Paper 3 (21.10) [27 marks]

1. [Maximum mark: 27]

This question asks you to investigate conditions for the existence of complex roots of polynomial equations of degree 3 and 4.

The cubic equation $x^3 + px^2 + qx + r = 0$, where $p,\ q,\ r \ \in\ \mathbb{R}$, has roots $\alpha,\ \beta$ and γ.

(a) By expanding $(x-\alpha)(x-\beta)(x-\gamma)$ show that:

$$p = -(\alpha + \beta + \gamma)$$

$$q = \alpha\beta + \beta\gamma + \gamma\alpha$$

$$r = -\alpha\beta\gamma.$$ [3]

(b.i) Show that $p^2 - 2q = \alpha^2 + \beta^2 + \gamma^2$. [3]

(b.ii) Hence show that $(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 = 2p^2 - 6q$. [3]

(c) Given that $p^2 < 3q$, deduce that $\alpha,\ \beta$ and γ cannot all be real. [2]

Consider the equation $x^3 - 7x^2 + qx + 1 = 0$, where $q \in \mathbb{R}$.

(d) Using the result from part (c), show that when $q = 17$, this equation has at least one complex root. [2]

Noah believes that if $p^2 \geq 3q$ then $\alpha,\ \beta$ and γ are all real.

(e.i) By varying the value of q in the equation $x^3 - 7x^2 + qx + 1 = 0$, determine the smallest positive integer value of q required to show that Noah is incorrect.

[2]

(e.ii) Explain why the equation will have at least one real root for all values of q. [1]

Now consider polynomial equations of degree 4.

The equation $x^4 + px^3 + qx^2 + rx + s = 0$, where $p,\ q,\ r,\ s \in \mathbb{R}$, has roots $\alpha,\ \beta,\ \gamma$ and δ.

In a similar way to the cubic equation, it can be shown that:

$$p = -(\alpha + \beta + \gamma + \delta)$$

$$q = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$$

$$r = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$$

$$s = \alpha\beta\gamma\delta.$$

(f.i) Find an expression for $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ in terms of p and q. [3]

(f.ii) Hence state a condition in terms of p and q that would imply $x^4 + px^3 + qx^2 + rx + s = 0$ has at least one complex root. [1]

(g) Use your result from part (f)(ii) to show that the equation $x^4 - 2x^3 + 3x^2 - 4x + 5 = 0$ has at least one complex root. [1]

The equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$, has one integer root.

(h.i) State what the result in part (f)(ii) tells us when considering this equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$. [1]

(h.ii) Write down the integer root of this equation. [1]

(h.iii) By writing $x^4 - 9x^3 + 24x^2 + 22x - 12$ as a product of one linear and one cubic factor, prove that the equation has at least one complex root. [4]