Trigonometry [134 marks]

1. [Maximum mark: 5] 22M.1.SL.TZ2.5

Find the least positive value of x for which $\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$. [5]

Markscheme

determines $\frac{\pi}{4}$ (or 45°) as the first quadrant (reference) angle ***(A1)***

attempts to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}$ ***(M1)***

Note: Award ***M1*** for attempting to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}, \frac{7\pi}{4}(,\ldots)$

$\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}\Rightarrow x<0$ and so $\frac{\pi}{4}$ is rejected ***(R1)***

$\frac{x}{2}+\frac{\pi}{3}=2\pi-\frac{\pi}{4}\left(=\frac{7\pi}{4}\right)$ ***A1***

$x=\frac{17\pi}{6}$ (must be in radians) ***A1***

[5 marks]

2. [Maximum mark: 6] 23M.1.AHL.TZ1.3

Solve $\cos 2x=\sin x$, where $-\pi\leq x\leq\pi$. [6]

Markscheme

$1-2\sin^2 x=\sin x$ ***A1***

$2\sin^2 x + \sin x - 1 = 0$

valid attempt to solve quadratic ***(M1)***

$(2\sin x - 1)(\sin x + 1)$ OR $\frac{-1 \pm \sqrt{1 - 4(2)(-1)}}{2(2)}$

recognition to solve for $\sin x$ ***(M1)***

$\sin x = \frac{1}{2}$ OR $\sin x = -1$

any correct solution from $\sin x = -1$ ***A1***

any correct solution from $\sin x = \frac{1}{2}$ ***A1***

Note: The previous two marks may be awarded for degree or radian values, irrespective of domain.

$x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ ***A1***

Note: If no working shown, award no marks for a final value(s).

Award ***A0*** for $-\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ if additional values also given.

[6 marks]

3. [Maximum mark: 5] 18M.2.AHL.TZ1.H_3

Let $f(x) = \tan(x + \pi)\cos\left(x - \frac{\pi}{2}\right)$ where $0 < x < \frac{\pi}{2}$.

Express $f(x)$ in terms of sin x and cos x. [5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\tan(x+\pi) = \tan x \left(= \frac{\sin x}{\cos x}\right)$ ***(M1)A1***

$\cos\left(x - \frac{\pi}{2}\right) = \sin x$ ***(M1)A1***

Note: The two ***M1***s can be awarded for observation or for expanding.

$\tan(x+\pi)\cos\left(x - \frac{\pi}{2}\right) = \frac{\sin^2 x}{\cos x}$ ***A1***

[5 marks]

4. [Maximum mark: 7] 21N.1.SL.TZ0.6

(a) Show that $2x - 3 - \frac{6}{x-1} = \frac{2x^2-5x-3}{x-1},\ x \in \mathbb{R},\ x \neq 1$. [2]

Markscheme

METHOD 1

attempt to write all LHS terms with a common denominator of $x - 1$ ***(M1)***

$2x - 3 - \frac{6}{x-1} = \frac{2x(x-1)-3(x-1)-6}{x-1}$ OR $\frac{(2x-3)(x-1)}{x-1} - \frac{6}{x-1}$

$= \frac{2x^2-2x-3x+3-6}{x-1}$ OR $\frac{2x^2-5x+3}{x-1} - \frac{6}{x-1}$ ***A1***

$= \frac{2x^2-5x-3}{x-1}$ ***AG***

METHOD 2

attempt to use algebraic division on RHS ***(M1)***

correctly obtains quotient of $2x - 3$ and remainder -6 ***A1***

$= 2x - 3 - \frac{6}{x-1}$ as required. ***AG***

[2 marks]

(b) Hence or otherwise, solve the equation

$2 \sin 2\theta - 3 - \frac{6}{\sin 2\theta - 1} = 0$ for $0 \leq \theta \leq \pi,\ \theta \neq \frac{\pi}{4}$. [5]

Markscheme

consider the equation $\frac{2 \sin^2 2\theta - 5 \sin 2\theta - 3}{\sin 2\theta - 1} = 0$ ***(M1)***

$\Rightarrow 2 \sin^2 2\theta - 5 \sin 2\theta - 3 = 0$

EITHER

attempt to factorise in the form $(2 \sin 2\theta + a)(\sin 2\theta + b)$ ***(M1)***

Note: Accept any variable in place of $\sin 2\theta$.

$(2 \sin 2\theta + 1)(\sin 2\theta - 3) = 0$

OR

attempt to substitute into quadratic formula ***(M1)***

$\sin 2\theta = \frac{5 \pm \sqrt{49}}{4}$

THEN

$\sin 2\theta = -\frac{1}{2}$ or $\sin 2\theta = 3$ ***(A1)***

Note: Award ***A1*** for $\sin 2\theta = -\frac{1}{2}$ only.

one of $\frac{7\pi}{6}$ OR $\frac{11\pi}{6}$ (accept 210 or 330) ***(A1)***

$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}$ (must be in radians) ***A1***

Note: Award ***A0*** if additional answers given.

[5 marks]

5. [Maximum mark: 7] 22M.1.SL.TZ1.4

Consider the functions $f\left(x\right) = \sqrt{3}\sin x + \cos x$ where $0 \le x \le \pi$ and $g(x) = 2x$ where $x \in \mathbb{R}$.

(a) Find $(f \circ g)(x)$. [2]

Markscheme

$(f \circ g)(x) = f(2x)$ ***(A1)***

$f(2x) = \sqrt{3}\sin 2x + \cos 2x$ ***A1***

[2 marks]

(b) Solve the equation $(f \circ g)(x) = 2\cos 2x$ where $0 \le x \le \pi$. [5]

Markscheme

$\sqrt{3}\sin 2x + \cos 2x = 2\cos 2x$

$\sqrt{3}\sin 2x = \cos 2x$

recognising to use $\tan$ or $\cot$ ***M1***

$\tan 2x = \frac{1}{\sqrt{3}}$ OR $\cot 2x = \sqrt{3}$ (values may be seen in right triangle) ***(A1)***

$\left(\arctan\left(\frac{1}{\sqrt{3}}\right) =\right) \frac{\pi}{6}$ (seen anywhere) (accept degrees) ***(A1)***

$2x = \frac{\pi}{6}, \frac{7\pi}{6}$

$x = \frac{\pi}{12}, \frac{7\pi}{12}$ ***A1A1***

Note: Do not award the final ***A1*** if any additional solutions are seen.
Award ***A1A0*** for correct answers in degrees.
Award ***A0A0*** for correct answers in degrees with additional values.

[5 marks]

6. [Maximum mark: 8] 21M.1.SL.TZ1.6

(a) Show that
$\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$.

[2]

Markscheme

Note: Do not award the final ***A1*** for proofs which work from both sides to find a common expression other than $2\sin x\cos x - 2\sin^2 x$.

METHOD 1 (LHS to RHS)

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

LHS $= 2\sin x\cos x + \cos 2x - 1$ OR

$\sin 2x + 1 - 2\sin^2 x - 1$ OR

$2\sin x\cos x + 1 - 2\sin^2 x - 1$

$= 2\sin x\cos x - 2\sin^2 x$ ***A1***

$\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x) =$ RHS ***AG***

METHOD 2 (RHS to LHS)

RHS $= 2\sin x\cos x - 2\sin^2 x$

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

$= \sin 2x + 1 - 2\sin^2 x - 1$ ***A1***

$= \sin 2x + \cos 2x - 1 =$ LHS ***AG***

[2 marks]

(b) Hence or otherwise, solve
$\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$ for
$0 < x < 2π$.

[6]

Markscheme

attempt to factorise ***M1***

$(\cos x - \sin x)(2 \sin x + 1) = 0$ ***A1***

recognition of $\cos x = \sin x \Rightarrow \frac{\sin x}{\cos x} = \tan x = 1$ OR
$\sin x = -\frac{1}{2}$ ***(M1)***

one correct reference angle seen anywhere, accept degrees ***(A1)***

$\frac{π}{4}$ OR $\frac{π}{6}$ (accept $-\frac{π}{6}, \frac{7π}{6}$)

Note: This ***(M1)(A1)*** is independent of the previous ***M1A1***.

$x = \frac{7π}{6}, \frac{11π}{6}, \frac{π}{4}, \frac{5π}{4}$ ***A2***

Note: Award ***A1*** for any two correct (radian) answers.
Award ***A1A0*** if additional values given with the four correct (radian) answers.
Award ***A1A0*** for four correct answers given in degrees.

[6 marks]

7. [Maximum mark: 5] 18M.1.AHL.TZ1.H_8

Let $a = \sin b,\ 0 < b < \frac{\pi}{2}$.

Find, in terms of b, the solutions of $\sin 2x = -a,\ 0 \leqslant x \leqslant \pi$. [5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

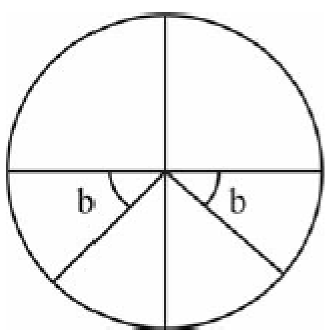
$\sin 2x = -\sin b$

EITHER

$\sin 2x = \sin(-b)$ or $\sin 2x = \sin(\pi + b)$ or
$\sin 2x = \sin(2\pi - b) \ldots$ ***(M1)(A1)***

Note: Award ***M1*** for any one of the above, ***A1*** for having final two.

OR



(M1)(A1)

Note: Award ***M1*** for one of the angles shown with b clearly labelled, ***A1*** for both angles shown. Do not award ***A1*** if an angle is shown in the second quadrant and subsequent ***A1*** marks not awarded.

THEN

$2x = \pi + b$ or $2x = 2\pi - b$ ***(A1)(A1)***

$x = \frac{\pi}{2} + \frac{b}{2},\ x = \pi - \frac{b}{2}$ ***A1***

[5 marks]

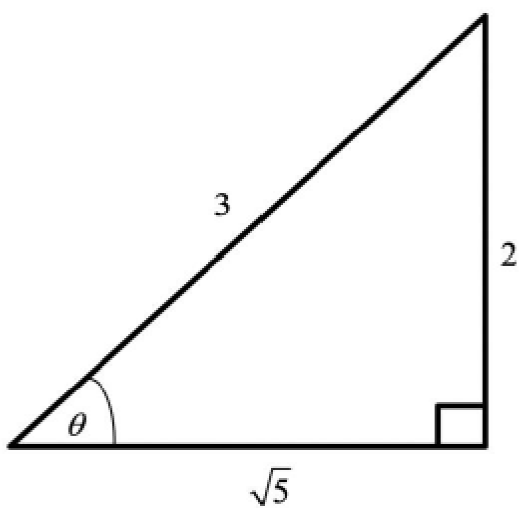
8. [Maximum mark: 4] 21M.1.AHL.TZ1.6

It is given that $\operatorname{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot\theta$. [4]

Markscheme

METHOD 1

attempt to use a right angled triangle ***M1***



correct placement of all three values and θ seen in the triangle ***(A1)***

$\cot\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer
The ***R1*** should be awarded independently for a negative value only given as a final answer.

METHOD 2

Attempt to use $1+\cot^2\theta=\operatorname{cosec}^2\theta$ ***M1***

$1+\cot^2\theta=\frac{9}{4}$

$\cot^2\theta=\frac{5}{4}$ ***(A1)***

$\cot\theta=\pm\frac{\sqrt{5}}{2}$

$\cot\theta<0$ (since $\operatorname{cosec}\theta>0$ puts θ in the second quadrant) ***R1***

$\cot\theta=-\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta=\frac{\sqrt{5}}{2}$ seen as the final answer
The ***R1*** should be awarded independently for a negative value only given as a final answer.

METHOD 3

$\sin\theta=\frac{2}{3}$

attempt to use $\sin^2\theta+\cos^2\theta=1$ ***M1***

$\frac{4}{9}+\cos^2\theta=1$

$\cos^2\theta=\frac{5}{9}$ ***(A1)***

$\cos\theta=\pm\frac{\sqrt{5}}{3}$

$\cos\theta<0$ (since $\operatorname{cosec}\theta>0$ puts θ in the second quadrant) ***R1***

$\cos\theta=-\frac{\sqrt{5}}{3}$

$\cot\theta=-\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

[4 marks]

9. [Maximum mark: 7] 21M.1.AHL.TZ2.2

Solve the equation $2\cos^2 x + 5\sin x = 4,\ 0 \le x \le 2\pi$. [7]

Markscheme

attempt to use $\cos^2 x = 1 - \sin^2 x$ ***M1***

$2\sin^2 x - 5\sin x + 2 = 0$ ***A1***

EITHER

attempting to factorise ***M1***

$(2\sin x - 1)(\sin x - 2)$ ***A1***

OR

attempting to use the quadratic formula ***M1***

$\sin x = \frac{5 \pm \sqrt{5^2 - 4\times2\times2}}{4} \left(= \frac{5\pm3}{4}\right)$ ***A1***

THEN

$\sin x = \frac{1}{2}$ ***(A1)***

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ ***A1A1***

[7 marks]

10. [Maximum mark: 8] 19N.1.SL.TZ0.S_6

Let $f(x) = 4\cos\left(\frac{x}{2}\right) + 1$, for $0 \leqslant x \leqslant 6\pi$. Find the values of x for which $f(x) > 2\sqrt{2} + 1$. [8]

Markscheme

METHOD 1 – FINDING INTERVALS FOR *x*

$4\cos\left(\frac{x}{2}\right) + 1 > 2\sqrt{2} + 1$

correct working ***(A1)***

eg $4\cos\left(\frac{x}{2}\right) = 2\sqrt{2}$, $\cos\left(\frac{x}{2}\right) > \frac{\sqrt{2}}{2}$

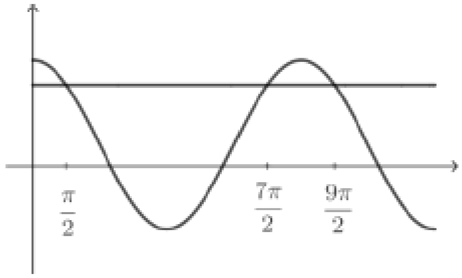
recognizing $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ ***(A1)***

one additional correct value for $\frac{x}{2}$ (ignoring domain and equation/inequalities) ***(A1)***

eg $-\frac{\pi}{4}, \frac{7\pi}{4}, 315^\circ, \frac{9\pi}{4}, -45^\circ, \frac{15\pi}{4}$

three correct values for x ***A1A1***

eg $\frac{\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$

valid approach to find intervals ***(M1)*** eg 

correct intervals (must be in radians) ***A1A1 N2***

$0 \leqslant x < \frac{\pi}{2}, \frac{7\pi}{2} < x < \frac{9\pi}{2}$

Note: If working shown, award ***A1A0*** if inclusion/exclusion of endpoints is incorrect. If no working shown award ***N1***.
If working shown, award ***A1A0*** if both correct intervals are given, **and** additional intervals are given. If no working shown award ***N1***.
Award ***A0A0*** if inclusion/exclusion of endpoints are incorrect **and** additional intervals are given.

METHOD 2 – FINDING INTERVALS FOR $\frac{x}{2}$

$4\cos\left(\frac{x}{2}\right) + 1 > 2\sqrt{2} + 1$

correct working ***(A1)***

eg $4\cos\left(\frac{x}{2}\right) = 2\sqrt{2}$, $\cos\left(\frac{x}{2}\right) > \frac{\sqrt{2}}{2}$

recognizing $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ ***(A1)***

one additional correct value for $\frac{x}{2}$ (ignoring domain and equation/inequalities) ***(A1)***

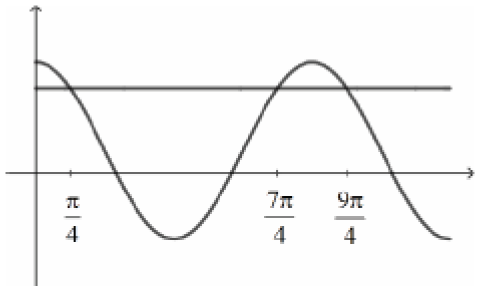
eg $-\frac{\pi}{4}, \frac{7\pi}{4}, 315°, \frac{9\pi}{4}, -45°, \frac{15\pi}{4}$

three correct values for $\frac{x}{2}$ ***A1***

eg $\frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}$

valid approach to find intervals ***(M1)***

eg



one correct interval for $\frac{x}{2}$ ***A1***

eg $0 \leqslant \frac{x}{2} < \frac{\pi}{4}, \frac{7\pi}{4} < \frac{x}{2} < \frac{9\pi}{4}$

correct intervals (must be in radians) ***A1A1 N2***

$0 \leqslant x < \frac{\pi}{2}, \frac{7\pi}{2} < x < \frac{9\pi}{2}$

Note: If working shown, award ***A1A0*** if inclusion/exclusion of endpoints is incorrect. If no working shown award ***N1***.
If working shown, award ***A1A0*** if both correct intervals are given, **and** additional intervals are given. If no working shown award ***N1***.
Award ***A0A0*** if inclusion/exclusion of endpoints are incorrect **and** additional intervals are given.

[8 marks]

11. [Maximum mark: 7] 19N.1.AHL.TZ0.H_4

A and B are acute angles such that $\cos A = \frac{2}{3}$ and $\sin B = \frac{1}{3}$.

Show that $\cos(2A + B) = -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$. [7]

Markscheme

attempt to use $\cos(2A+B) = \cos 2A \cos B - \sin 2A \sin B$ (may be seen later) ***M1***

attempt to use any double angle formulae (seen anywhere) ***M1***

attempt to find either $\sin A$ or $\cos B$ (seen anywhere) ***M1***

$\cos A = \frac{2}{3} \Rightarrow \sin A \left(= \sqrt{1 - \frac{4}{9}}\right) = \frac{\sqrt{5}}{3}$ ***(A1)***

$\sin B = \frac{1}{3} \Rightarrow \cos B \left(= \sqrt{1 - \frac{1}{9}} = \frac{\sqrt{8}}{3}\right) = \frac{2\sqrt{2}}{3}$ ***A1***

$\cos 2A \left(= 2\cos^2 A - 1\right) = -\frac{1}{9}$ ***A1***

$\sin 2A \left(= 2 \sin A \cos A\right) = \frac{4\sqrt{5}}{9}$ ***A1***

So $\cos(2A+B) = \left(-\frac{1}{9}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{4\sqrt{5}}{9}\right)\left(\frac{1}{3}\right)$

$= -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$ ***AG***

[7 marks]

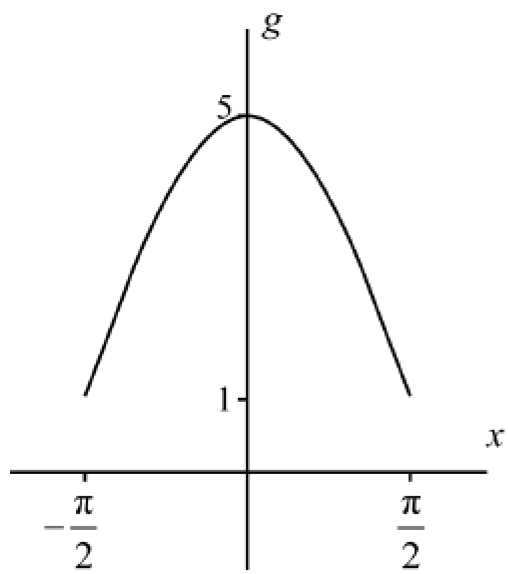
12. [Maximum mark: 7] 18N.1.AHL.TZ0.H_3

Consider the function $g(x) = 4\cos x + 1$, $a \leqslant x \leqslant \frac{\pi}{2}$ where $a < \frac{\pi}{2}$.

(a) For $a = -\frac{\pi}{2}$, sketch the graph of $y = g(x)$. Indicate clearly the maximum and minimum values of the function. [3]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.



concave down and symmetrical over correct domain ***A1***

indication of maximum and minimum values of the function (correct range) ***A1A1***

[3 marks]

(b) Write down the least value of a such that g has an inverse. [1]

Markscheme

$a = 0$ ***A1***

Note: Award ***A1*** for $a = 0$ only if consistent with their graph.

[1 mark]

(c.i) For the value of a found in part (b), write down the domain of g^{-1}. [1]

Markscheme

$1 \leqslant x \leqslant 5$ ***A1***

Note: Allow FT from their graph.

[1 mark]

(c.ii) For the value of a found in part (b), find an expression for $g^{-1}(x)$. [2]

Markscheme

$y = 4\cos x + 1$

$x = 4\cos y + 1$

$\frac{x-1}{4} = \cos y$ ***(M1)***

$\Rightarrow y = \arccos\left(\frac{x-1}{4}\right)$

$\Rightarrow g^{-1}(x) = \arccos\left(\frac{x-1}{4}\right)$ ***A1***

[2 marks]

13. [Maximum mark: 19] SPM.2.AHL.TZ0.12

(a) Show that $\cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$. [1]

Markscheme

stating the relationship between cot and $\tan$ and stating the identity for $\tan 2\theta$ ***M1***

$\cot 2\theta = \frac{1}{\tan 2\theta}$ and $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$ ***AG***

[1 mark]

(b) Verify that $x = \tan\theta$ and $x = -\cot\theta$ satisfy the equation $x^2 + (2\cot 2\theta)x - 1 = 0.$ [7]

Markscheme

METHOD 1

attempting to substitute $\tan\theta$ for x and using the result from (a) ***M1***

LHS = $\tan^2\theta + 2\tan\theta\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$\tan^2\theta + 1 - \tan^2\theta - 1 = 0$(= RHS) ***A1***

so $x = \tan\theta$ satisfies the equation ***AG***

attempting to substitute $-\cot\theta$ for x and using the result from (a) ***M1***

LHS = $\cot^2\theta - 2\cot\theta\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$= \frac{1}{\tan^2\theta} - \left(\frac{1-\tan^2\theta}{\tan^2\theta}\right) - 1$ ***A1***

$\frac{1}{\tan^2\theta} - \frac{1}{\tan^2\theta} + 1 - 1 = 0$(= RHS) ***A1***

so $x = -\cot\theta$ satisfies the equation ***AG***

METHOD 2

let $\alpha = \tan\theta$ and $\beta = -\cot\theta$

attempting to find the sum of roots ***M1***

$\alpha + \beta = \tan\theta - \frac{1}{\tan\theta}$

$= \frac{\tan^2\theta - 1}{\tan\theta}$ ***A1***

$= -2\cot 2\theta$ (from part (a)) ***A1***

attempting to find the product of roots ***M1***

$\alpha\beta = \tan\theta \times (-\cot\theta)$ ***A1***

= −1 ***A1***

the coefficient of x and the constant term in the quadratic are $2\cot 2\theta$ and −1 respectively ***R1***

hence the two roots are $\alpha = \tan\theta$ and $\beta = -\cot\theta$ ***AG***

[7 marks]

(c) Hence, or otherwise, show that the exact value of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. [5]

Markscheme

METHOD 1

$x = \tan\frac{\pi}{12}$ and $x = -\cot\frac{\pi}{12}$ are roots of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$ ***R1***

Note: Award ***R1*** if only $x = \tan\frac{\pi}{12}$ is stated as a root of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$.

$x^2 + 2\sqrt{3}x - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$x = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ $(-\cot\frac{\pi}{12} < 0)$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

METHOD 2

attempting to substitute $\theta = \frac{\pi}{12}$ into the identity for $\tan 2\theta$ ***M1***

$\tan\frac{\pi}{6} = \frac{2\tan\frac{\pi}{12}}{1-\tan^2\frac{\pi}{12}}$

$\tan^2\frac{\pi}{12} + 2\sqrt{3}\tan\frac{\pi}{12} - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$\tan\frac{\pi}{12} = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

[5 marks]

(d) Using the results from parts (b) and (c) find the exact value of $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$.

Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$. [6]

Markscheme

$\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$ is the sum of the roots of $x^2 + \left(2\cot\frac{\pi}{12}\right)x - 1 = 0$ ***R1***

$\tan\frac{\pi}{24} - \cot\frac{\pi}{24} = -2\cot\frac{\pi}{12}$ ***A1***

$= \frac{-2}{2-\sqrt{3}}$ ***A1***

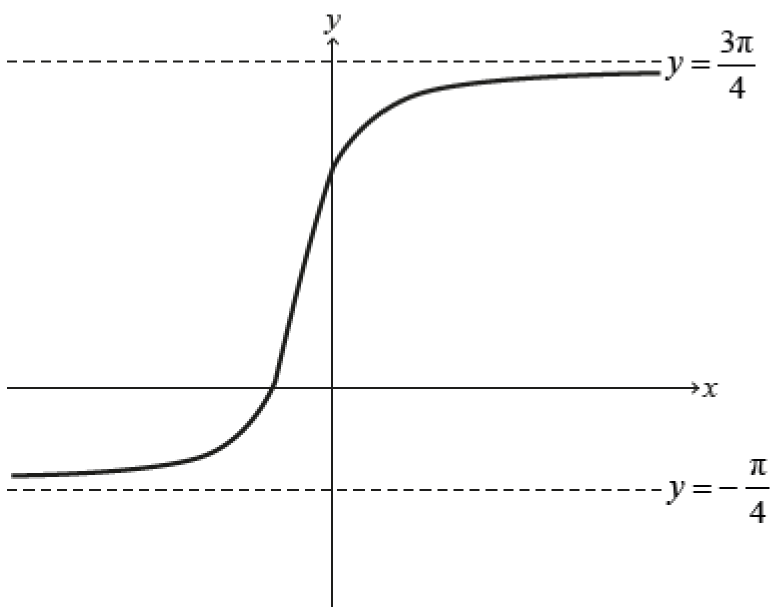
attempting to rationalise **their** denominator ***(M1)***

$= -4 - 2\sqrt{3}$ ***A1A1***

[6 marks]

14. [Maximum mark: 19] 21M.1.AHL.TZ2.12

The following diagram shows the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.



(a) Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. [3]

Markscheme

EITHER

horizontal stretch/scaling with scale factor $\frac{1}{2}$

Note: Do not allow 'shrink' or 'compression'

followed by a horizontal translation/shift $\frac{1}{2}$ units to the left ***A2***

Note: Do not allow 'move'

OR

horizontal translation/shift 1 unit to the left

followed by horizontal stretch/scaling with scale factor $\frac{1}{2}$ ***A2***

THEN

vertical translation/shift up by $\frac{\pi}{4}$ (or translation through $\begin{pmatrix} 0 \\ \frac{\pi}{4} \end{pmatrix}$ ***A1***

(may be seen anywhere)

[3 marks]

(b) Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where $p,\ q > 0$ and $pq < 1$. [4]

Markscheme

let $\alpha = \arctan p$ and $\beta = \arctan q$ ***M1***

$p = \tan\alpha$ and $q = \tan\beta$ ***(A1)***

$\tan(\alpha+\beta) = \frac{p+q}{1-pq}$ ***A1***

$\alpha+\beta = \arctan\left(\frac{p+q}{1-pq}\right)$ ***A1***

so $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where $p,\ q > 0$ and $pq < 1.$ ***AG***

[4 marks]

(c) Verify that $\arctan(2x+1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R},\ x > 0.$ [3]

Markscheme

METHOD 1

$\frac{\pi}{4} = \arctan 1$ (or equivalent) ***A1***

$\arctan\left(\frac{x}{x+1}\right) + \arctan 1 = \arctan\left(\frac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}(1)}\right)$ ***A1***

$= \arctan\left(\frac{\frac{x+x+1}{x+1}}{\frac{x+1-x}{x+1}}\right)$ ***A1***

$= \arctan(2x+1)$ ***AG***

METHOD 2

$\tan\frac{\pi}{4} = 1$ (or equivalent) ***A1***

Consider $\arctan(2x+1) - \arctan\left(\frac{x}{x+1}\right) = \frac{\pi}{4}$

$\tan\left(\arctan(2x+1) - \arctan\left(\frac{x}{x+1}\right)\right)$

$= \arctan\left(\frac{2x+1-\frac{x}{x+1}}{1+\frac{x(2x+1)}{x+1}}\right)$ ***A1***

$= \arctan\left(\frac{(2x+1)(x+1)-x}{x+1+x(2x+1)}\right)$ ***A1***

$= \arctan 1$ ***AG***

METHOD 3

$\tan\left(\arctan(2x+1)\right) = \tan\left(\arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}\right)$

$\tan\frac{\pi}{4} = 1$ (or equivalent) ***A1***

$\text{LHS} = 2x+1$ ***A1***

$\text{RHS} = \frac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}} (= 2x+1)$ ***A1***

[3 marks]

(d) Using mathematical induction and the result from part (b), prove that $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. [9]

Markscheme

let $\mathrm{P}(n)$ be the proposition that $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$

consider $\mathrm{P}(1)$

when $n=1,\ \sum\limits_{r=1}^{1}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{1}{2}\right)=\mathrm{RHS}$ and so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true, ie.

$\sum\limits_{r=1}^{k}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{k}{k+1}\right)\ (k\in\mathbb{Z}^+)$ ***M1***

Note: Award ***M0*** for statements such as "let $n=k$".

Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

consider $\mathrm{P}(k+1)$:

$\sum\limits_{r=1}^{k+1}\arctan\left(\frac{1}{2r^2}\right)=\sum\limits_{r=1}^{k}\arctan\left(\frac{1}{2r^2}\right)+\arctan\left(\frac{1}{2(k+1)^2}\right)$
(M1)

$=\arctan\left(\frac{k}{k+1}\right)+\arctan\left(\frac{1}{2(k+1)^2}\right)$ ***A1***

$=\arctan\left(\frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\left(\frac{k}{k+1}\right)\left(\frac{1}{2(k+1)^2}\right)}\right)$ ***M1***

$=\arctan\left(\frac{(k+1)\left(2k^2+2k+1\right)}{2(k+1)^3-k}\right)$ ***A1***

Note: Award ***A1*** for correct numerator, with $(\mathrm{k}+1)$ factored. Denominator does not need to be simplified

$=\arctan\left(\frac{(k+1)\left(2k^2+2k+1\right)}{2k^3+6k^2+5k+2}\right)$ ***A1***

Note: Award ***A1*** for denominator correctly expanded. Numerator does not need to be simplified. These two ***A*** marks may be awarded in any order

$$= \arctan\left(\frac{(k+1)(2k^2+2k+1)}{(k+2)(2k^2+2k+1)}\right) = \arctan\left(\frac{k+1}{k+2}\right)$$ ***A1***

Note: The word 'arctan' must be present to be able to award the last three A marks

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so

$\mathrm{P}(n)$ is true for for $n \in \mathbb{Z}^+$ ***R1***

Note: Award the final ***R1*** mark provided at least four of the previous marks have been awarded.
Note: To award the final ***R1***, the truth of $\mathrm{P}(k)$ must be mentioned. '$\mathrm{P}(k)$ implies $\mathrm{P}(k+1)$' is insufficient to award the mark.

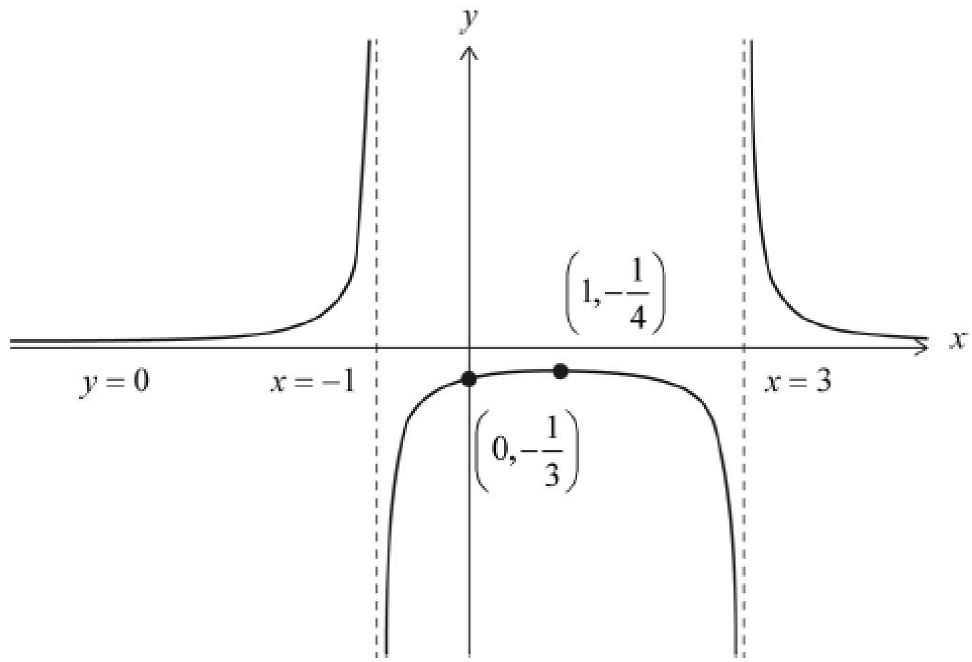
[9 marks]

15. [Maximum mark: 20] 22M.1.AHL.TZ2.11

A function f is defined by $f(x) = \frac{1}{x^2-2x-3}$, where $x \in \mathbb{R},\ x \neq -1,\ x \neq 3$.

(a) Sketch the curve $y = f(x)$, clearly indicating any asymptotes with their equations. State the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [6]

Markscheme



y-intercept $\left(0, -\frac{1}{3}\right)$ ***A1***

Note: Accept an indication of $-\frac{1}{3}$ on the y-axis.

vertical asymptotes $x = -1$ and $x = 3$ ***A1***

horizontal asymptote $y = 0$ ***A1***

uses a valid method to find the x-coordinate of the local maximum point ***(M1)***

Note: For example, uses the axis of symmetry or attempts to solve $f\prime(x) = 0$.

local maximum point $\left(1, -\frac{1}{4}\right)$ ***A1***

Note: Award ***(M1)A0*** for a local maximum point at $x = 1$ and coordinates not given.

three correct branches with correct asymptotic behaviour and the key features in approximately correct relative positions to each other ***A1***

[6 marks]

A function g is defined by $g(x) = \frac{1}{x^2-2x-3}$, where $x \in \mathbb{R},\ x > 3$.

The inverse of g is g^{-1}.

(b.i) Show that $g^{-1}(x) = 1 + \frac{\sqrt{4x^2+x}}{x}$. [6]

Markscheme

$x = \frac{1}{y^2-2y-3}$ ***M1***

Note: Award ***M1*** for interchanging x and y (this can be done at a later stage).

EITHER

attempts to complete the square ***M1***

$y^2 - 2y - 3 = (y-1)^2 - 4$ ***A1***

$x = \frac{1}{(y-1)^2-4}$

$(y-1)^2 - 4 = \frac{1}{x} \left((y-1)^2 = 4 + \frac{1}{x}\right)$ ***A1***

$y - 1 = \pm\sqrt{4 + \frac{1}{x}} \left(= \pm\sqrt{\frac{4x+1}{x}}\right)$

OR

attempts to solve $xy^2 - 2xy - 3x - 1 = 0$ for y ***M1***

$y = \frac{-(-2x)\pm\sqrt{(-2x)^2+4x(3x+1)}}{2x}$ ***A1***

Note: Award ***A1*** even if $-$ (in $\pm$) is missing

$= \frac{2x\pm\sqrt{16x^2+4x}}{2x}$ ***A1***

THEN

$= 1 \pm \frac{\sqrt{4x^2+x}}{x}$ ***A1***

$y > 3$ and hence $y = 1 - \frac{\sqrt{4x^2+x}}{x}$ is rejected ***R1***

Note: Award ***R1*** for concluding that the expression for y must have the '$+$' sign.
The ***R1*** may be awarded earlier for using the condition $x > 3$.

$y = 1 + \frac{\sqrt{4x^2+x}}{x}$

$g^{-1}(x) = 1 + \frac{\sqrt{4x^2+x}}{x}$ ***AG***

[6 marks]

(b.ii) State the domain of g^{-1}. [1]

Markscheme

domain of g^{-1} is $x > 0$ ***A1***

[1 mark]

A function h is defined by $h(x) = \arctan \frac{x}{2}$, where $x \in \mathbb{R}$.

(c) Given that $(h \circ g)(a) = \frac{\pi}{4}$, find the value of a.

Give your answer in the form $p + \frac{q}{2}\sqrt{r}$, where $p,\ q,\ r \in \mathbb{Z}^+$. [7]

Markscheme

attempts to find $(h \circ g)(a)$ ***(M1)***

$(h \circ g)(a) = \arctan\left(\frac{g(a)}{2}\right)$ $\left((h \circ g)(a) = \arctan\left(\frac{1}{2(a^2-2a-3)}\right)\right)$ ***(A1)***

$\arctan\left(\frac{g(a)}{2}\right)=\frac{\pi}{4}\quad\left(\arctan\left(\frac{1}{2(a^2-2a-3)}\right)=\frac{\pi}{4}\right)$

attempts to solve for $g(a)$ ***M1***

$\Rightarrow g(a)=2\quad\left(\frac{1}{(a^2-2a-3)}=2\right)$

EITHER

$\Rightarrow a=g^{-1}(2)$ ***A1***

attempts to find their $g^{-1}(2)$ ***M1***

$a=1+\frac{\sqrt{4(2)^2+2}}{2}$ ***A1***

Note: Award all available marks to this stage if x is used instead of a.

OR

$\Rightarrow 2a^2-4a-7=0$ ***A1***

attempts to solve their quadratic equation ***M1***

$a=\frac{-(-4)\pm\sqrt{(-4)^2+4(2)(7)}}{4}\quad\left(=\frac{4\pm\sqrt{72}}{4}\right)$ ***A1***

Note: Award all available marks to this stage if x is used instead of a.

THEN

$a=1+\frac{3}{2}\sqrt{2}$ (as $a>3$) ***A1***

$(p=1,\ q=3,\ r=2)$

Note: Award ***A1*** for $a = 1 + \frac{1}{2}\sqrt{18}\ (p = 1,\ q = 1,\ r = 18)$

[7 marks]