Harmonic form

The goal of this presentation is to understand how to turn a sum/difference of sine and cosine into a single trigonometric function.

Before you start make sure that you remember the compound angle formulae:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha$$
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When we want to calculate for example $\sin\frac{\pi}{12}$, we will write it as $\sin(\frac{\pi}{3}-\frac{\pi}{4})$ and apply the second formula. Now we want to learn to use the formulae in the opposite direction.

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Sidenote

Notation.

Recall that $\sin^{-1}(x)$ denotes the inverse of sine. Similarly $\cos^{-1}(x)$ and $\tan^{-1}(x)$ are inverses of cosine and tangent respectively. Another way to write these is $\arcsin(x)$, $\arccos(x)$ and $\arctan(x)$.

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Example 1

Let's start with a simple question. Consider a function

$$f(x) = 3\cos x + 4\sin x$$

What is the range of this function? Think about it for a moment.

I will use the following argument: the range of $\cos x$ is $[-1, 1]$, so the range of $3\cos x$ is $[-3, 3]$, similarly the range of $4\sin x$ is $[-4, 4]$, so the range of $3\cos x + 4\sin x$ is $[-7, 7]$. Correct? **NO!**

The range of $f(x) = 3\cos x + 4\sin x$ is **not** $[-7, 7]$. The reason the above argument is wrong is that $\cos x$ and $\sin x$ are maximal/minimal for different values of x (there is no x for which $\cos x = 1$ and $\sin x = 1$ simultaneously).

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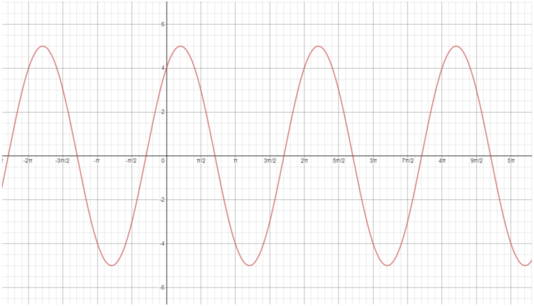

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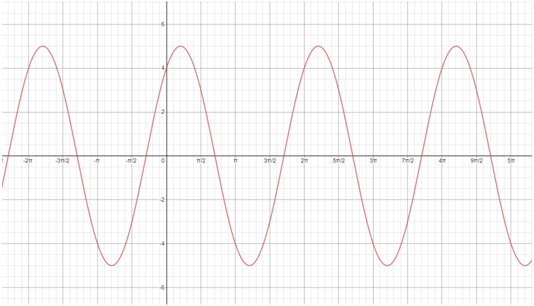

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Let's draw a triangle with the angle θ. We want to turn 3 into $\cos\theta$, so we will make the adjacent side equal to 3 (and opposite side equal to 4).

The hypotenuse is then 5, so we have $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$. Also $\theta = \arctan\left(\frac{4}{3}\right)$. We can actually calculate that $\theta \approx 0.927$, but I'll stick with $\theta = \arctan\left(\frac{4}{3}\right)$.

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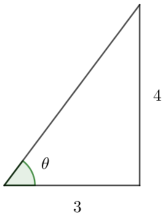

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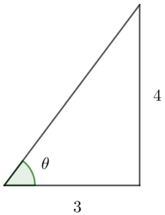

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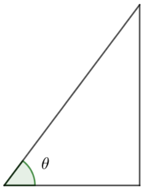

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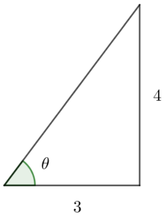

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All of this allows us to write:

$$3\cos x + 4\sin x = 5\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$$

which gives:

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Example 2

Find the range of $f(x) = 2\sin x - \cos x$.

We will try to write $f(x)$ in the form $R\sin(x - \theta)$. So we want to change the 2 into cos and 1 into sin. We can draw a triangle with adjacent side 2 and the opposite side 1.

The hypotenuse is $\sqrt{5}$ and $\theta = \arctan\left(\frac{1}{2}\right)$.

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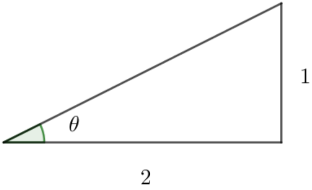

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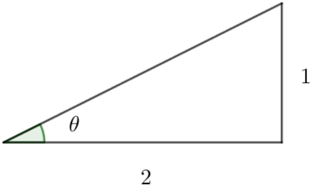

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We will try to write $f(x)$ in the form $R\sin(x+\theta)$. So we want to change the 2 into cos and $\sqrt{5}$ into sin. We will draw a triangle with adjacent side 2 and opposite side $\sqrt{5}$:

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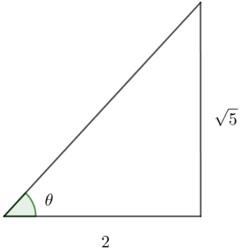

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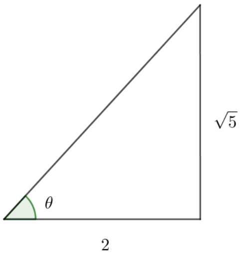

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