Paper 3 (21.10) [27 marks]

1. [Maximum mark: 27]

This question asks you to investigate conditions for the existence of complex roots of polynomial equations of degree 3 and 4.

The cubic equation $x^3 + px^2 + qx + r = 0$, where $p,\ q,\ r\ \in\ \mathbb{R}$, has roots $\alpha,\ \beta$ and γ.

(a) By expanding $(x - \alpha)(x - \beta)(x - \gamma)$ show that:

$p = -(\alpha + \beta + \gamma)$

$q = \alpha\beta + \beta\gamma + \gamma\alpha$

$r = -\alpha\beta\gamma.$ [3]

Markscheme

attempt to expand $(x - \alpha)(x - \beta)(x - \gamma)$ ***M1***

$= \left(x^2 - (\alpha + \beta)x + \alpha\beta\right)(x - \gamma)$ **OR**
$= (x - \alpha)\left(x^2 - (\beta + \gamma)x + \beta\gamma\right)$ ***A1***

$\left(x^3 + px^2 + qx + r\right) = x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$ ***A1***

comparing coefficients:

$p = -(\alpha + \beta + \gamma)$ ***AG***

$q = (\alpha\beta + \beta\gamma + \gamma\alpha)$ ***AG***

$r = -\alpha\beta\gamma$ ***AG***

Note: For candidates who do not include the ***AG*** lines award full marks.

[3 marks]

(b.i) Show that $p^2 - 2q = \alpha^2 + \beta^2 + \gamma^2$. [3]

Markscheme

$p^2 - 2q = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ ***(A1)***

attempt to expand $(\alpha + \beta + \gamma)^2$ ***(M1)***

$= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ or equivalent ***A1***

$= \alpha^2 + \beta^2 + \gamma^2$ ***AG***

Note: Accept equivalent working from RHS to LHS.

[3 marks]

(b.ii) Hence show that $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = 2p^2 - 6q$. [3]

Markscheme

EITHER

attempt to expand $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2$ ***(M1)***

$= (\alpha^2 + \beta^2 - 2\alpha\beta) + (\beta^2 + \gamma^2 - 2\beta\gamma) + (\gamma^2 + \alpha^2 - 2\gamma\alpha)$ ***A1***

$= 2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= 2(p^2 - 2q) - 2q$ or equivalent ***A1***

$= 2p^2 - 6q$ ***AG***

OR

attempt to write $2p^2 - 6q$ in terms of $\alpha,\ \beta,\ \gamma$ ***(M1)***

$= 2\left(p^2 - 2q\right) - 2q$

$= 2\left(\alpha^2 + \beta^2 + \gamma^2\right) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ ***A1***

$= \left(\alpha^2 + \beta^2 - 2\alpha\beta\right) + \left(\beta^2 + \gamma^2 - 2\beta\gamma\right) + \left(\gamma^2 + \alpha^2 - 2\gamma\alpha\right)$ ***A1***

$= (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2$ ***AG***

Note: Accept equivalent working where LHS and RHS are expanded to identical expressions.

[3 marks]

(c) Given that $p^2 < 3q$, deduce that $\alpha,\ \beta$ and γ cannot all be real. [2]

Markscheme

$p^2 < 3q \Rightarrow 2p^2 - 6q < 0$

$\Rightarrow (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 < 0$ ***A1***

if all roots were real $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 \geq 0$ ***R1***

Note: Condone strict inequality in the ***R1*** line.
Note: Do not award ***A0R1***.

$\Rightarrow$roots cannot all be real ***AG***

[2 marks]

Consider the equation $x^3 - 7x^2 + qx + 1 = 0$, where $q \in \mathbb{R}$.

(d) Using the result from part (c), show that when $q = 17$, this equation has at least one complex root. [2]

Markscheme

$p^2 = (-7)^2 = 49$ and $3q = 51$ ***A1***

so $p^2 < 3q \Rightarrow$ the equation has at least one complex root ***R1***

Note: Allow equivalent comparisons; e.g. checking $p^2 < 6q$

[2 marks]

Noah believes that if $p^2 \geq 3q$ then $\alpha,\ \beta$ and γ are all real.

(e.i) By varying the value of q in the equation $x^3 - 7x^2 + qx + 1 = 0$, determine the smallest positive integer value of q required to show that Noah is incorrect. [2]

Markscheme

use of GDC (*eg* graphs or tables) ***(M1)***

$q = 12$ ***A1***

[2 marks]

(e.ii) Explain why the equation will have at least one real root for all values of q. [1]

Markscheme

complex roots appear in conjugate pairs (so if complex roots occur the other root will be real OR all 3 roots will be real).

OR

a cubic curve always crosses the x-axis at at least one point. ***R1***

[1 mark]

Now consider polynomial equations of degree 4.

The equation $x^4 + px^3 + qx^2 + rx + s = 0$, where $p,\ q,\ r,\ s \in \mathbb{R}$, has roots $\alpha,\ \beta,\ \gamma$ and δ.

In a similar way to the cubic equation, it can be shown that:

$p = -(\alpha + \beta + \gamma + \delta)$

$q = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

$r = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$

$s = \alpha\beta\gamma\delta.$

(f.i) Find an expression for $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ in terms of p and q. [3]

Markscheme

attempt to expand $(\alpha + \beta + \gamma + \delta)^2$ ***(M1)***

$(\alpha + \beta + \gamma + \delta)^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$ ***(A1)***

$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$(\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 =)p^2 - 2q$ ***A1***

[3 marks]

(f.ii) Hence state a condition in terms of p and q that would imply $x^4 + px^3 + qx^2 + rx + s = 0$ has at least one complex root. [1]

Markscheme

$p^2 < 2q$ **OR** $p^2 - 2q < 0$ ***A1***

Note: Allow ***FT*** on their result from part (f)(i).

[1 mark]

(g) Use your result from part (f)(ii) to show that the equation $x^4 - 2x^3 + 3x^2 - 4x + 5 = 0$ has at least one complex root. [1]

Markscheme

$4 < 6$ OR $2^2 - 2 \times 3 < 0$ ***R1***

hence there is at least one complex root. ***AG***

Note: Allow ***FT*** from part (f)(ii) for the ***R*** mark provided numerical reasoning is seen.

[1 mark]

The equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$, has one integer root.

(h.i) State what the result in part (f)(ii) tells us when considering this equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$. [1]

Markscheme

$(p^2 > 2q)$ $(81 > 2 \times 24)$ (so) nothing can be deduced ***R1***

Note: Do not allow ***FT*** for the ***R*** mark.

[1 mark]

(h.ii) Write down the integer root of this equation. [1]

Markscheme

-1 ***A1***

[1 mark]

(h.iii) By writing $x^4 - 9x^3 + 24x^2 + 22x - 12$ as a product of one linear and one cubic factor, prove that the equation has at least one complex root. [4]

Markscheme

attempt to express as a product of a linear and cubic factor ***M1***

$(x + 1)(x^3 - 10x^2 + 34x - 12)$ ***A1A1***

Note: Award ***A1*** for each factor. Award at most ***A1A0*** if not written as a product.

since for the cubic, $p^2 < 3q$ $(100 < 102)$ ***R1***

there is at least one complex root ***AG***

[4 marks]