Revision questions [170 marks]

1. [Maximum mark: 6] 23M.1.AHL.TZ1.7

Consider $P(z) = 4m - mz + \frac{36}{m}z^2 - z^3$, where $z \in \mathbb{C}$ and $m \in \mathbb{R}^+$.

Given that $z - 3\mathrm{i}$ is a factor of $P(z)$, find the roots of $P(z) = 0$. [6]

Markscheme

METHOD 1

$3\mathrm{i}$ (is a root) ***A1***

(other complex root is) $-3\mathrm{i}$ ***A1***

Note: Award ***A1A1*** for $P(3\mathrm{i})$ and $P(-3\mathrm{i}) = 0$ seen in their working.

Award ***A1*** for each correct root seen in sum or product of their roots.

EITHER

attempt to find $P(3\mathrm{i}) = 0$ or $P(-3\mathrm{i}) = 0$ ***(M1)***

$4m - 3m\mathrm{i} + \frac{36}{m}(3\mathrm{i})^2 - (3\mathrm{i})^3 = 0$

$4m - 3m\mathrm{i} - \frac{36}{m}(-9) + 27\mathrm{i} = 0$

attempt to equate the real or imaginary parts ***(M1)***

$27 - 3m = 0$ OR $9 \times \frac{36}{m} = 4m$

OR

attempt to equate sum of three roots to $\frac{36}{m}$ ***(M1)***

Note: Accept sum of three roots set to $-\frac{36}{m}$.

Award ***M0*** for stating sum of roots is $\pm\frac{36}{m}$.

$3\mathrm{i} - 3\mathrm{i} + r = \frac{36}{m} \left(\Rightarrow r = \frac{36}{m}\right)$

substitute their r into product of roots ***(M1)***

$\left(3\mathrm{i}\right)\left(-3\mathrm{i}\right)\left(\frac{36}{m}\right) = 4m$ OR $\left(z^2+9\right)\left(\frac{36}{m} - z\right)$

$9 \times \frac{36}{m} = 4m$ OR $\frac{4m}{9} = \frac{36}{m}$

OR

attempt to equate product of three roots to $4m$ ***(M1)***

Note: Accept product of three roots set to $-4m$.

Award ***M0*** for stating product of roots is $\pm 4m$.

$\left(3\mathrm{i}\right)\left(-3\mathrm{i}\right) \times r = 4m \left(\Rightarrow r = \frac{4m}{9}\right)$

substitute their r into sum of roots ***(M1)***

$3\mathrm{i} - 3\mathrm{i} + \frac{4m}{9} = \frac{36}{m}$ OR $\left(z^2+9\right)\left(\frac{4m}{9} - z\right)$

$\frac{4m}{9} = \frac{36}{m}$

THEN

$m = 9$ ***(A1)***

third root is 4 ***A1***

METHOD 2

$3\mathrm{i}$ (is a root) ***A1***

(other complex root is) $-3\mathrm{i}$ ***A1***

recognition that the other factor is $(z+3\mathrm{i})$ and attempt to write $P(z)$ as product of three linear factors or as product of a quadratic and a linear factor ***(M1)***

$P(z)=(z-3\mathrm{i})(z+3\mathrm{i})(r-z)$ OR
$\left(z-3\mathrm{i}\right)\left(z+3\mathrm{i}\right)=z^2+9\Rightarrow P\left(z\right)=\left(z^2+9\right)\left(\frac{4m}{9}-z\right)$

Note: Accept any attempt at long division of $P(z)$ by z^2+9.

Award ***M0*** for stating other factor is $(z+3\mathrm{i})$ or obtaining z^2+9 with no further working.

Attempt to compare their coefficients ***(M1)***

$-9=-m$ OR $\frac{4m}{9}=\frac{36}{m}$

$m=9$ ***(A1)***

third root is 4 ***A1***

Note: Award a maximum of ***A0A0(M1)(M1)(A1)A1*** for a final answer $P(z)=(z-3\mathrm{i})(z+3\mathrm{i})(4-z)$ seen or stating all three correct factors with no evidence of roots throughout their working.

[6 marks]

2. [Maximum mark: 7] 22N.1.AHL.TZ0.5

Consider the equation $z^4 + pz^3 + 54z^2 - 108z + 80 = 0$ where $z \in \mathbb{C}$ and $p \in \mathbb{R}$.

Three of the roots of the equation are $3 + \mathrm{i},\ \alpha$ and α^2, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all the roots of the equation, find the value of α. [4]

Markscheme

product of roots $= 80$ ***(A1)***

$3 - \mathrm{i}$ is a root ***(A1)***

attempt to set up an equation involving the product of their four roots and ± 80 ***(M1)***

$(3 + \mathrm{i})(3 - \mathrm{i})\alpha^3 = 80 \Rightarrow 10\alpha^3 = 80$

$\alpha = 2$ ***A1***

[4 marks]

(b) Find the value of p. [3]

Markscheme

METHOD 1

sum of roots $= -p$ *(A1)*

$-p = 3 + \mathrm{i} + 3 - \mathrm{i} + 2 + 4$ *(M1)*

Note: Accept $p = 3 + \mathrm{i} + 3 - \mathrm{i} + 2 + 4$ for *(M1)*

$p = -12$ *A1*

METHOD 2

$(z - (3 + \mathrm{i}))(z - (3 - \mathrm{i}))(z - 2)(z - 4)$ *(M1)*

$((z - 3) - \mathrm{i})((z - 3) + \mathrm{i})(z - 2)(z - 4)$ *(A1)*

$\left(z^2 - 6z + 10\right)\left(z^2 - 6z + 8\right) = z^4 - 12z^3 + \ldots$

$p = -12$ *A1*

[3 marks]

3. [Maximum mark: 16] 22N.1.AHL.TZ0.11

Consider a three-digit code abc, where each of a, b and c is assigned one of the values $1, 2, 3, 4$ or 5.

Find the total number of possible codes

(a.i) assuming that each value can be repeated (for example, 121 or 444). [2]

Markscheme

5^3 *(A1)*

$= 125$ *A1*

[2 marks]

(a.ii) assuming that no value is repeated. [2]

Markscheme

${}^5P_3 = 5 \times 4 \times 3$ *(A1)*

$= 60$ *A1*

[2 marks]

Let $P(x) = x^3 + ax^2 + bx + c$, where each of a, b and c is assigned one of the values $1, 2, 3, 4$ or 5. Assume that no value is repeated.

Consider the case where $P(x)$ has a factor of $(x^2 + 3x + 2)$.

(b.i) Find an expression for b in terms of a. [6]

Markscheme

METHOD 1

$x^2 + 3x + 2 = (x + 1)(x + 2)$ *(A1)*

correct use of factor theorem for at least one of their factors *(M1)*

$P(-1) = 0$ or $P(-2) = 0$

attempt to find two equations in $a,\ b$ and c *(M1)*

$(-1)^3 + a(-1)^2 + b(-1) + c = 0 (\Rightarrow -1 + a - b + c = 0)$

$(-2)^3 + a(-2)^2 + b(-2) + c = 0$

$-8 + 4a - 2b + c = 0$ and $-1 + a - b + c = 0$ ***A1***

attempt to combine their two equations in $-8 + 4a - 2b + c = 0$ to eliminate c *(M1)*

$b = 3a - 7$ ***A1***

Note: Award at most ***A1M1M1A0M1A0*** for $b = -3a - 7$ from $P(1) = P(2) = 0$

METHOD 2

$P(x) = x^3 + ax^2 + bx + c = \left(x^2 + 3x + 2\right)(x + d)$ *(M1)*

$= x^3 + (3 + d)x^2 + (2 + 3d)x + 2d$ *(A1)*

attempt to compare coefficients of x^2 and x *(M1)*

$a = 3 + d$ and $b = 2 + 3d$ ***A1***

attempt to eliminate d *(M1)*

$\Rightarrow b = 3a - 7$ ***A1***

METHOD 3

attempt to divide $x^3 + ax^2 + bx + c$ by $x^2 + 3x + 2$ ***M1***

$\frac{x^3+ax^2+bx+c}{x^2+3x+2} = (x + a - 3) + \frac{(-3a+b+7)x+(c-2a+6)}{x^2+3x+2}$
A1A1A1

Note: Award ***A1*** for $x + a - 3$, ***A1*** for $(-3a + b + 7)x$ and ***A1*** for $c - 2a + 6$

recognition that, if $\left(x^2 + 3x + 2\right)$ is a factor of $P(x)$, then $-3a + b + 7 = 0$ ***(M1)***

leading to $b = 3a - 7$ ***A1***

METHOD 4

$x^2 + 3x + 2 = (x + 1)(x + 2)$ ***(A1)***

attempt to use Vieta's formulae for a cubic with roots $-1, \ -2$ and "p" ***(M1)***

$(-1) + (-2) + p = -a(\Rightarrow p = 3 - a)$ ***A1***

$(-1)(-2) + (-1)p + (-2)p = b$ ***A1***

Attempt to eliminate "p" ***(M1)***

$2 - (3 - a) - 2(3 - a) = b$

$b = 3a - 7$ ***A1***

Note: Award at most ***A1M1A0A0M1A0*** for $b = -3a - 7$ from roots $1, \ 2$ and "p"

[6 marks]

(b.ii) Hence show that the only way to assign the values is $a = 4,\ b = 5$ and $c = 2$. [2]

Markscheme

METHOD 1

$a = 1, 2, 5$ lead to invalid values for b ***R1***

$a = 3\,, b = 2 \Rightarrow c = 0$ so not possible ***R1***

so $a = 4,\ b = 5,\ c = 2$ is the only solution ***AG***

METHOD 2

$c = 2a - 6$ ***R1***

correctly argues $a = 4$ is the only possibility ***R1***

so $a = 4,\ b = 5,\ c = 2$ is the only solution ***AG***

[2 marks]

(b.iii) Express $P(x)$ as a product of linear factors. [1]

Markscheme

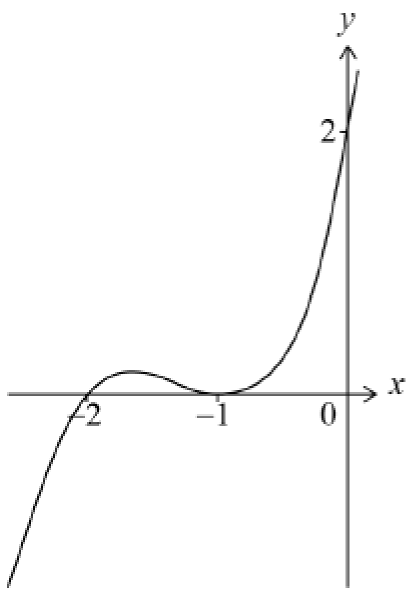
$x^3 + 4x^2 + 5x + 2 = \left(x^2 + 3x + 2\right)(x + 1)$

$= (x + 2)(x + 1)(x + 1)$ ***A1***

[1 mark]

(b.iv) Hence or otherwise, sketch the graph of $y = P(x)$, clearly showing the coordinates of any intercepts with the axes. [3]

Markscheme



positive cubic shape with y-intercept at $(0, 2)$ ***A1***

x-intercept at $(-2, 0)$ and local maximum point anywhere between $x = -2$ and $x = -1$ ***A1***

local minimum point at $(-1, 0)$ ***A1***

Note: Accept answers from an approach based on calculus.

[3 marks]

4. [Maximum mark: 5] 21M.1.AHL.TZ2.7

The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots $\alpha,\ \beta$ and $\alpha + \beta$.

Given that $\alpha\beta = -\frac{k^2}{4}$, find the value of k. [5]

Markscheme

$\alpha + \beta + \alpha + \beta = k$ ***(A1)***

$\alpha + \beta = \frac{k}{2}$

$\alpha\beta(\alpha + \beta) = -3k$ ***(A1)***

$\left(-\frac{k^2}{4}\right)\left(\frac{k}{2}\right) = -3k\ \left(-\frac{k^3}{8} = -3k\right)$ ***M1***

attempting to solve $-\frac{k^3}{8} + 3k = 0$ (or equivalent) for k ***(M1)***

$k = 2\sqrt{6}\left(= \sqrt{24}\right)(k > 0)$ ***A1***

Note: Award ***A0*** for $k = \pm 2\sqrt{6}\left(\pm\sqrt{24}\right)$.

[5 marks]

5. [Maximum mark: 17] 19M.2.AHL.TZ1.H_11

Consider the equation $x^5 - 3x^4 + mx^3 + nx^2 + px + q = 0$, where $m, n, p, q \in \mathbb{R}$.

The equation has three distinct real roots which can be written as $\log_2 a$, $\log_2 b$ and $\log_2 c$.

The equation also has two imaginary roots, one of which is $d\mathrm{i}$ where $d \in \mathbb{R}$.

(a) Show that $abc = 8$. [5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

recognition of the other root $= -d\mathrm{i}$ ***(A1)***

$\log_2 a + \log_2 b + \log_2 c + d\mathrm{i} - d\mathrm{i} = 3$ ***M1A1***

Note: Award ***M1*** for sum of the roots, ***A1*** for 3. Award ***A0M1A0*** for just $\log_2 a + \log_2 b + \log_2 c = 3$.

$\log_2 abc = 3$ ***(M1)***

$\Rightarrow abc = 2^3$ ***A1***

$abc = 8$ ***AG***

[5 marks]

The values a, b, and c are consecutive terms in a geometric sequence.

(b) Show that one of the real roots is equal to 1. [3]

Markscheme

METHOD 1

let the geometric series be $u_1, u_1 r, u_1 r^2$

$(u_1 r)^3 = 8$ ***M1***

$u_1 r = 2$ ***A1***

hence one of the roots is $\log_2 2 = 1$ ***R1***

METHOD 2

$\frac{b}{a} = \frac{c}{b}$

$b^2 = ac \Rightarrow b^3 = abc = 8$ ***M1***

$b = 2$ ***A1***

hence one of the roots is $\log_2 2 = 1$ ***R1***

[3 marks]

(c) Given that $q = 8d^2$, find the other two real roots. [9]

Markscheme

METHOD 1

product of the roots is $r_1 \times r_2 \times 1 \times d\mathrm{i} \times -d\mathrm{i} = -8d^2$ ***(M1)(A1)***

$r_1 \times r_2 = -8$ ***A1***

sum of the roots is $r_1 + r_2 + 1 + d\mathrm{i} + -d\mathrm{i} = 3$ ***(M1)(A1)***

$r_1 + r_2 = 2$ ***A1***

solving simultaneously ***(M1)***

$r_1 = -2, r_2 = 4$ ***A1A1***

METHOD 2

product of the roots $\log_2 a \times \log_2 b \times \log_2 c \times d\mathrm{i} \times -d\mathrm{i} = -8d^2$
M1A1

$\log_2 a \times \log_2 b \times \log_2 c = -8$ ***A1***

EITHER

a, b, c can be written as $\frac{2}{r}, 2, 2r$ ***M1***

$\left(\log_2 \frac{2}{r}\right)(\log_2 2)(\log_2 2r) = -8$

attempt to solve ***M1***

$(1 - \log_2 r)(1 + \log_2 r) = -8$

$\log_2 r = \pm 3$

$r = \frac{1}{8}, 8$ ***A1A1***

OR

a, b, c can be written as $a, 2, \frac{4}{a}$ ***M1***

$(\log_2 a)(\log_2 2)\left(\log_2 \frac{4}{a}\right) = -8$

attempt to solve ***M1***

$a = \frac{1}{4}, 16$ ***A1A1***

THEN

a and c are $\frac{1}{4}, 16$ ***(A1)***

roots are −2, 4 ***A1***

[9 marks]

6. [Maximum mark: 7] 18N.1.AHL.TZ0.H_8

Consider the equation $z^4 + az^3 + bz^2 + cz + d = 0$, where $a, b,$ $c, d \in \mathbb{R}$ and $z \in \mathbb{C}$.

Two of the roots of the equation are $\log_2 6$ and $i\sqrt{3}$ and the sum of all the roots is $3 + \log_2 3$.

Show that $6a + d + 12 = 0$. [7]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$-i\sqrt{3}$ is a root ***(A1)***

$3 + \log_2 3 - \log_2 6 \left(= 3 + \log_2 \frac{1}{2} = 3 - 1 = 2\right)$ is a root ***(A1)***

sum of roots: $-a = 3 + \log_2 3 \Rightarrow a = -3 - \log_2 3$ ***M1***

Note: Award M1 for use of $-a$ is equal to the sum of the roots, do not award if minus is missing.

Note: If expanding the factored form of the equation, award ***M1*** for equating a to the coefficient of z^3.

product of roots: $(-1)^4 d = 2\left(\log_2 6\right)\left(i\sqrt{3}\right)\left(-i\sqrt{3}\right)$ ***M1***

$= 6 \log_2 6$ ***A1***

Note: Award ***M1A0*** for $d = -6 \log_2 6$

$$6a + d + 12 = -18 - 6\log_2 3 + 6\log_2 6 + 12$$

EITHER

$= -6 + 6\log_2 2 = 0$ ***M1A1AG***

Note: ***M1*** is for a correct use of one of the log laws.

OR

$= -6 - 6\log_2 3 + 6\log_2 3 + 6\log_2 2 = 0$ ***M1A1AG***

Note: ***M1*** is for a correct use of one of the log laws.

[7 marks]

7. [Maximum mark: 5] 18M.1.AHL.TZ1.H_1

Let $f(x) = x^4 + px^3 + qx + 5$ where p, q are constants.

The remainder when $f(x)$ is divided by $(x + 1)$ is 7, and the remainder when $f(x)$ is divided by $(x - 2)$ is 1. Find the value of p and the value of q. [5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to substitute $x = -1$ or $x = 2$ or to divide polynomials ***(M1)***

$1 - p - q + 5 = 7$, $16 + 8p + 2q + 5 = 1$ or equivalent ***A1A1***

attempt to solve their two equations ***M1***

$p = -3$, $q = 2$ ***A1***

[5 marks]

8. [Maximum mark: 5] 18M.2.AHL.TZ2.H_2

The polynomial $x^4 + px^3 + qx^2 + rx + 6$ is exactly divisible by each of $(x - 1)$, $(x - 2)$ and $(x - 3)$.

Find the values of p, q and r. [5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

substitute each of x = 1,2 and 3 into the quartic and equate to zero ***(M1)***

$p + q + r = -7$

$4p + 2q + r = -11$ or equivalent ***(A2)***

$9p + 3q + r = -29$

Note: Award ***A2*** for all three equations correct, ***A1*** for two correct.

attempting to solve the system of equations ***(M1)***

$p = -7, q = 17, r = -17$ ***A1***

Note: Only award ***M1*** when some numerical values are found when solving algebraically or using GDC.

METHOD 2

attempt to find fourth factor ***(M1)***

$(x - 1)$ ***A1***

attempt to expand $(x - 1)^2 (x - 2) (x - 3)$ ***M1***

$x^4 - 7x^3 + 17x^2 - 17x + 6$ ($p = -7, q = 17, r = -17$) ***A2***

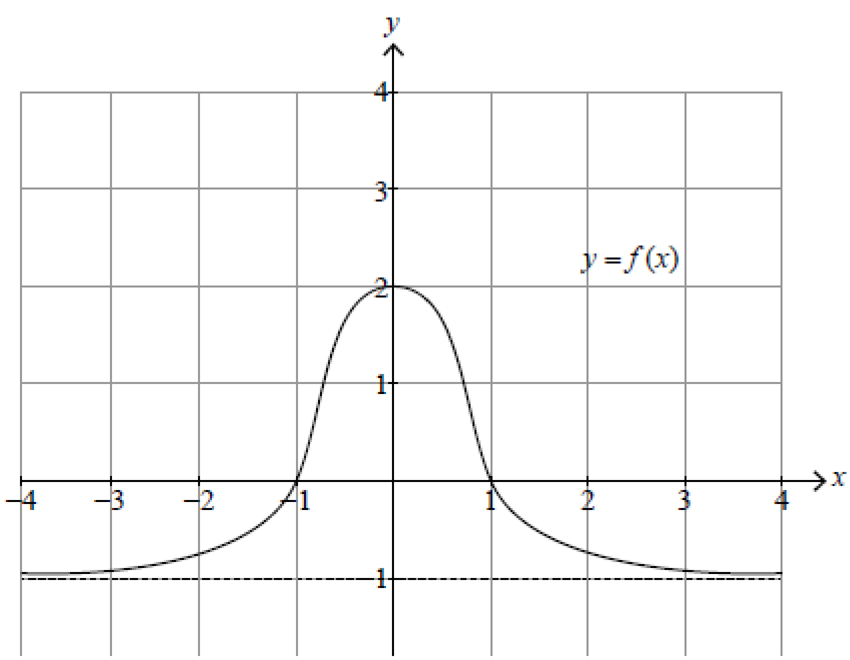
Note: Award ***A2*** for all three values correct, ***A1*** for two correct.

Note: Accept long / synthetic division.

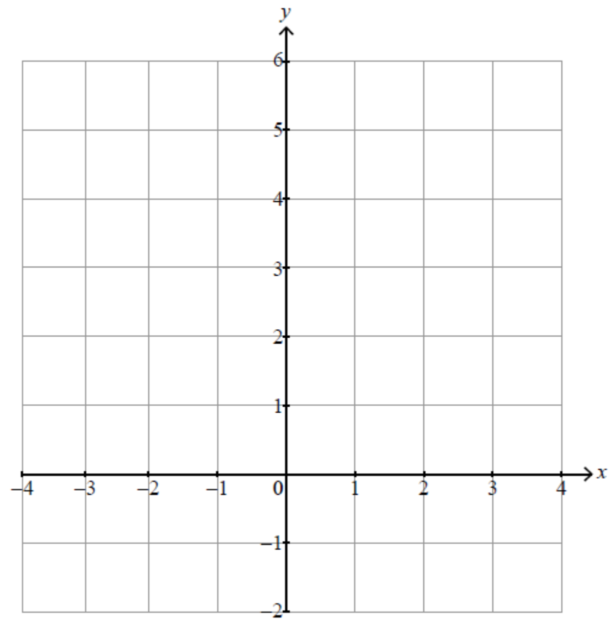
[5 marks]

9. [Maximum mark: 5] SPM.1.AHL.TZ0.4

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the x-axis at $x = -1$ and $x = 1$, and the y-axis at $y = 2$.

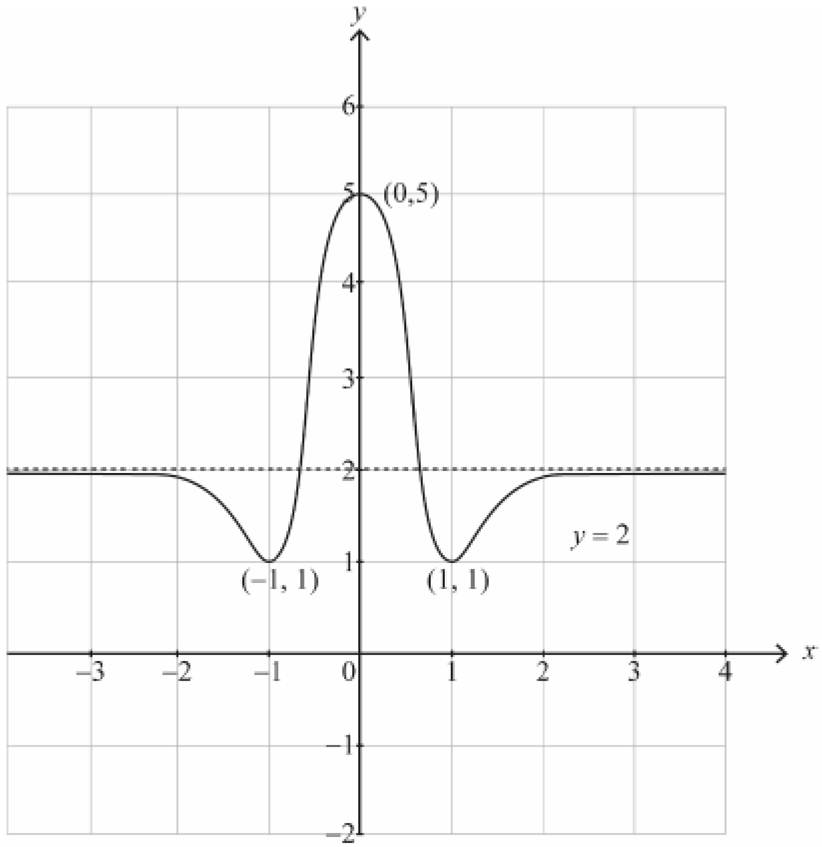


On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.



[5]

Markscheme



no y values below 1 ***A1***

horizontal asymptote at $y = 2$ with curve approaching from below as $x \to \pm\infty$ ***A1***

(±1,1) local minima ***A1***

(0,5) local maximum ***A1***

smooth curve and smooth stationary points ***A1***

[5 marks]

10. [Maximum mark: 5] SPM.2.AHL.TZ0.9

Consider the graphs of $y = \frac{x^2}{x-3}$ and $y = m\left(x+3\right), m \in \mathbb{R}$.

Find the set of values for m such that the two graphs have no intersection points. [5]

Markscheme

METHOD 1

sketching the graph of $y = \frac{x^2}{x-3}$ $(y = x + 3 + \frac{9}{x-3})$ ***M1***

the (oblique) asymptote has a gradient equal to 1

and so the maximum value of m is 1 ***R1***

consideration of a straight line steeper than the horizontal line joining (−3, 0) and (0, 0) ***M1***

so $m > 0$ ***R1***

hence $0 < m \le 1$ ***A1***

METHOD 2

attempting to eliminate y to form a quadratic equation in x ***M1***

$x^2 = m\left(x^2 - 9\right)$

$\Rightarrow (m-1)x^2 - 9m = 0$ ***A1***

EITHER

attempting to solve $-4\left(m-1\right)\left(-9m\right) < 0$ for m ***M1***

OR

attempting to solve $x^2 < 0$ *ie* $\frac{9m}{m-1} < 0\ (m \neq 1)$ for m ***M1***

THEN

$\Rightarrow 0 < m < 1$ ***A1***

a valid reason to explain why $m = 1$ gives no solutions *eg* if $m = 1$,

$(m-1)x^2 - 9m = 0 \Rightarrow -9 = 0$ and so $0 < m \leq 1$ ***R1***

[5 marks]

11. [Maximum mark: 9] EXM.1.AHL.TZ0.5

Let $f(x) = \frac{2x^2-5x-12}{x+2},\ x \in \mathbb{R},\ x \neq -2$.

(a) Find all the intercepts of the graph of $f(x)$ with both the x and y axes. [4]

Markscheme

$x = 0 \Rightarrow y = -6$ intercept on the y axes is (0, −6) ***A1***

$2x^2 - 5x - 12 = 0 \Rightarrow (2x+3)(x-4) = 0 \Rightarrow x = \frac{-3}{2} \text{ or } 4$ ***M1***

intercepts on the x axes are $\left(\frac{-3}{2},\ 0\right) \text{ and } (4,\ 0)$ ***A1A1***

[4 marks]

(b) Write down the equation of the vertical asymptote. [1]

Markscheme

$x = -2$ ***A1***

[1 mark]

(c) As $x \to \pm\infty$ the graph of $f(x)$ approaches an oblique straight line asymptote.

Divide $2x^2 - 5x - 12$ by $x + 2$ to find the equation of this asymptote. [4]

Markscheme

$f(x) = 2x - 9 + \frac{6}{x+2}$ ***M1A1***

So equation of asymptote is $y = 2x - 9$ ***M1A1***

[4 marks]

12. [Maximum mark: 19] 23N.2.AHL.TZ1.11

Consider the function defined by $f(x)\frac{x^2-14x+24}{2x+6}$, where $x \in \mathbb{R}, x \neq -3$.

(a) State the equation of the vertical asymptote on the graph of f. [1]

Markscheme

(vertical asymptote equation) $x = -3$ ***A1***

Note: Accept $2x + 6 = 0$ or equivalent.

[1 mark]

(b) Find the coordinates of the points where the graph of f crosses the x-axis. [2]

Markscheme

$(2, 0)$ and $(12, 0)$ ***A1A1***

Note: Award ***A1*** for $(2, 0)$ and ***A1*** for $(12, 0)$.

Award ***A1A0*** if only x values are given.

[2 marks]

The graph of f also has an oblique asymptote of the form $y = ax + b$, where $a,\ b \in \mathbb{Q}$.

(c) Find the value of a and the value of b. [4]

Markscheme

METHOD 1

$a = \frac{1}{2}$ ***A1***

attempt at 'long division' on $\frac{x^2 - 14x + 24}{2x + 6}$ ***(M1)***

$\frac{x^2 - 14x + 24}{2x + 6}$

$= \frac{1}{2}x - \frac{17}{2}\left(+\frac{\ldots}{2x+6}\right)$ ***(A1)***

$b = -\frac{17}{2}$ ***A1***

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

METHOD 2

$a = \frac{1}{2}$ ***A1***

$\frac{x^2 - 14x + 24}{2x + 6} \equiv \frac{1}{2}x + b + \frac{c}{2x + 6}$ ***(A1)***

$x^2 - 14x + 24 \equiv \frac{1}{2}x(2x + 6) + b(2x + 6) + c$

attempt to equate coefficients of x: ***(M1)***

$-14 = 3 + 2b$

$b = -\frac{17}{2}$

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

METHOD 3

$a = \frac{1}{2}$ ***A1***

$\frac{x^2 - 14x + 24}{2x + 6} - \frac{1}{2}x \equiv \frac{-17 + 24}{2x + 6}$ ***(A1)***

attempt to find the limit of $f(x) - ax$ as $x \to \infty$ ***(M1)***

$b = \lim_{x \to \infty} \frac{-17x + 24}{2x + 6}$

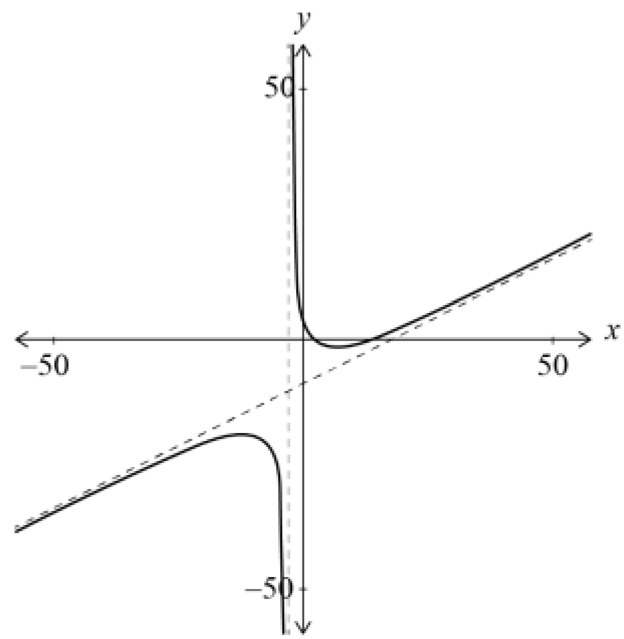
$= -\frac{17}{2}$ ***A1***

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

[4 marks]

(d) Sketch the graph of f for $-50 \leq x \leq 50$, showing clearly the asymptotes and any intersections with the axes. [4]

Markscheme



two branches with approximately correct shape (for $-50 \leq x \leq 50$)
A1

Note: For this **A1** the graph must be a function.

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1A1***

Note: Award ***A1*** for vertical asymptote and behaviour and ***A1*** for oblique asymptote and behaviour. If only top half of the graph seen only award ***A1A0*** if both asymptotes and behaviour are seen.

their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes do not need to be labelled. Ignore incorrect labels

[4 marks]

(e) Find the range of f. [4]

Markscheme

$\left(-10 - 5\sqrt{3} =\right) - 18.6602\ldots$ OR
$\left(-10 + 5\sqrt{3} =\right) - 1.33974\ldots$ seen anywhere ***(A1)***

attempt to write the range using at least one value in an interval or an inequality in y or $f(x)$ ***(M1)***

$y \leq -18.7,\ y \geq -1.34$ ***A1A1***

Note: Award ***A1*** for each inequality. Award ***A1A0*** for strict inequalities in both.

Do not award FT from (d).

Accept equivalent set notation.

[4 marks]

(f) Solve the inequality $f(x) > x$. [4]

Markscheme

$\left(-10 - 2\sqrt{31} =\right) - 21.1355\ldots$ OR
$\left(-10 + 2\sqrt{31} =\right) 1.13522\ldots$ seen anywhere ***(A1)***

$x < -21.1,\ -3 < x < 1.14$ ***A1A1A1***

Note: Award ***A1*** for $x < -21.1$, ***A1*** for correct endpoints of a single interval -3 and 1.14 and for ***A1*** for $-3 < x < 1.14$.

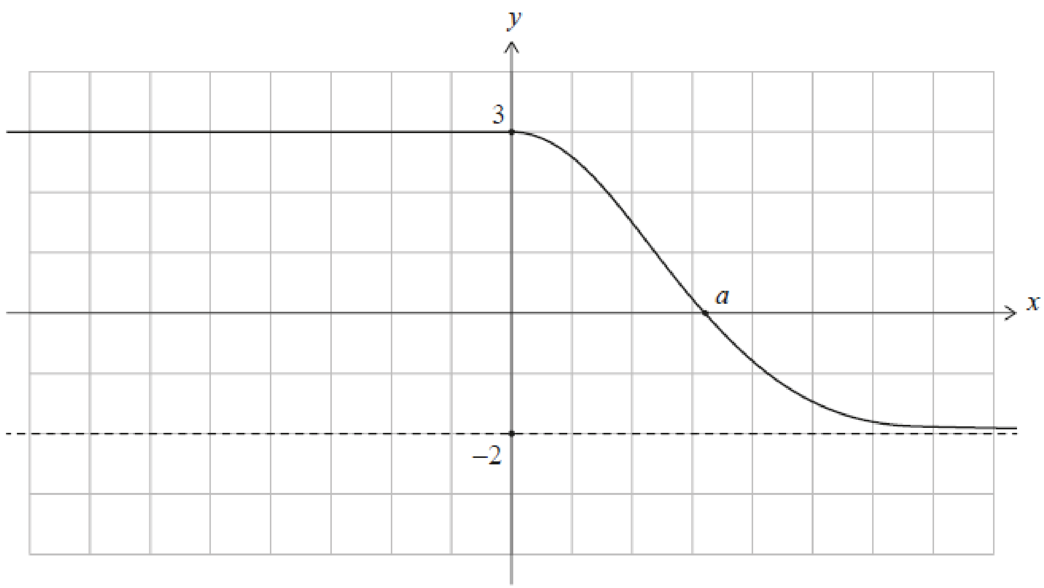
Do not award FT from (d).

Accept equivalent set notation.

[4 marks]

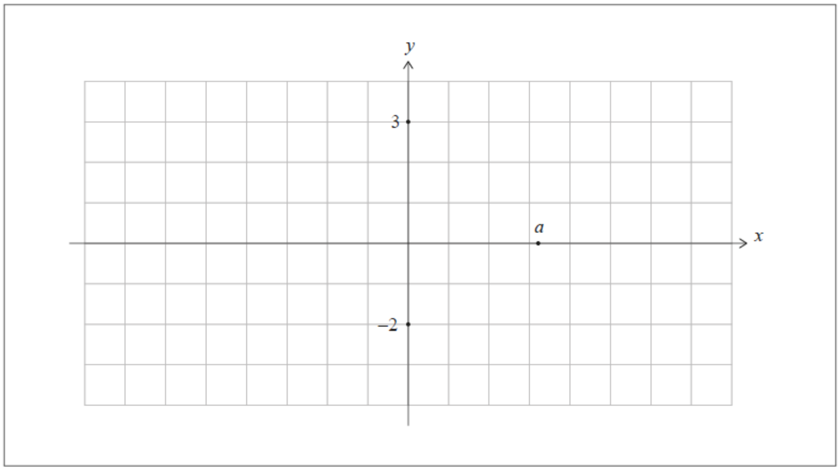
13. [Maximum mark: 7] 23M.1.AHL.TZ1.8

Part of the graph of a function, f, is shown in the following diagram. The graph of $y = f(x)$ has a y-intercept at $(0,\ 3)$, an x-intercept at $(a,\ 0)$ and a horizontal asymptote $y = -2$.



Consider the function $g(x) = |f(|x|)|$.

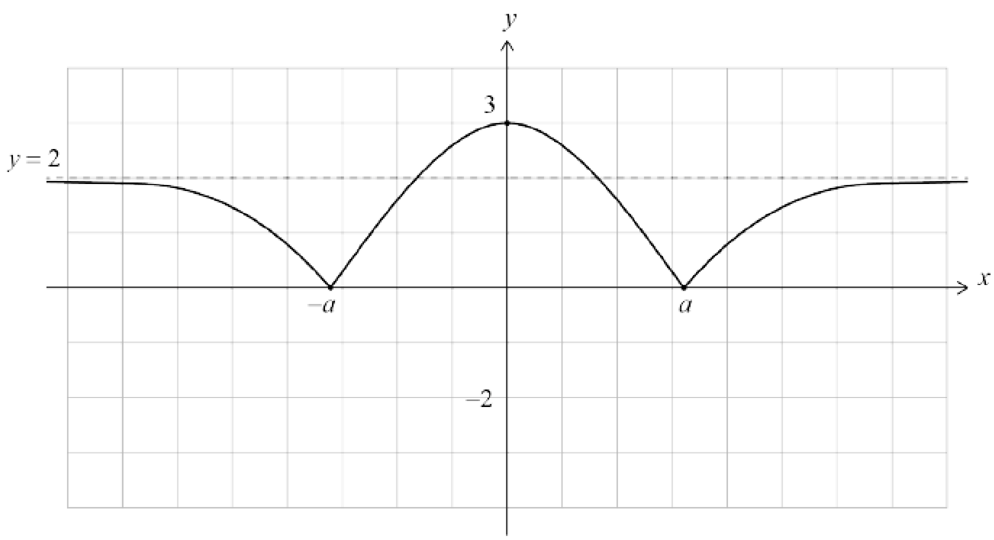
(a) On the following grid, sketch the graph of $y = g(x)$, labelling any axis intercepts and giving the equation of the asymptote.



[4]

Markscheme

attempt to reflect f in the x OR y axis ***(M1)***



A1A1A1

Note: For a curve with an approximately correct shaped right-hand branch, award:

A1 for correct asymptotic behaviour at $y = 2$ (either side)

A1 for correctly reflected RHS of the graph in the y-axis with smooth maximum at $(0,\ 3)$.

A1 for labelled x-intercept at $(-a, 0)$ and labelled asymptote at $y = 2$ with sharp points (cusps) at the x-intercepts.

[4 marks]

(b) Find the possible values of k such that $\left(g(x)\right)^2 = k$ has exactly two solutions.

[3]

Markscheme

$k = 0$ ***A1***

$4 \leq k < 9$ ***A2***

Note: If final answer incorrect, award ***A1*** for critical values 4 and 9 seen anywhere.

Exception to FT:
Award a maximum of ***A0A2FT*** if their graph from (a) is not symmetric about the y-axis.

[3 marks]

14. [Maximum mark: 8] 22M.1.AHL.TZ2.3

A function f is defined by $f(x) = \frac{2x-1}{x+1}$, where $x \in \mathbb{R},\ x \neq -1.$

The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

(a.i) Write down the equation of the vertical asymptote. [1]

Markscheme

$x = -1$ ***A1***

[1 mark]

(a.ii) Write down the equation of the horizontal asymptote. [1]

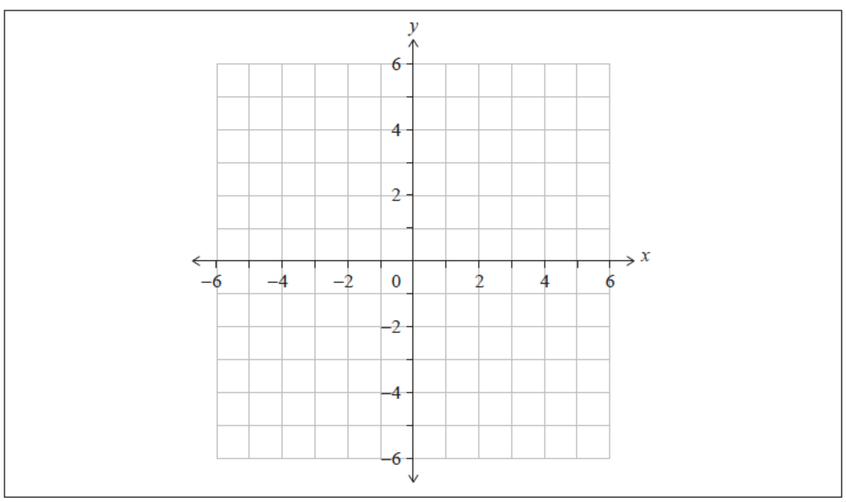
Markscheme

$y = 2$ ***A1***

[1 mark]

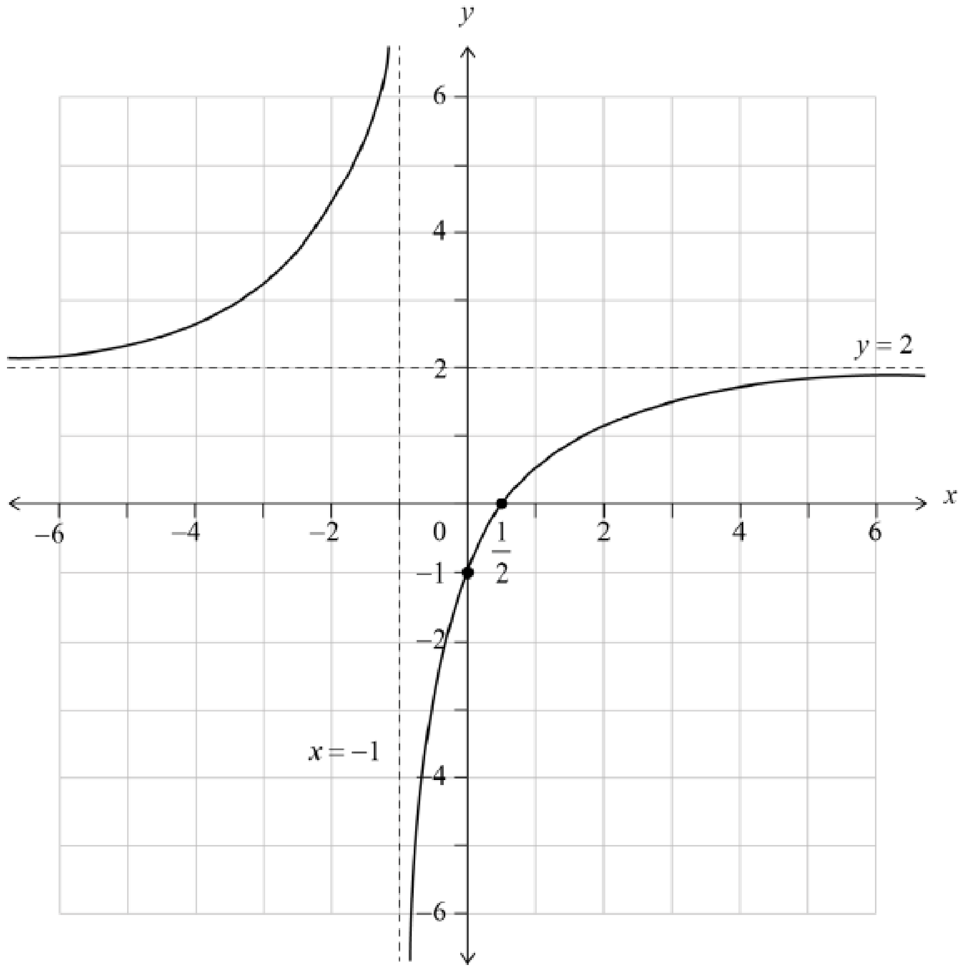
(b) On the set of axes below, sketch the graph of $y = f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes.



[3]

Markscheme



rational function shape with two branches in opposite quadrants, with two correctly positioned asymptotes and asymptotic behaviour shown ***A1***

axes intercepts clearly shown at $x = \frac{1}{2}$ and $y = -1$ ***A1A1***

[3 marks]

(c) Hence, solve the inequality $0 < \frac{2x-1}{x+1} < 2$. [1]

Markscheme

$x > \frac{1}{2}$ ***A1***

Note: Accept correct alternative correct notation, such as $\left(\frac{1}{2},\ \infty\right)$ and $\left]\frac{1}{2}, \infty\right[$.

[1 mark]

(d) Solve the inequality $0 < \frac{2|x|-1}{|x|+1} < 2$. [2]

Markscheme

EITHER

attempts to sketch $y = \frac{2|x|-1}{|x|+1}$ ***(M1)***

OR

attempts to solve $2|x| - 1 = 0$ ***(M1)***

Note: Award the ***(M1)*** if $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ are identified.

THEN

$x < -\frac{1}{2}$ or $x > \frac{1}{2}$ ***A1***

Note: Accept the use of a comma. Condone the use of 'and'. Accept correct alternative notation.

[2 marks]

15. [Maximum mark: 11] 21N.2.AHL.TZ0.10

Consider the function $f(x) = \frac{x^2 - x - 12}{2x - 15}, \ x \in \mathbb{R}, \ x \neq \frac{15}{2}$.

Find the coordinates where the graph of f crosses the

(a.i) x-axis. [2]

Markscheme

Note: In part (a), penalise once only, if correct values are given instead of correct coordinates.

attempts to solve $x^2 - x - 12 = 0$ ***(M1)***

$(-3, 0)$ and $(4, 0)$ ***A1***

[2 marks]

(a.ii) y-axis. [1]

Markscheme

Note: In part (a), penalise once only, if correct values are given instead of correct coordinates.

$\left(0, \frac{4}{5}\right)$ ***A1***

[1 mark]

(a.iii) Write down the equation of the vertical asymptote of the graph of f. [1]

Markscheme

$x = \frac{15}{2}$ ***A1***

Note: Award ***A0*** for $x \neq \frac{15}{2}$.
Award ***A1*** in part (b), if $x = \frac{15}{2}$ is seen on their graph in part (d).

[1 mark]

(a.iiii) The oblique asymptote of the graph of f can be written as $y = ax + b$ where $a,\ b \in \mathbb{Q}$.

Find the value of a and the value of b. [4]

Markscheme

METHOD 1

$(ax + b)(2x - 15) \equiv x^2 - x - 12$

attempts to expand $(ax + b)(2x - 15)$ ***(M1)***

$2ax^2 - 15ax + 2bx - 15b \equiv x^2 - x - 12$

$a = \frac{1}{2}$ ***A1***

equates coefficients of x ***(M1)***

$-1 = -\frac{15}{2} + 2b$

$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

METHOD 2

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x}{2} + \frac{13}{4} + \ldots$ ***M1***

$a = \frac{1}{2}$ ***A1***

$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

METHOD 3

$a = \frac{1}{2}$ ***A1***

$\frac{x^2-x-12}{2x-15} \equiv \frac{x}{2} + b + \frac{c}{2x-15}$ ***M1***

$x^2 - x - 12 \equiv \frac{(2x-15)x}{2} + (2x-15)b + c$

equates coefficients of x : ***(M1)***

$-1 = -\frac{15}{2} + 2b$

$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

METHOD 4

attempts division on $\frac{x^2 - x - 12}{2x - 15}$ ***M1***

$\frac{x^2 - x - 12}{2x - 15} = \frac{x}{2} + \frac{\frac{13x}{2} - 12}{2x - 15}$

$a = \frac{1}{2}$ ***A1***

$\frac{\frac{13x}{2} - 12}{2x - 15} = \frac{13}{4} + \ldots$ ***M1***

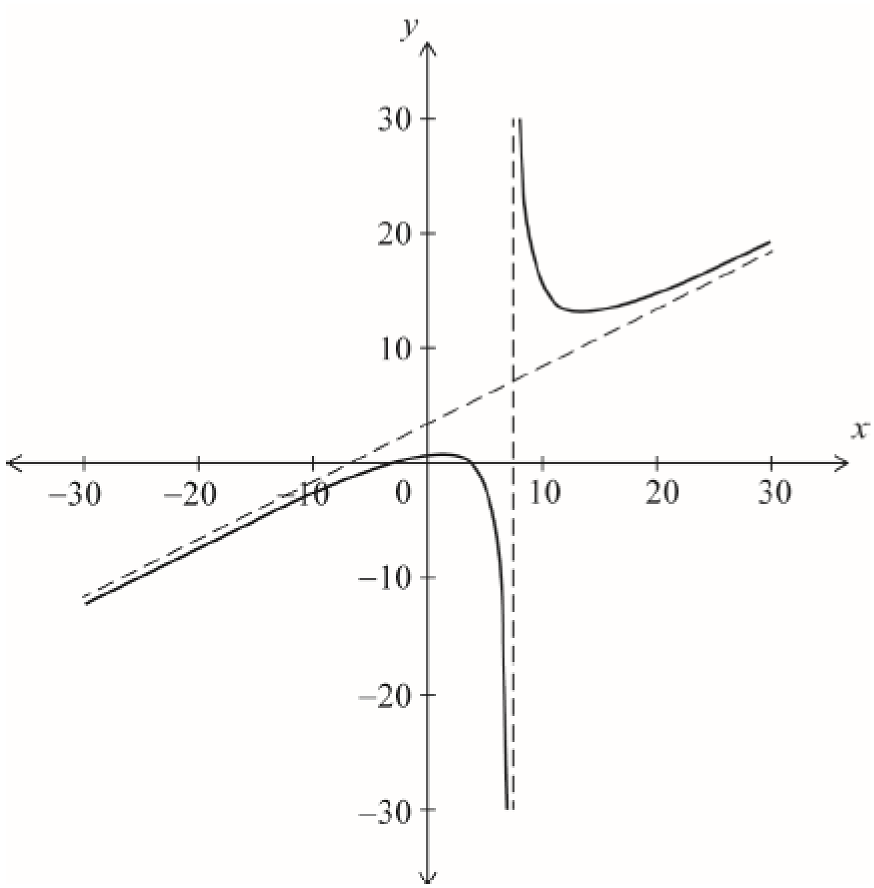
$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

[4 marks]

(a.iiiii) Sketch the graph of f for $-30 \leq x \leq 30$, clearly indicating the points of intersection with each axis and any asymptotes. [3]

Markscheme



two branches with approximately correct shape (for $-30 \leq x \leq 30$) ***A1***

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1***

their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes are not required to be labelled.

[3 marks]

16. [Maximum mark: 7] SPM.2.AHL.TZ0.8

The complex numbers w and z satisfy the equations

$\frac{w}{z} = 2\mathrm{i}$

$\mathrm{z}^* - 3w = 5 + 5\mathrm{i}$.

Find w and z in the form $a + b\mathrm{i}$ where $a, \mathrm{b} \in \mathbb{Z}$. [7]

Markscheme

substituting $w = 2\mathrm{i}z$ into $\mathrm{z}^* - 3w = 5 + 5\mathrm{i}$ ***M1***

$\mathrm{z}^* - 6\mathrm{i}z = 5 + 5\mathrm{i}$ ***A1***

let $z = x + y\mathrm{i}$

comparing real and imaginary parts of
$(x - y\mathrm{i}) - 6\mathrm{i}\,(x + y\mathrm{i}) = 5 + 5\mathrm{i}$ ***M1***

to obtain $x + 6y = 5$ and $-6x - y = 5$ ***A1***

attempting to solve for x and y) ***M1***

$x = -1$ and $y = 1$ so $z = -1 + \mathrm{i}$ ***A1***

hence $w = -2 - 2\mathrm{i}$ ***A1***

[7 marks]

17. [Maximum mark: 5] 23N.1.AHL.TZ1.7

It is given that $z = 5 + q\mathrm{i}$ satisfies the equation
$z^2 + \mathrm{i}z = -p + 25\mathrm{i}$, where $p,\ q \in \mathbb{R}$.

Find the value of p and the value of q. [5]

Markscheme

METHOD 1

attempt to substitute solution into given equation ***(M1)***

$(5+q\text{i})^2+\text{i}(5+q\text{i})=-p+25\text{i}$

$25-q^2+10q\text{i}-q+5\text{i}+p-25\text{i}=0$ OR
$25-q^2+10q\text{i}-q+5\text{i}=-p+25\text{i}$ ***A1***

$25-q^2+p-q+(10q-20)\text{i}=0$

attempt to equate real or imaginary parts: ***(M1)***

$10q-20=0$ OR $25-q^2+p-q=0$

$q=2,\ p=-19$ ***A1A1***

METHOD 2

$z^2+\text{i}z+p-25\text{i}=0$

sum of roots $=-\text{i}$, product of roots $=p-25\text{i}$ ***M1***

one root is $(5+q\text{i})$ so other root is $(-5-q\text{i}-\text{i})$ ***A1***

product
$(5+q\text{i})(-5-q\text{i}-\text{i})=-25-5q\text{i}-5\text{i}-5q\text{i}+q^2+q=p-25\text{i}$

equating real and imaginary parts for product of roots ***(M1)***

Im:$-25=10q-5$ Re:$p=-25+q^2+q$

$q=2,\ p=-19$ ***A1A1***

[5 marks]

18. [Maximum mark: 8] 21M.1.AHL.TZ1.7

Consider the quartic equation

$z^4 + 4z^3 + 8z^2 + 80z + 400 = 0,\ z \in \mathbb{C}.$

Two of the roots of this equation are $a + b\mathrm{i}$ and $b + a\mathrm{i}$, where $a,\ b \in \mathbb{Z}$.

Find the possible values of a. [8]

Markscheme

METHOD 1

other two roots are $a - b\mathrm{i}$ and $b - a\mathrm{i}$ ***A1***

sum of roots $= -4$ and product of roots $= 400$ ***A1***

attempt to set sum of four roots equal to -4 or 4 OR
attempt to set product of four roots equal to 400 ***M1***

$a + b\mathrm{i} + a - b\mathrm{i} + b + a\mathrm{i} + b - ai = -4$

$2a + 2b = -4 (\Rightarrow a + b = -2)$ ***A1***

$(a + b\mathrm{i})(a - b\mathrm{i})\ (b + a\mathrm{i})(b - a\mathrm{i}) = 400$

$\left(a^2 + b^2\right)^2 = 400$ ***A1***

$a^2 + b^2 = 20$

attempt to solve simultaneous equations ***(M1)***

$a = 2$ or $a = -4$ ***A1A1***

METHOD 2

other two roots are $a - b\mathrm{i}$ and $b - a\mathrm{i}$ ***A1***

$(z-(a+b\mathrm{i}))(z-(a-b\mathrm{i}))(z-(b+a\mathrm{i}))(z-(b-a\mathrm{i}))(=0)$ ***A1***

$\left((z-a)^2+b^2\right)\left((z-b)^2+a^2\right)(=0)$

$\left(z^2-2az+a^2+b^2\right)\left(z^2-2bz+b^2+a^2\right)(=0)$ ***A1***

Attempt to equate coefficient of z^3 and constant with the given quartic equation ***M1***

$-2a-2b=4$ and $\left(a^2+b^2\right)^2=400$ ***A1***

attempt to solve simultaneous equations ***(M1)***

$a=2$ or $a=-4$ ***A1A1***

[8 marks]

19. [Maximum mark: 5] 20N.1.AHL.TZ0.H_4

Consider the equation $\frac{2z}{3-z^*}=\mathrm{i}$, where $z=x+\mathrm{i}y$ and $x,\ y\in\mathbb{R}$.

Find the value of x and the value of y. [5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

substituting $z=x+\mathrm{i}y$ **and** $z^*=x-\mathrm{i}y$ ***M1***

$\frac{2(x+\mathrm{i}y)}{3-(x-\mathrm{i}y)}=\mathrm{i}$

$2x + 2iy = -y + i(3 - x)$

equate real and imaginary: ***M1***

$y = -2x$ AND $2y = 3 - x$ ***A1***

Note: If they multiply top and bottom by the conjugate, the equations $6x - 2x^2 + 2y^2 = 0$ and $6y - 4xy = (3 - x)^2 + y^2$ may be seen. Allow for ***A1***.

solving simultaneously:

$x = -1,\ y = 2\ \ (z = -1 + 2i)$ ***A1A1***

[5 marks]

20. [Maximum mark: 7] 19N.1.AHL.TZ0.H_5

Consider the equation $z^4 = -4$, where $z \in \mathbb{C}$.

(a) Solve the equation, giving the solutions in the form $a + ib$, where $a,\ b \in \mathbb{R}$. [5]

Markscheme

METHOD 1

$|z| = \sqrt[4]{4}\left(= \sqrt{2}\right)$ ***(A1)***

$\arg(z_1) = \frac{\pi}{4}$ ***(A1)***

first solution is $1 + i$ ***A1***

valid attempt to find all roots (De Moivre or +/– their components) ***(M1)*** other solutions are $-1+\mathrm{i}$, $-1-\mathrm{i}$, $1-\mathrm{i}$ ***A1***

METHOD 2

$z^4 = -4$

$(a+\mathrm{i}b)^4 = -4$

attempt to expand and equate **both** reals and imaginaries. ***(M1)***

$a^4 + 4a^3b\mathrm{i} - 6a^2b^2 - 4ab^3\mathrm{i} + b^4 = -4$

$(a^4 - 6a^4 + a^4 = -4 \Rightarrow)a = \pm 1$ ***and***
$(4a^3b - 4ab^3 = 0 \Rightarrow)a = \pm b$ ***(A1)***

first solution is $1+\mathrm{i}$ ***A1***

valid attempt to find all roots (De Moivre or +/– their components) ***(M1)***

other solutions are $-1+\mathrm{i}$, $-1-\mathrm{i}$, $1-\mathrm{i}$ ***A1***

[5 marks]

(b) The solutions form the vertices of a polygon in the complex plane. Find the area of the polygon. [2]

Markscheme

complete method to find area of 'rectangle' ***(M1)***

$= 4$ ***A1***

[2 marks]

21. [Maximum mark: 6] 19N.2.AHL.TZ0.H_6

Let $P\left(z\right)=az^{3}-37z^{2}+66z-10$, where $z\in\mathbb{C}$ and $a\in\mathbb{Z}$.

One of the roots of $P\left(z\right)=0$ is $3+\mathrm{i}$. Find the value of a. [6]

Markscheme

METHOD 1

one other root is $3-\mathrm{i}$ ***A1***

let third root be α ***(M1)***

considering sum or product of roots ***(M1)***

sum of roots $=6+\alpha=\frac{37}{a}$ ***A1***

product of roots $=10\alpha=\frac{10}{a}$ ***A1***

hence $a=6$ ***A1***

METHOD 2

one other root is $3-\mathrm{i}$ ***A1***

quadratic factor will be $z^{2}-6z+10$ ***(M1)A1***

$P\left(z\right)=az^{3}-37z^{2}+66z-10=\left(z^{2}-6z+10\right)\left(az-1\right)$ ***M1***

comparing coefficients ***(M1)***

hence $a=6$ ***A1***

METHOD 3

substitute $3+\mathrm{i}$ into $P\left(z\right)$ ***(M1)***

$a\left(18+26\mathrm{i}\right)-37\left(8+6\mathrm{i}\right)+66\left(3+\mathrm{i}\right)-10=0$ ***(M1)A1***

equating real or imaginary parts or dividing ***M1***

$18a-296+198-10=0$ or $26a-222+66=0$ or $\frac{10-66(3+\mathrm{i})+37(8+6\mathrm{i})}{18+26\mathrm{i}}$ ***A1***

hence $a=6$ ***A1***

[6 marks]