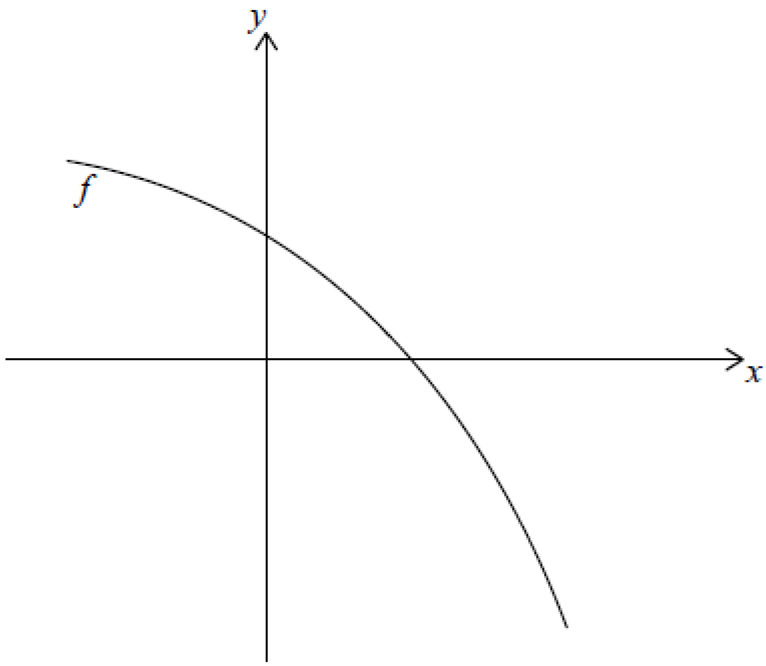
Volume of solid of revolution [60 marks]

1. [Maximum mark: 5] 19M.2.SL.TZ2.S_2

Let $f(x) = 4 - 2\mathrm{e}^{x}$. The following diagram shows part of the graph of f.



(a) Find the x-intercept of the graph of f. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

valid approach ***(M1)***

eg $f(x) = 0,\ 4 - 2\mathrm{e}^{x} = 0$

0.693147

x = ln 2 (exact), 0.693 ***A1 N2***

[2 marks]

(b) The region enclosed by the graph of f, the x-axis and the y-axis is rotated 360° about the x-axis. Find the volume of the solid formed. [3]

Markscheme

attempt to substitute either their correct limits or the function into formula ***(M1)***

involving f^2

eg $\int_0^{0.693} f^2$, $\pi \int (4 - 2e^x)^2 dx$, $\int_0^{\ln 2} (4 - 2e^x)^2$

3.42545

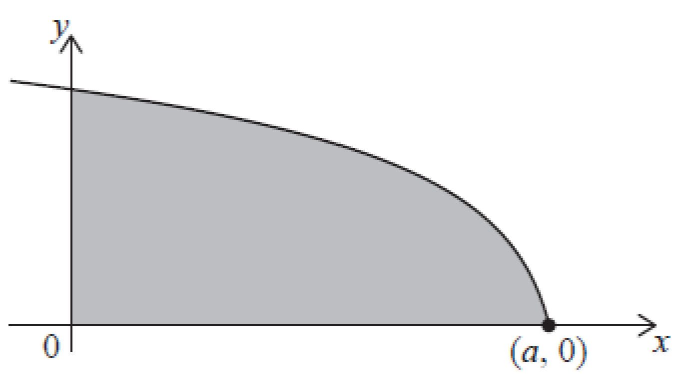
volume = 3.43 ***A2 N3***

[3 marks]

2. [Maximum mark: 7] 20N.1.SL.TZ0.S_3

Let $f(x) = \sqrt{12 - 2x},\ x \le a$. The following diagram shows part of the graph of f.

The shaded region is enclosed by the graph of f, the x-axis and the y-axis.



The graph of f intersects the x-axis at the point $(a,\ 0)$.

(a) Find the value of a. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

recognize $f(x) = 0$ ***(M1)***

eg $\sqrt{12 - 2x} = 0,\ \ 2x = 12$

$a = 6$ (accept $x = 6$, $(6,\ 0)$) ***A1 N2***

[2 marks]

(b) Find the volume of the solid formed when the shaded region is revolved 360° about the x-axis. [5]

Markscheme

attempt to substitute either **their** limits or the function into volume formula (must involve f^2) ***(M1)***

eg $\int_0^6 f^2\ \mathrm{d}\ x$, $\pi \int \left(\sqrt{12 - 2x}\right)^2$, $\pi \int_0^6 12 - 2x\ \mathrm{d}\ x$

correct integration of each term ***A1 A1***

eg $12x - x^2$, $12x - x^2 + c$, $\left[12x - x^2\right]_0^6$

substituting limits into **their integrated** function and subtracting (in any order) ***(M1)***

eg

$\pi\left(12(6) - (6)^2\right) - \pi(0)$, $72\pi - 36\pi$, $\left(12(6) - (6)^2\right) - (0)$

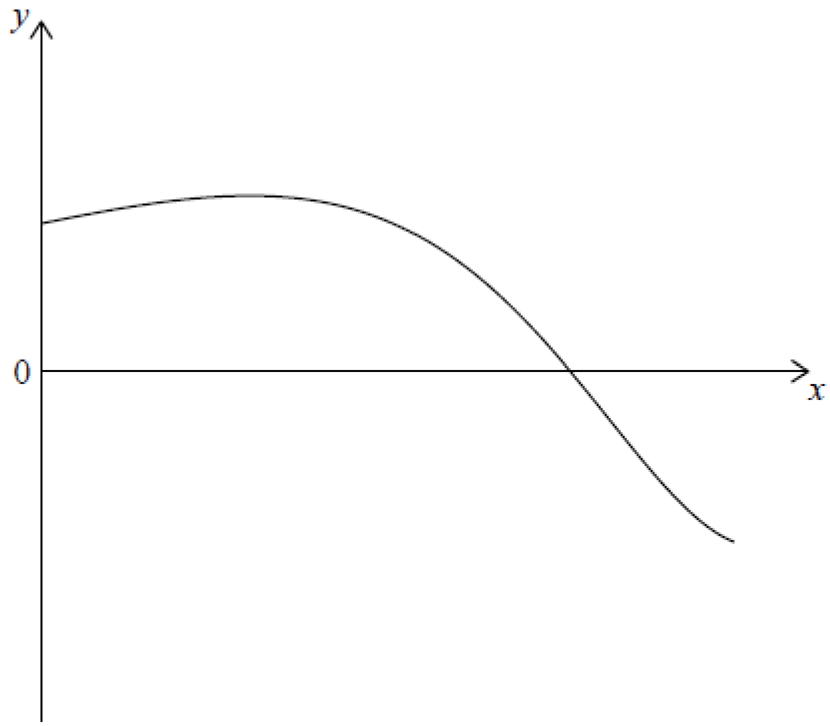
Note: Award ***M0*** if candidate has substituted into f, f^2 or $f\prime$.

volume$= 36\pi$ ***A1 N2***

[5 marks]

3. [Maximum mark: 5] 18M.2.SL.TZ2.S_3

Let $f(x) = \sin(e^x)$ for $0 \le x \le 1.5$. The following diagram shows the graph of f.



(a) Find the *x*-intercept of the graph of f. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

valid approach ***(M1)***

eg $f(x) = 0, \quad e^x = 180$ or 0…

1.14472

$x = \ln \pi$ (exact), 1.14 ***A1 N2***

[2 marks]

(b) The region enclosed by the graph of f, the *y*-axis and the *x*-axis is rotated 360° about the *x*-axis.

Find the volume of the solid formed. [3]

Markscheme

attempt to substitute either their **limits** or the function into formula involving f^2. ***(M1)***

eg $\int_0^{1.14} f^2, \ \pi \int (\sin\ (e^x))^2 dx, \ 0.795135$

2.49799

volume = 2.50 ***A2 N3***

[3 marks]

4. [Maximum mark: 8] SPM.1.AHL.TZ0.14

The graph of $y = -x^3$ is transformed onto the graph of $y = 33 - 0.08x^3$ by a translation of a units vertically and a stretch parallel to the x-axis of scale factor b.

(a.i) Write down the value of a. [1]

Markscheme

$a = 33$ ***A1***

[1 mark]

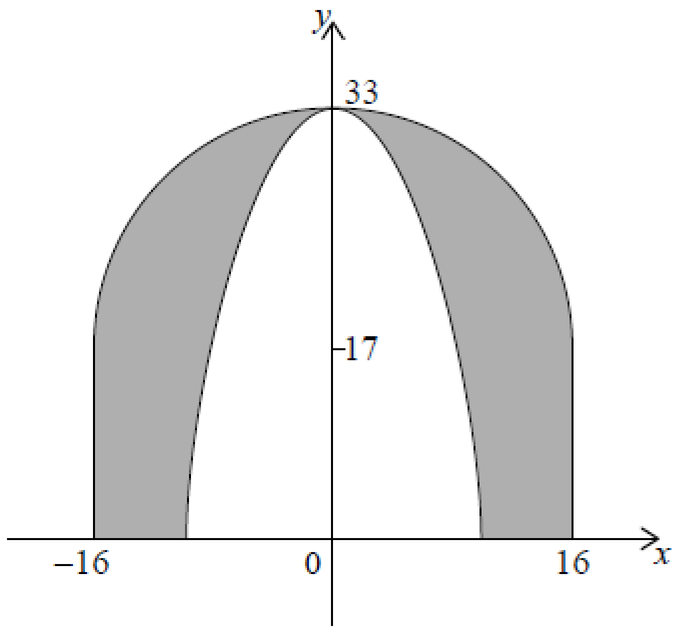
(a.ii) Find the value of b. [2]

Markscheme

$\frac{1}{\sqrt[3]{0.08}} = 2.32$ ***M1A1***

[2 marks]

(b) The outer dome of a large cathedral has the shape of a hemisphere of diameter 32 m, supported by vertical walls of height 17 m. It is also supported by an inner dome which can be modelled by rotating the curve $y = 33 - 0.08x^3$ through 360° about the y-axis between $y = 0$ and $y = 33$, as indicated in the diagram.



[5]

Find the volume of the space between the two domes.

Markscheme

volume within outer dome

$\frac{2}{3}\pi + 16^3 + \pi \times 16^2 \times 17 = 22\,250.85$ ***M1A1***

volume within inner dome

$\pi \int_0^{33} \left(\frac{33-y}{0.08}\right)^{\frac{2}{3}} \mathrm{d}y = 3446.92$ ***M1A1***

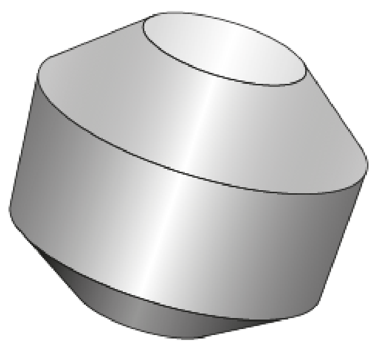
volume between = 22 250.85 − 3446.92 = 18 803.93 m^3 ***A1***

[5 marks]

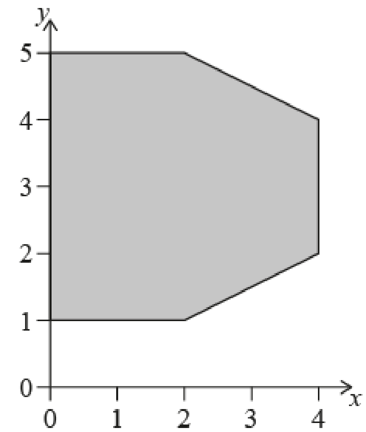
5. [Maximum mark: 6] 23M.1.AHL.TZ1.15

The solid shown is formed by rotating the hexagon with vertices $(2,\ 1)$, $(0,\ 1)$, $(0,\ 5)$, $(2,\ 5)$, $(4,\ 4)$ and $(4,\ 2)$ about the y-axis.

Solid



Hexagon



Find the volume of this solid. [6]

Markscheme

METHOD 1 Using the volume formula

volume of a "full" or "half" cylinder (seen anywhere) ***(A1)***

$\pi \int_2^4 4^2 \mathrm{d}y, \quad \pi \times 4^2 \times 2, \quad 32\pi \ \left(100.53\ldots\right)$ **OR**

$\pi \int_2^3 4^2 \mathrm{d}y, \quad \pi \times 4^2 \times 1, \quad 16\pi \ \left(50.265\ldots\right)$

one correct equation for the diagonal lines (seen anywhere) ***(A1)***

$y = \frac{1}{2}x$ or $y = 6 - \frac{1}{2}x$

attempt to write one equation x in terms of y ***(M1)***

$x = 2y, \ x = \ 12 - 2y$

EITHER (symmetry plus the volume of the "half" cylinder)

recognition of symmetry between $y = 1$ and $y = 3$ ***(M1)***

$2\pi\left(\int_1^2 (2y)^2 \,\mathrm{d}y + \int_2^3 4^2 \,\mathrm{d}y\right)$ ***(A1)***

OR (symmetry plus volume of the "full" cylinder)

recognition of symmetry between $y = 1$ and $y = 2$ ***(M1)***

$2\pi\left(\int_1^2 (2y)^2 \,\mathrm{d}y\right) + \int_2^4 4^2 \,\mathrm{d}y$ ***(A1)***

OR (calculation of separate parts) ***(M1)***

$\pi\left(\int_1^2 (2y)^2 \,\mathrm{d}y + \int_2^4 4^2 \,\mathrm{d}y + \int_4^5 (-2y + 12)^2 \,\mathrm{d}y\right)$ ***(A1)***

THEN

(volume of the solid $=$) $159 \ \left(159.174\ldots,\ \frac{152\pi}{3}\right)$ ***A1***

METHOD 2 Geometric approach using cones and cylinders

volume of a cylinder (seen anywhere) ***(A1)***

$\pi \times 4^2 \times 2,\ 32\pi\ (100.53\ldots)$ (a full cylinder) **OR**

$\pi \times 4^2 \times 1,\ 16\pi\ (50.265\ldots)$ (a half cylinder)

using volume of cone formula to find the volume of the truncated cone ***(M1)***

correct expression to find the volume of the truncated cone (seen anywhere) ***(A1)***

$\frac{1}{3}\left(\pi \times 4^2 \times 2 - \pi \times 2^2 \times 1\right)$

attempt to find an expression for total volume using symmetry or individual parts ***(M1)***

correct expression for total volume ***(A1)***

$2\left(\frac{1}{3}\left(\pi 4^2 \times 2 - \pi 2^2 \times 1\right) + \pi 4^2 \times 1\right)$ **OR**
$\frac{1}{3}\left(\pi 4^2 \times 2 - \pi 2^2 \times 1\right) + \pi 4^2 \times 2 + \frac{1}{3}\left(\pi 4^2 \times 2 - \pi 2^2 \times 1\right)$

(volume of the solid $=$) 159 $\left(159.174\ldots,\ \frac{152\pi}{3}\right)$ ***A1***

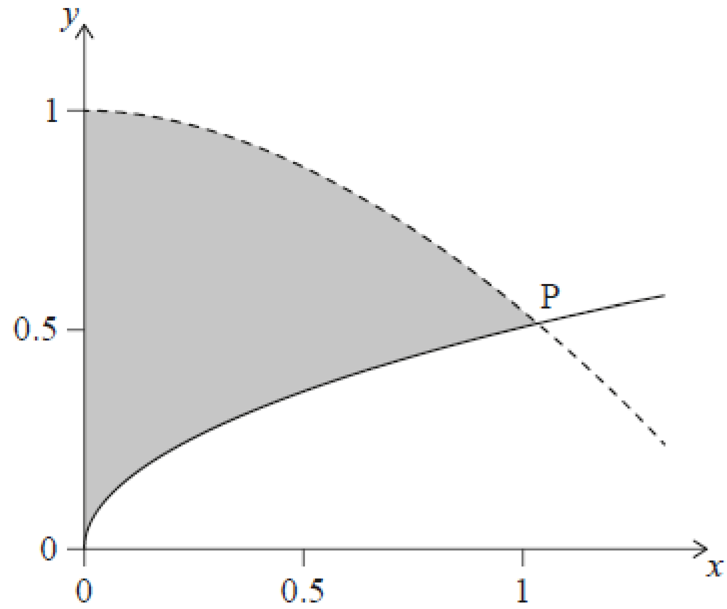
Note: There are other valid approaches possible.

[6 marks]

6. [Maximum mark: 9] 23M.1.AHL.TZ2.16

The following diagram shows parts of the curves of $y = \cos x$ and $y = \frac{\sqrt{x}}{2}$.

P is the point of intersection of the two curves.



(a) Use your graphic display calculator to find the coordinates of P. [2]

Markscheme

$(1.04,\ 0.509)$ $((1.03667\ldots,\ 0.509085\ldots))$ ***A1A1***

[2 marks]

The shaded region is rotated 360° **about the y-axis** to form a volume of revolution V.

(b) Express V as the sum of two definite integrals. [5]

Markscheme

attempt to make x the subject for either function ***(M1)***

$x = 4y^2,\ x = \cos^{-1} y$ ***A1A1***

attempt to use $V = \pi \int x^2 \mathrm{d}y$ ***(M1)***

$$V = \pi \int_0^{0.509085\ldots} \left(4y^2\right)^2 \mathrm{d}y + \pi \int_{0.509085\ldots}^{1} \left(\cos^{-1} y\right)^2 \mathrm{d}y$$ ***A1***

[5 marks]

(c) Hence find the value of V. [2]

Markscheme

$= 1.15$ (units^3) ***A2***

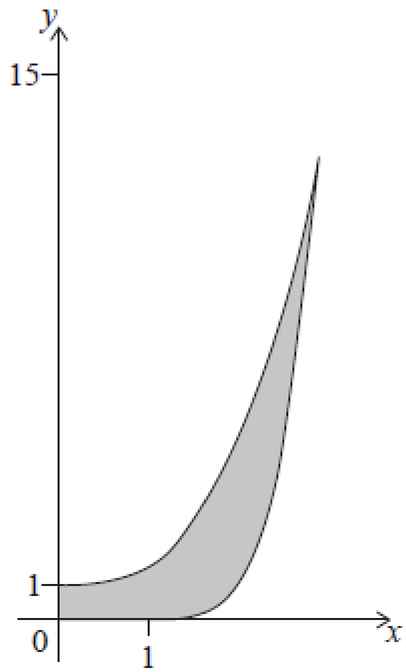
Note: Do not ***FT*** from part (b) to part (c).

Award ***A1*** for $1.1.$ with no previous working.

[2 marks]

7. [Maximum mark: 13] 22N.2.AHL.TZ0.5

Adesh is designing a glass. The glass has an inner surface and an outer surface. Part of the cross section of his design is shown in the following graph, where the shaded region represents the glass. The two surfaces meet at the top of the glass. 1 unit represents $1\,\text{cm}$.



The inner surface is modelled by $f(x) = \frac{1}{2}x^3 + 1$ for $0 \leq x \leq p$.

The outer surface is modelled by $g(x)\begin{cases} 0 & \text{for } 0 \leq x < 1 \\ (x-1)^4 & \text{for } 1 \leq x \leq p \end{cases}$.

(a) Find the value of p. [2]

Markscheme

$\frac{1}{2}x^3 + 1 = (x-1)^4$ *(M1)*

$(p =)\ 2.91\,\text{cm}\ \ (2.91082\ldots)$ *A1*

[2 marks]

The glass design is finished by rotating the shaded region in the diagram through 360° about the y-axis.

(b) Find the volume of liquid that can be contained inside the finished glass. [5]

Markscheme

attempt to make x (or x^2) the subject of $y = \frac{1}{2}x^3 + 1$ ***(M1)***

$x = \sqrt[3]{2(y-1)}$ (or $x^2 = (2(y-1))^{\frac{2}{3}}$) ***(A1)***

(upper limit =) $13.3(315\ldots)$ ***(A1)***

$V = \int_1^{13.3315\ldots} \pi(2(y-1))^{\frac{2}{3}}\,\mathrm{d}\,y$ ***(M1)***

Note: Award ***(M1)*** for setting up correct integral squaring their expression for x with both correct lower limit and their upper limit, and π. Condone omission of $\mathrm{d}\,y$.

$= 197\,\mathrm{cm}^3 \ \ (196.946\ldots)$ ***A1***

[5 marks]

(c) Find the volume of the region between the two surfaces of the finished glass. [6]

Markscheme

$x = y^{\frac{1}{4}} + 1$ (or $x^2 = \left(y^{\frac{1}{4}} + 1\right)^2$) ***(A1)***

$V_2 = \int_0^{13.3315\ldots} \pi\left(y^{\frac{1}{4}} + 1\right)^2\,\mathrm{d}\,y$ ***(M1)(A1)***

Note: Award ***(M1)*** for setting up correct integral squaring their expression for x with their upper limit, and π. Award ***(A1)*** for lower limit of 0, dependent on ***M1***. Condone omission of $\mathrm{d}\ y$. If a candidate found an area in part (b), do not award ***FT*** for another area calculation seen in part (c).

$= 271.87668\ldots$ ***(A1)***

Note: Accept $271.038\ldots$ from use of 3sf in the upper limit.

subtracting their volumes ***(M1)***

$271.87668\ldots - 196.946\ldots$

$= 74.9\,\mathrm{cm}^3 \quad (74.93033\ldots)$ ***A1***

Note: Accept any answer that rounds to $75\ (\mathrm{cm}^3)$. If a candidate found an area in part (b), do not award ***FT*** for another area calculation seen in part (c).

[6 marks]

8. [Maximum mark: 7] 18M.1.SL.TZ1.S_5

Let $f\left(x\right) = \frac{1}{\sqrt{2x-1}}$, for $x > \frac{1}{2}$.

(a) Find $\int \left(f\left(x\right)\right)^2 \mathrm{d}x$. [3]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

correct working ***(A1)***

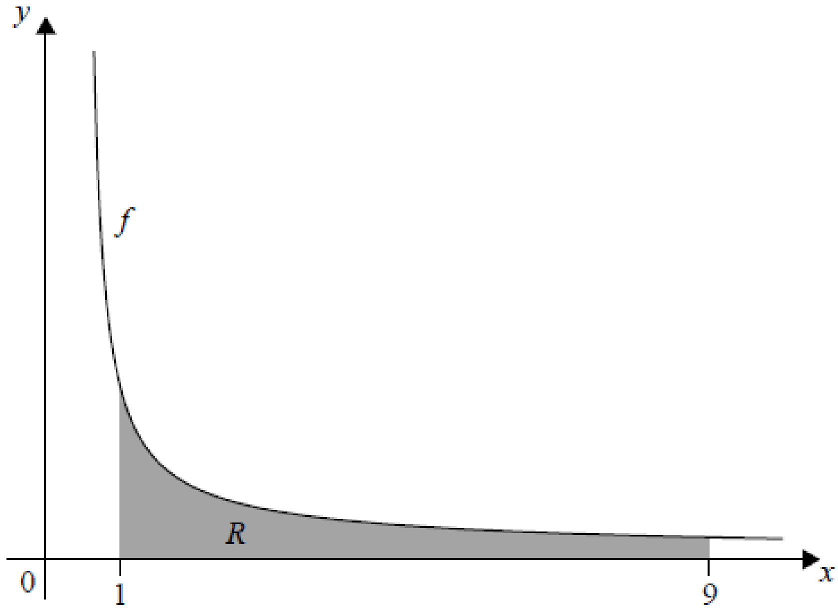
eg $\int \frac{1}{2x-1}\mathrm{d}x,\ \int (2x-1)^{-1},\ \frac{1}{2x-1},\ \int \left(\frac{1}{\sqrt{u}}\right)^2 \frac{\mathrm{d}u}{2}$

$\int (f(x))^2 \mathrm{d}x = \frac{1}{2}\ln(2x-1) + c$ ***A2 N3***

Note: Award ***A1*** for $\frac{1}{2}\ln(2x-1)$.

[3 marks]

(b) Part of the graph of *f* is shown in the following diagram.



The shaded region *R* is enclosed by the graph of *f*, the *x*-axis, and the lines *x* = 1 and *x* = 9 . Find the volume of the solid formed when *R* is revolved 360° about the *x*-axis. [4]

Markscheme

attempt to substitute either limits or the function into formula involving f^2 (accept absence of π / d*x*) ***(M1)***

eg $\int_1^9 y^2 \mathrm{d}x,\ \pi \int \left(\frac{1}{\sqrt{2x-1}}\right)^2 \mathrm{d}x,\ \left[\frac{1}{2}\ln(2x-1)\right]_1^9$

substituting limits into **their** integral and subtracting (in any order) ***(M1)***

eg $\frac{\pi}{2}(\ln(17) - \ln(1)),\ \pi\left(0 - \frac{1}{2}\ln(2 \times 9 - 1)\right)$

correct working involving calculating a log value or using log law ***(A1)***

eg $\ln(1) = 0,\ \ln\left(\frac{17}{1}\right)$

$\frac{\pi}{2}\ln 17 \quad \left(\text{accept } \pi \ln\sqrt{17}\right)$ ***A1 N3***

Note: Full ***FT*** may be awarded as normal, from their incorrect answer in part (a), however, do not award the final two ***A*** marks unless they involve logarithms.

[4 marks]