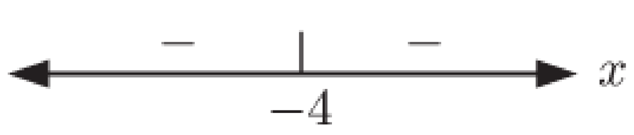
i 

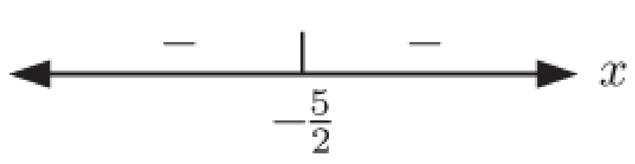
3 **a** 

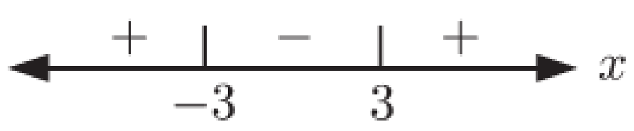
b 

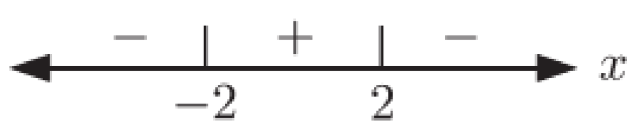
c 

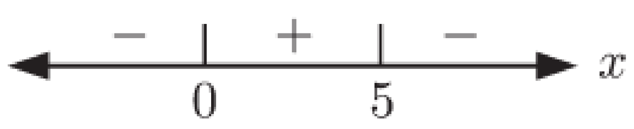
d 

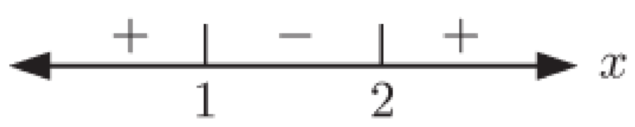
e 

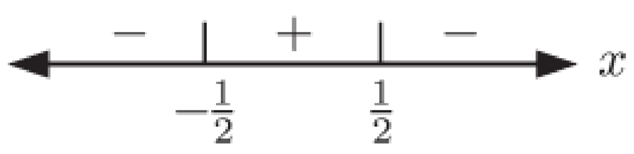
f 

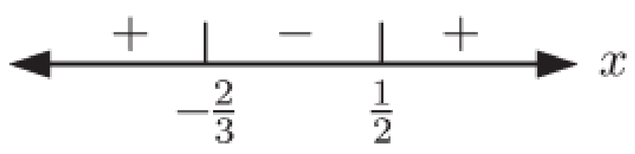
4 **a** 

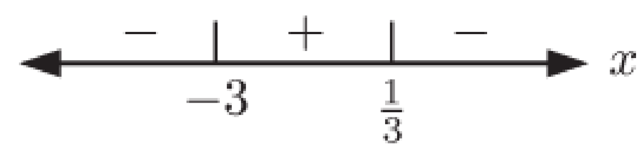
b 

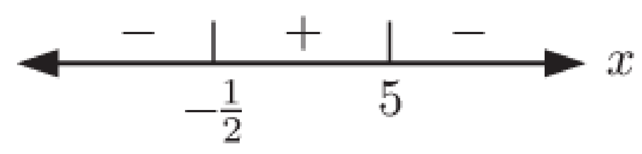
c 

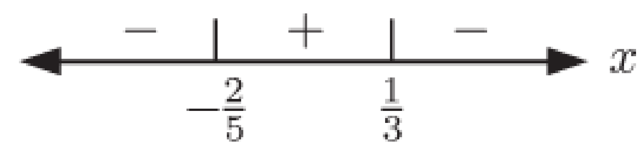
d 

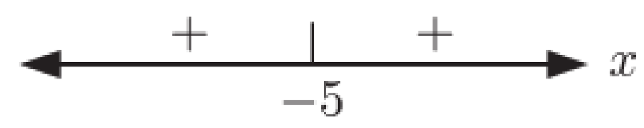
e 

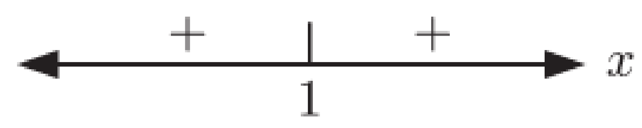
f 

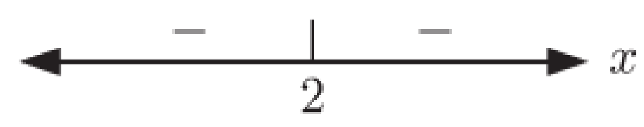
g 

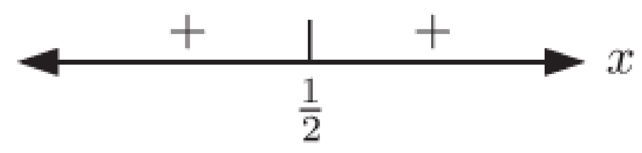
h 

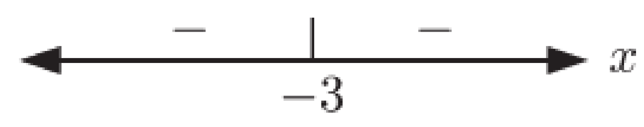
i 

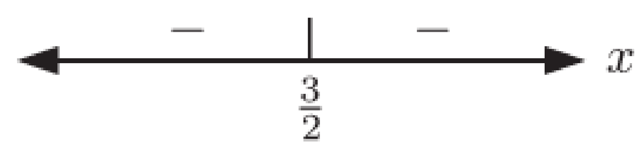
5 **a** 

b 

c 

d 

e 

f 

EXERCISE 14H.2

1 **a** $-5 \leqslant x \leqslant 2$ **b** $-3 \leqslant x \leqslant 2$ **c** no solutions
d all $x \in \mathbb{R}$ **e** $-\frac{1}{2} < x < 3$ **f** $-\frac{3}{2} < x < 4$

2 **a** $x \leqslant 0$ or $x \geqslant 1$ **b** $-\frac{2}{3} < x < 0$ **c** $x \neq -2$
d $-5 \leqslant x \leqslant 3$ **e** $x < -2$ or $x > 6$ **f** $-4 < x < 1$

3 **a** $x \leqslant 0$ or $x \geqslant 3$ **b** $-2 < x < 2$
c $x \leqslant -\sqrt{2}$ or $x \geqslant \sqrt{2}$ **d** $-3 \leqslant x \leqslant 7$
e $x < 5$ or $x > 6$ **f** $x < -6$ or $x > 7$
g $x \leqslant -1$ or $x \geqslant \frac{3}{2}$ **h** no solutions
i $-\frac{3}{2} < x < \frac{1}{3}$ **j** $x < -\frac{4}{3}$ or $x > 4$
k $x \neq 1$ **l** $\frac{1}{3} \leqslant x \leqslant \frac{1}{2}$ **m** $x < -\frac{1}{6}$ or $x > 1$
n $x \leqslant -\frac{1}{4}$ or $x \geqslant \frac{2}{3}$ **o** $x < \frac{3}{2}$ or $x > 3$

4 **a** **i** $k < -8$ or $k > 0$ **ii** $k = -8$ or 0
iii $-8 < k < 0$
b **i** $-1 < k < 1,\ k \neq 0$ **ii** $k = -1$ or 1
iii $k < -1$ or $k > 1$
c **i** $k < -6$ or $k > 2$ **ii** $k = -6$ or $k = 2$
iii $-6 < k < 2$

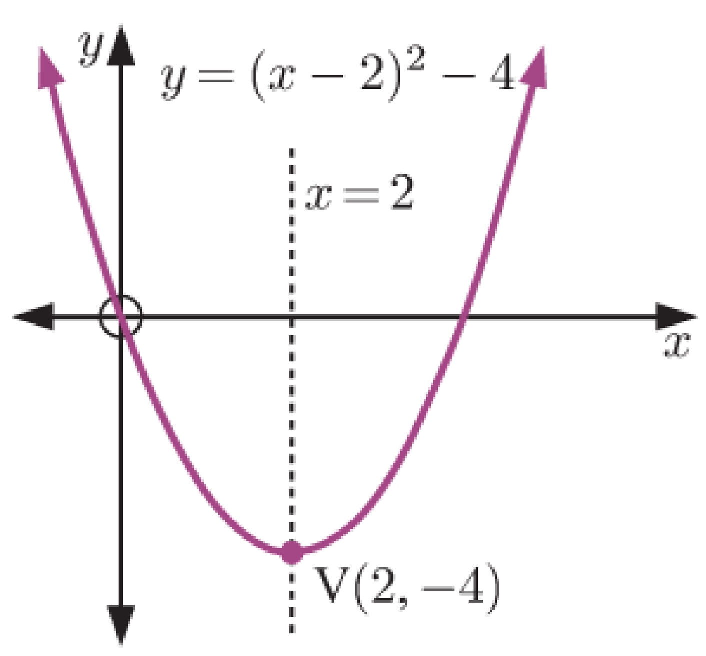
5 **a** **i** $k < -2$ or $k > 6$ **ii** $k = -2$ or $k = 6$
iii $-2 < k < 6$
b **i** $k < -\frac{13}{9}$ or $k > 3$ **ii** $k = -\frac{13}{9}$ or $k = 3$
iii $-\frac{13}{9} < k < 3$
c **i** $-\frac{4}{3} < k < 0,\ k \neq -1$ **ii** $k = -\frac{4}{3}$ or $k = 0$
iii $k < -\frac{4}{3}$ or $k > 0$

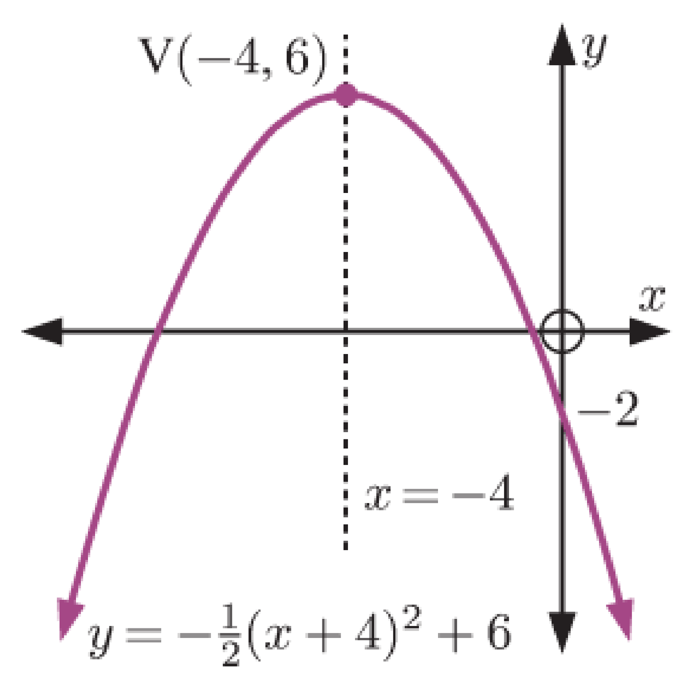
6 **a** $m > 3$ **b** $m < -1$

7 **a** $m < -1$ or $m > 7$ **b** $m = -1$ or $m = 7$
c $-1 < m < 7$

8 **a** $a < 6 - 2\sqrt{10}$ or $a > 6 + 2\sqrt{10}$ **b** $a = 6 \pm 2\sqrt{10}$
c $6 - 2\sqrt{10} < a < 6 + 2\sqrt{10}$

REVIEW SET 14A

1 **a** 

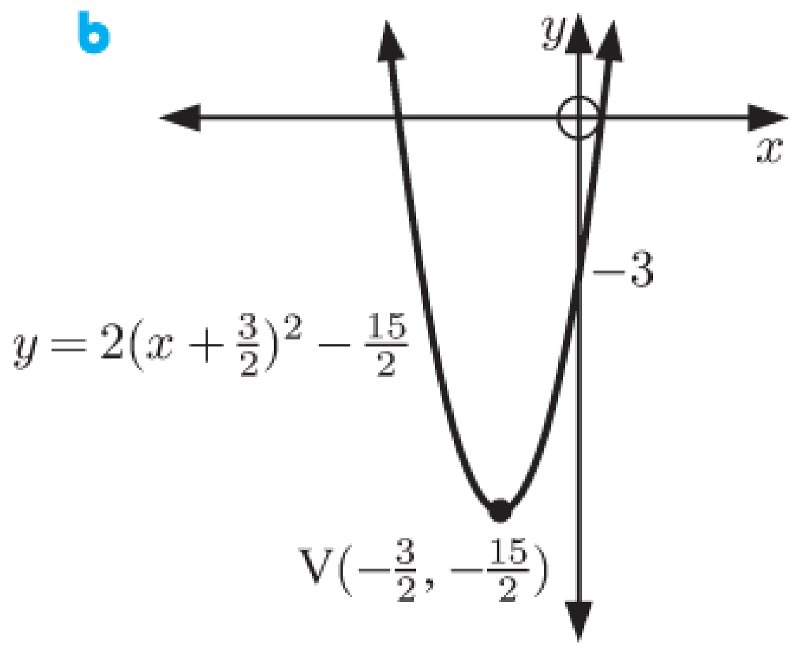
b 

2 (4, 4) and (−3, 18) **3** $k < -3\frac{1}{8}$

4 **a** $m = \frac{9}{8}$ **b** $m < \frac{9}{8}$ **c** $m > \frac{9}{8}$ **5** $\frac{6}{5}$ or $\frac{5}{6}$

6 **Hint:** Let the line have equation $y = mx + 10$.

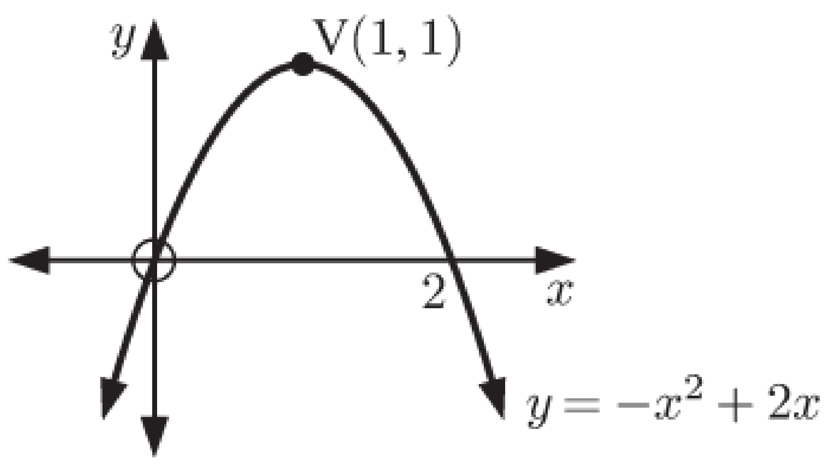
7 **a** $y = 2(x + \frac{3}{2})^2 - \frac{15}{2}$ **b**



8 **a** $y = \frac{20}{9}(x - 2)^2 - 20$ **b** $y = -\frac{2}{7}(x - 1)(x - 7)$

c $y = \frac{2}{9}(x + 3)^2$

9



10 $\frac{1}{2}$

11 **a** **i** $\Delta > 0$ **ii** $a < 0$

b **i** $A(-m, 0)$, $B(-n, 0)$ **ii** $x = \dfrac{-m - n}{2}$

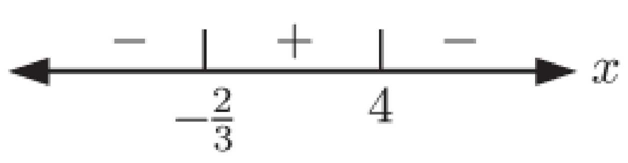
13 $y = -4x^2 + 4x + 24$ **14** $k = \frac{3}{2}$

15 **a** $c = 8$ **b** $3a + b = -3$, $a - b = -5$

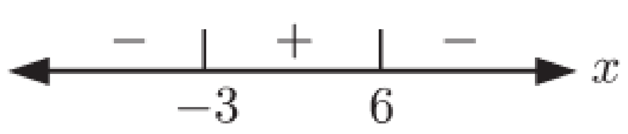
c $a = -2$, $b = 3$, $y = -2x^2 + 3x + 8$

16 $m = -5$ or 19 **17** 21 m

18 **a**



b



19 **a** $x < -2$ or $x > 3$ **b** $-1 \leqslant x \leqslant 5$

c $x < -\frac{5}{2}$ or $x > 2$

20 **a** $k < 6 - 2\sqrt{5}$ or $k > 6 + 2\sqrt{5}$ **b** $k = 6 \pm 2\sqrt{5}$

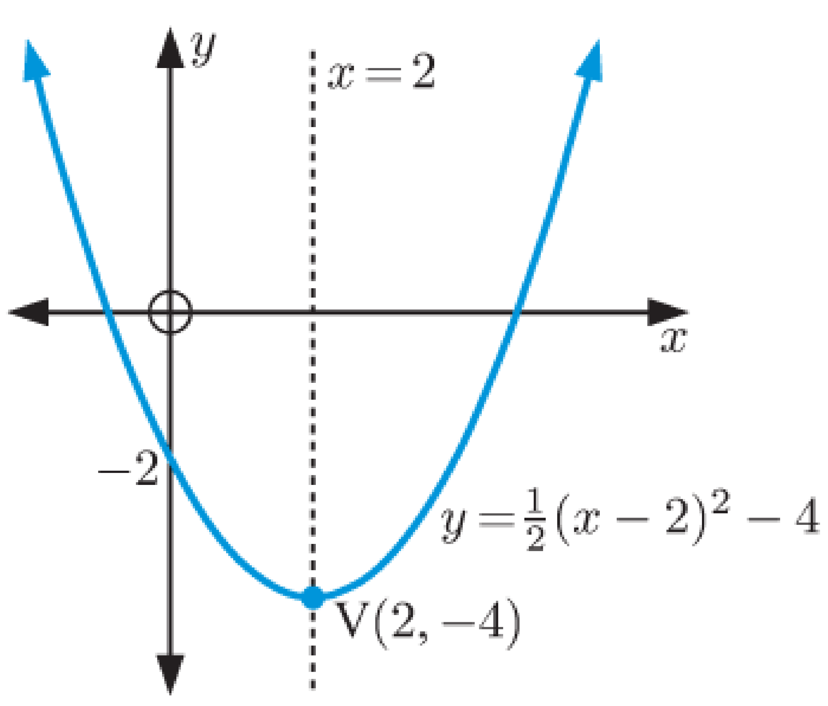
c $6 - 2\sqrt{5} < k < 6 + 2\sqrt{5}$

REVIEW SET 14B

1 **a** $x = 2$ **d**

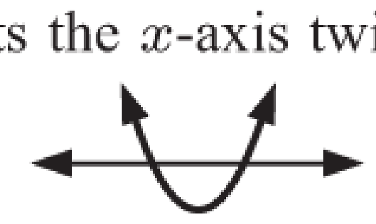
b (2, −4)

c −2

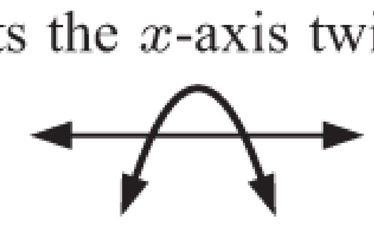


2 $x = \frac{4}{3}$, $V(1\frac{1}{3}, 12\frac{1}{3})$

3 **a** $\Delta = 65$, the graph cuts the x-axis twice



b $\Delta = 97$, the graph cuts the x-axis twice



4 $y = -6(x - 2)^2 + 25$

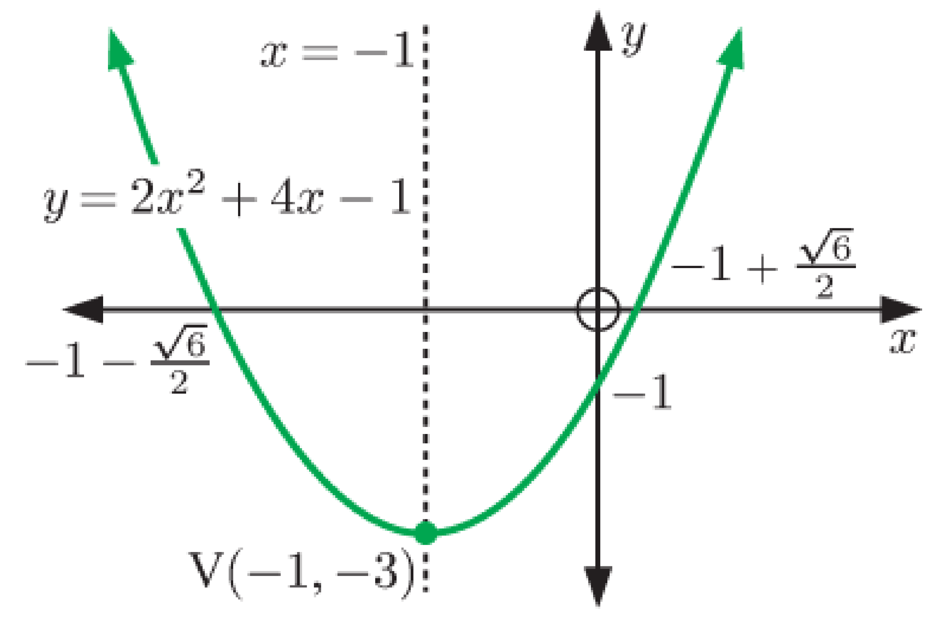
5 **a** $y = -\frac{2}{5}(x + 5)(x - 1)$ **b** $(-2, 3\frac{3}{5})$, $x = -2$

6 **a** $x = -1$ **d**

b (−1, −3)

c x-int. $-1 \pm \frac{\sqrt{6}}{2}$

y-intercept −1



7 **a** $y = 2x^2 - 12x + 18$ **b** $y = -\frac{1}{2}x^2 + \frac{1}{2}x + 3$

c $y = x^2 + 7x - 3$ **d** $y = -2x^2 + 12x - 3$

8 **a** $c > -6$

b For example, when $c = -2$, points of intersection are (−1, −5) and (3, 7).

9 **a** minimum is $5\frac{2}{3}$ when $x = -\frac{2}{3}$

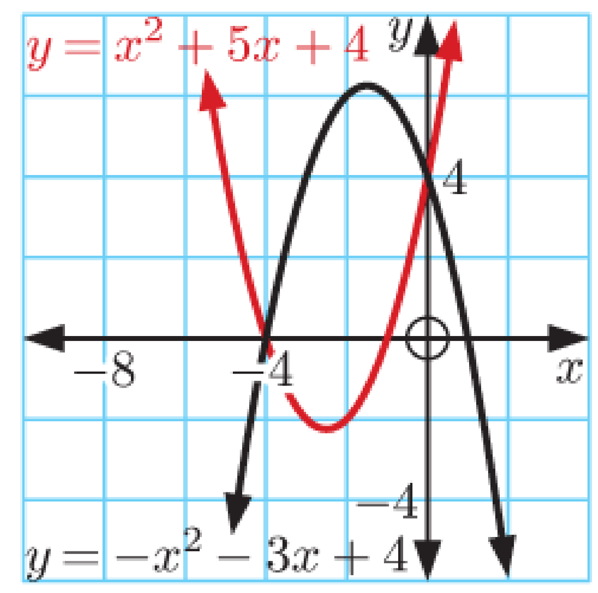
b maximum is $5\frac{1}{8}$ when $x = -\frac{5}{4}$

10 **a** $y = 3x^2 - 3x - 18$ **b** −18 **c** $(\frac{1}{2}, -18\frac{3}{4})$

11 **a** $m = -2$, $n = 4$ **b** $k = 7$ **12** ≈ 13.5 cm square

13 **a** $x = -4$ or 0 **b**

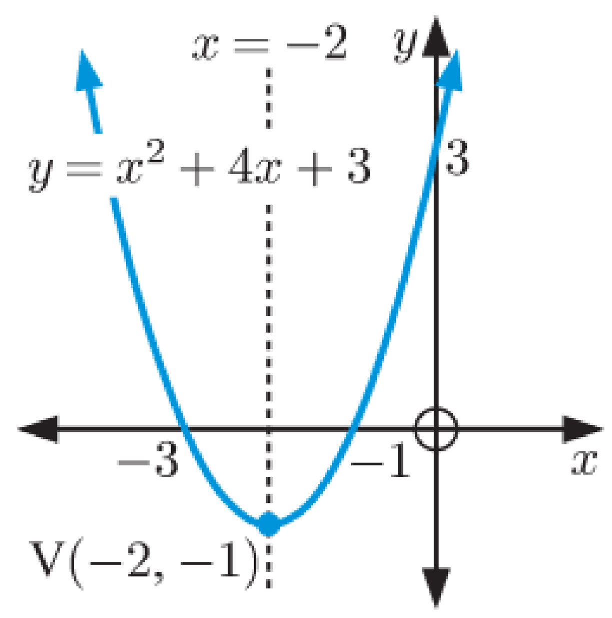
c $x < -4$ or $x > 0$



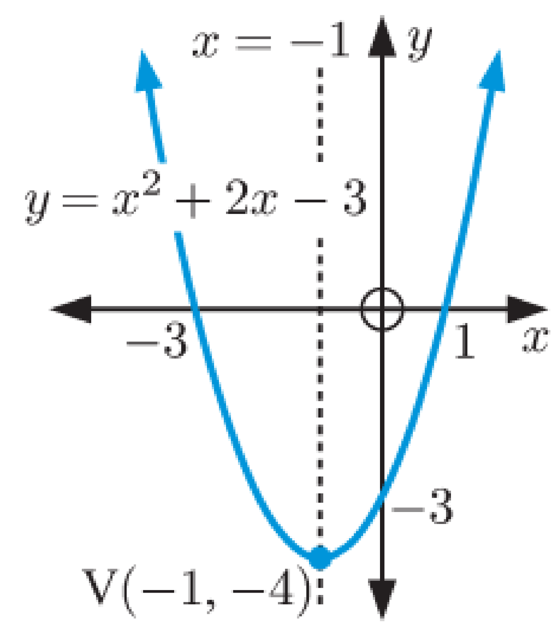
14 **a** **i** $y = (x + 2)^2 - 1$ **b** **i** $y = (x + 1)^2 - 4$

ii $y = (x + 3)(x + 1)$ **ii** $y = (x + 3)(x - 1)$

iii



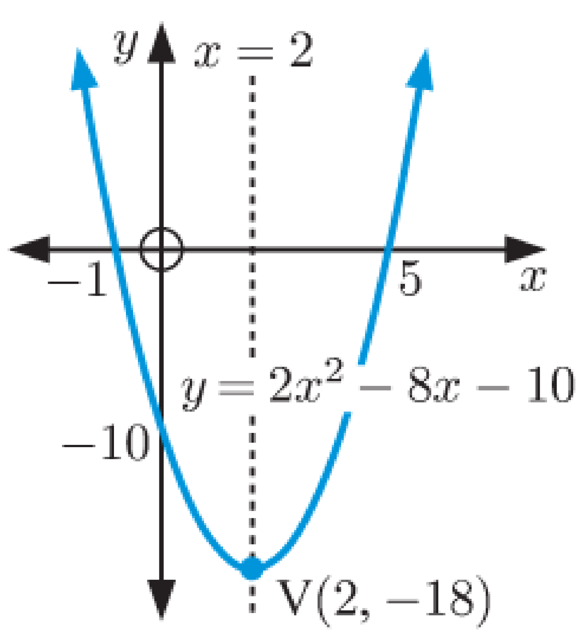
iii



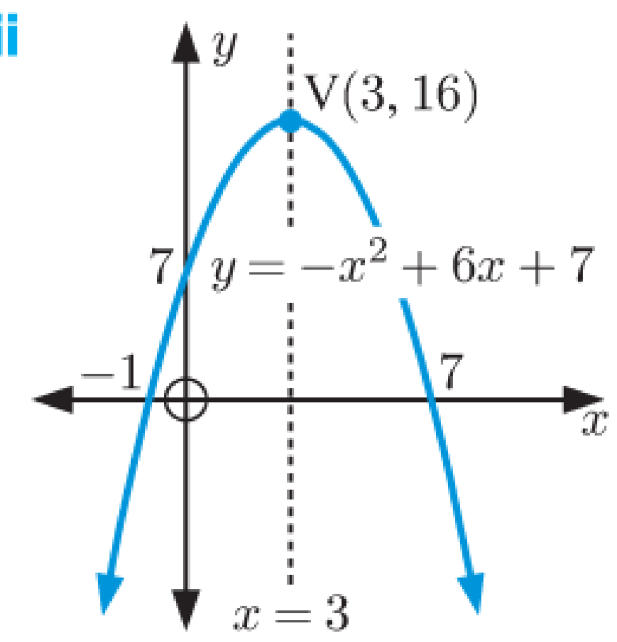
c **i** $y = 2(x - 2)^2 - 18$ **d** **i** $y = -(x - 3)^2 + 16$

ii $y = 2(x - 5)(x + 1)$ **ii** $y = -(x - 7)(x + 1)$

iii



iii

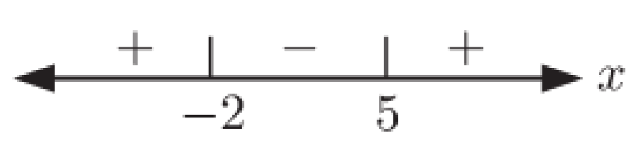


15 **a** $k = \pm 12$ **b** (0, 4)

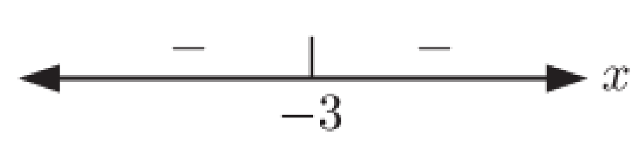
16 **b** $37\frac{1}{2}$ m by $33\frac{1}{3}$ m **c** 1250 m^2

17 **b** \$60, revenue is \$2400 per day

18 **a**



b



19 **a** $0 < x < \frac{3}{4}$ **b** $x \leqslant -1$ or $x \geqslant \frac{5}{2}$

c $x \leqslant \frac{1}{3}$ or $x \geqslant \frac{3}{2}$

20 **a** $-\frac{25}{2} < m < \frac{1}{2},\ m \neq 0$ **b** $m = -\frac{25}{2}$ or $m = \frac{1}{2}$
c $m < -\frac{25}{2}$ or $m > \frac{1}{2}$

EXERCISE 15A

1 **a** Is a function, since for every value of x there is only one corresponding value of y.
b Is not a function. When $x = 2,\ y = 1$ or 0.

2 **a** Is a function, since for any value of x there is at most one value of y.
b Is a function, since for any value of x there is at most one value of y.
c Is not a function. If $x^2 + y^2 = 9$, then $y = \pm\sqrt{9 - x^2}$. So, for example, for $x = 2,\ y = \pm\sqrt{5}$.

3 **a** function **b** not a function **c** function
d not a function

4 Not a function as a 2 year old child could pay \$0 or \$20.

5 No, because a vertical line (the y-axis) would cut the relation more than once.

6 No. A vertical line is not a function. It will not pass the "vertical line" test.

7 **a** $y^2 = x$ is a relation but not a function.
$y = x^2$ is a function (and a relation).
$y^2 = x$ has a horizontal axis of symmetry (the x-axis).
$y = x^2$ has a vertical axis of symmetry (the y-axis).
Both $y^2 = x$ and $y = x^2$ have vertex $(0, 0)$.
$y^2 = x$ is a rotation of $y = x^2$ clockwise through 90° about the origin *or* $y^2 = x$ is a reflection of $y = x^2$ in the line $y = x$.
b **i** The part of $y^2 = x$ in the first quadrant.
ii $y = \sqrt{x}$ is a function as any vertical line cuts the graph at most once.

8 **a** Both curves are functions since any vertical line will cut each curve at most once.
b $y = \sqrt[3]{x}$

EXERCISE 15B

1 **a** 2 **b** 2 **c** -16 **d** -68 **e** $\frac{17}{4}$

2 **a** -3 **b** 3 **c** 3 **d** -3 **e** $\frac{15}{2}$

3 **a** **i** $-\frac{7}{2}$ **ii** $-\frac{3}{4}$ **iii** $-\frac{4}{9}$ **b** $x = 4$ **c** $x = \frac{9}{5}$

4 **a** $7 - 3a$ **b** $7 + 3a$ **c** $-3a - 2$ **d** $7 - 6a$
e $1 - 3x$ **f** $7 - 3x - 3h$

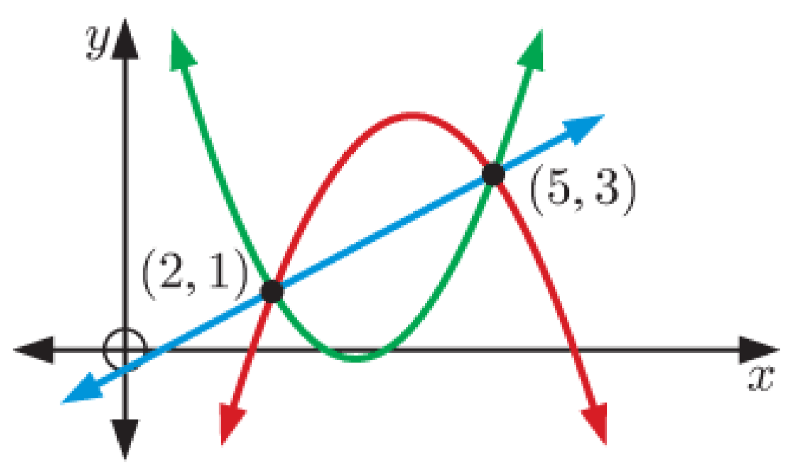
5 **a** $2x^2 + 19x + 43$ **b** $2x^2 - 11x + 13$
c $2x^2 - 3x - 1$ **d** $2x^4 + 3x^2 - 1$
e $18x^2 + 9x - 1$ **f** $2x^2 + (4h + 3)x + 2h^2 + 3h - 1$

6 **a** $9x^2$ **b** $\dfrac{x^2}{4}$ **c** $3x^2$ **d** $2x^2 - 4x + 7$

7 **a** $-\dfrac{1}{x}$ **b** $\dfrac{2}{x}$ **c** $\dfrac{2 + 3x}{x}$ **d** $\dfrac{2x + 1}{x - 1}$

8 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x.

9 **Note:** Other answers are possible.



10 $f(x) = -2x + 5$

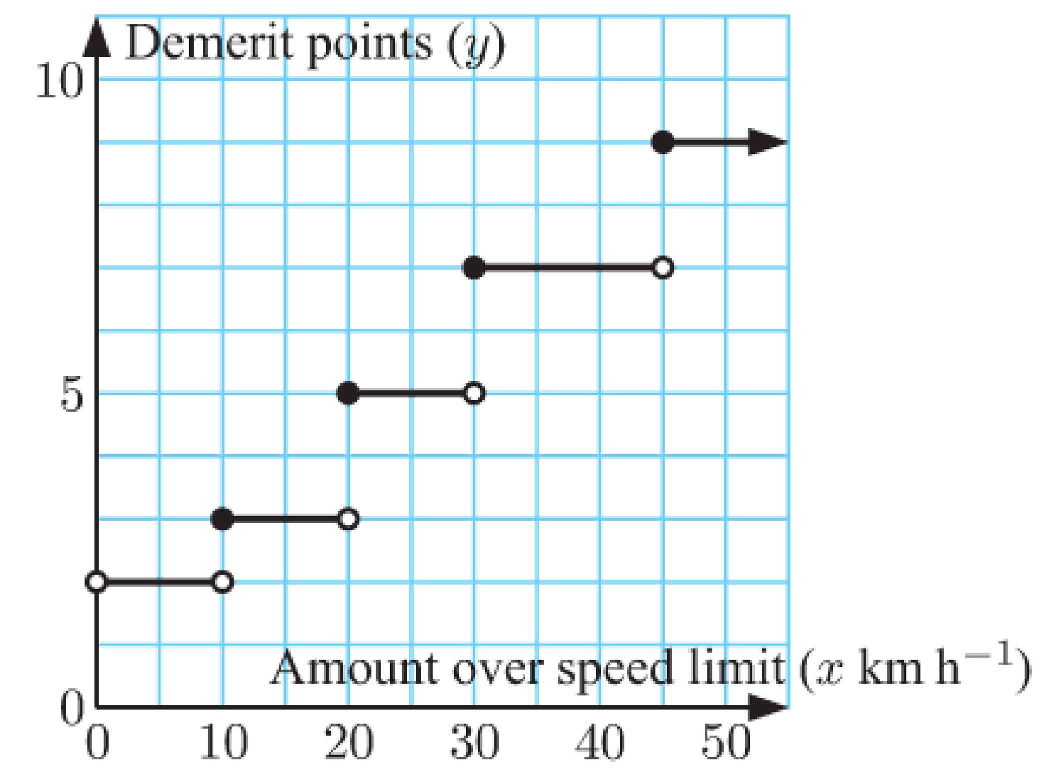
11 **a** $H(30) = 800$. After 30 minutes the balloon is 800 m high.
b $t = 20$ or 70. After 20 minutes and after 70 minutes the balloon is 600 m high.
c $0 \leqslant t \leqslant 80$ **d** 0 m to 900 m

12 $a = 3,\ b = -2$ **13** $a = 3,\ b = -1,\ c = -4$

14 **a** $V(4) = 5400$; $V(4)$ is the value of the photocopier in pounds after 4 years.
b $t = 6$. After 6 years the value of the photocopier is £3600.
c £9000 **d** $0 \leqslant t \leqslant 10$

EXERCISE 15C

1 **a**



b Yes, since for every value of x, there is at most one value of y.
c Domain is $\{x \mid x > 0\}$, Range is $\{2, 3, 5, 7, 9\}$

2 **a** At any moment in time there can be only one temperature, so the graph is a function.
b Domain is $\{t \mid 0 \leqslant t \leqslant 30\}$, Range is $\{T \mid 15 \leqslant T \leqslant 25\}$

3 **a** Domain is $\{x \mid -1 < x \leqslant 5\}$, Range is $\{y \mid 1 < y \leqslant 3\}$
b Domain is $\{x \mid x \neq 2\}$, Range is $\{y \mid y \neq -1\}$
c Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid 0 < y \leqslant 2\}$
d Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \leqslant \frac{25}{4}\}$
e Domain is $\{x \mid x \geqslant -4\}$, Range is $\{y \mid y \geqslant -3\}$
f Domain is $\{x \mid x \neq \pm 2\}$,
Range is $\{y \mid y \leqslant -1$ or $y > 0\}$

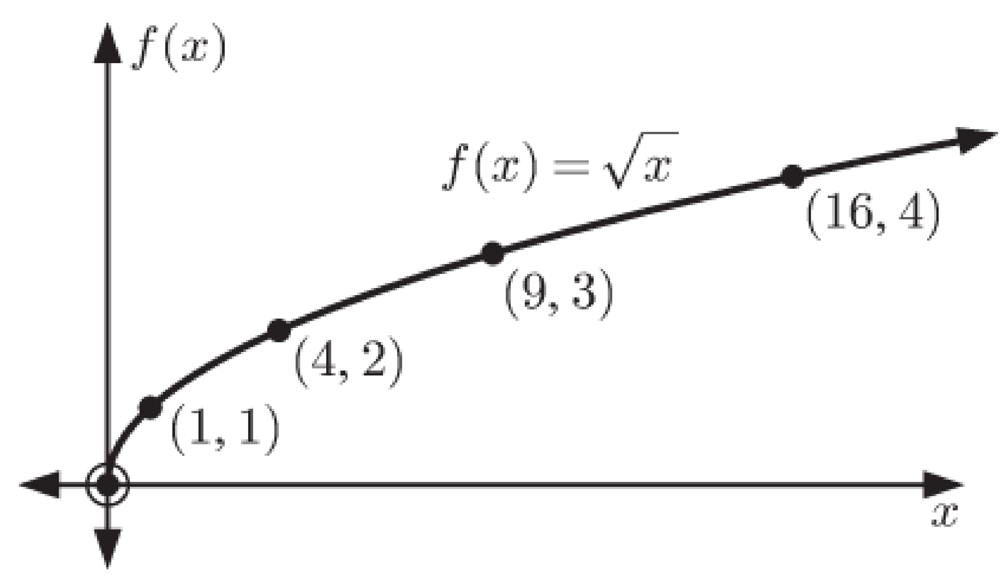
4 **a** true **b** false **c** true **d** true

5 **a** $\{y \mid y \geqslant 0\}$ **b** $\{y \mid y \leqslant 0\}$ **c** $\{y \mid y \geqslant 2\}$
d $\{y \mid y \leqslant 0\}$ **e** $\{y \mid y \leqslant 1\}$ **f** $\{y \mid y \geqslant 3\}$
g $\{y \mid y \geqslant -\frac{9}{4}\}$ **h** $\{y \mid y \leqslant 9\}$ **i** $\{y \mid y \leqslant \frac{25}{12}\}$

6 **a** $\{x \mid x \geqslant 0\}$ **b**

x	0	1	4	9	16
$f(x)$	0	1	2	3	4

c



d $\{y \mid y \geqslant 0\}$

7 **a** Domain is $\{x \mid x \geqslant -6\}$, Range is $\{y \mid y \geqslant 0\}$
b Domain is $\{x \mid x \neq 0\}$, Range is $\{y \mid y > 0\}$
c Domain is $\{x \mid x \neq -1\}$, Range is $\{y \mid y \neq 0\}$
d Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y < 0\}$
e Domain is $\{x \mid x \neq 3\}$, Range is $\{y \mid y \neq 0\}$
f Domain is $\{x \mid x \leqslant 4\}$, Range is $\{y \mid y \geqslant 0\}$