# Harmonic form

Tomasz Lechowski DP1 AA HL October 14, 2024 1/1

Before you start make sure that you remember the compound angle formulae:

 $\sin(\alpha - \beta) \equiv \sin \alpha \cos \beta - \sin \beta \cos \alpha$ 

 $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ 

 $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ 

When we want to calculate for example  $\sin \frac{\pi}{12}$ , we will write it as  $\sin \left(\frac{\pi}{3} - \frac{\pi}{4}\right)$  and apply the second formula. Now we want to learn to use the formulae in the opposite direction

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#### Sidenote

#### Notation.

Recall that  $\sin^{-1}(x)$  denotes the inverse of sine. Similarly  $\cos^{-1}(x)$  and  $\tan^{-1}(x)$  are inverses of cosine and tangent respectively. Another way to write these is  $\arcsin(x)$ ,  $\arccos(x)$  and  $\arctan(x)$ .

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Let's start with a simple question. Consider a function

$$f(x) = 3\cos x + 4\sin x$$

What is the range of this function?

I will use the following argument: the range of  $\cos x$  is [-1,1], so the range of  $3\cos x$  is [-3,3], similarly the range of  $4\sin x$  is [-4,4], so the range of  $3\cos x + 4\sin x$  is [-7,7]. Correct? **NO!** 

The range of  $f(x) = 3\cos x + 4\sin x$  is **not** [-7, 7]. The reason the above argument is wrong is that  $\cos x$  and  $\sin x$  are maximal/minimal for different values of x (there is no x for which  $\cos x = 1$  and  $\sin x = 1$  simultaneously).

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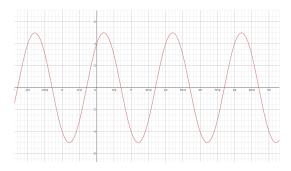
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We can actually see what the range is from the graph, but that won't always bessible. What's more important is that the graph is a trigonometric function. Sometimes that the graph is a trigonometric function.

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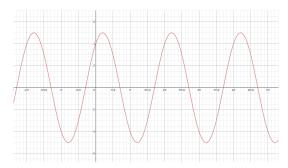
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If we look at the expression  $3\cos x + 4\sin x$  similar to the compound angle formula:  $\cos \theta \cos x + \sin \theta \sin x$ .

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If we look at the expression  $3\cos x + 4\sin x$  similar to the compound angle formula:  $\cos\theta\cos x + \sin\theta\sin x$ . What we want to do is to replace 3 with a  $\cos\theta$  and 4 with a  $\sin\theta$ . Of course this cannot be done, because  $\cos\theta$  cannot be equal to 3 (and similarly  $\sin\theta$  cannot be equal to 4).

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We will do a small trick.

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Let's draw a triangle with the angle  $\theta$ . We want to turn 3 into  $\cos \theta$ , so we will make the adjacent side equal to 3 (and opposite side equal to 4).

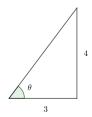
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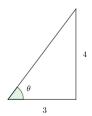
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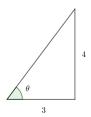
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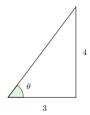
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All of this allows us to write:

$$3\cos x + 4\sin x = 5\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$$

which gives:

$$5\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right) = 5(\cos\theta\cos x + \sin\theta\sin x)$$

where  $\theta = \arctan\left(\frac{4}{3}\right)$ 

Now we use the compound angle formula to get:

 $5(\cos\theta\cos x + \sin\theta\sin x) = 5\cos(x - \theta)$ 

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where 
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Find the range of  $f(x) = 2 \sin x - \cos x$ .

We will try to write f(x) in the form  $R\sin(x-\theta)$ . So we want to change the into cos and 1 into sin. We can draw a triangle with adjacent side 2 and the opposite side 1.

The hypotenuse is  $\sqrt{5}$  and  $heta=\arctan\left(rac{1}{2}
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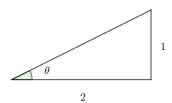
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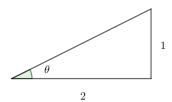
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The hypotenuse is  $\sqrt{5}$  and  $\theta = \arctan\left(\frac{1}{2}\right)$ .

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We can now write:

$$2\sin x - \cos x = \sqrt{5} \left( \frac{2}{\sqrt{5}} \sin x - \frac{1}{\sqrt{5}} \cos x \right) =$$
$$= \sqrt{5} \left( \cos \theta \sin x - \sin \theta \cos x \right)$$
$$= \sqrt{5} \sin(x - \theta)$$

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So  $f(x) = \sqrt{5}\sin(x-\theta)$ , which means that the range of f(x) is  $[-\sqrt{5}, \sqrt{5}]$ .

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Find the range of  $f(x) = 2\sin x + \sqrt{5}\cos x$ .

We will try to write f(x) in the form  $R\sin(x+\theta)$ . So we want to change the into  $\cos$  and  $\sqrt{5}$  into  $\sin$ . We will draw a triangle with adjacent side 2 and

The hypotenuse is 3 and  $\theta = \arctan(\sqrt{5})$ 

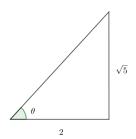
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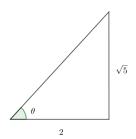
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The hypotenuse is 3 and  $\theta = \arctan(\frac{\sqrt{5}}{2})$ .

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We get:

$$2\sin x + \sqrt{5}\cos x = 3\left(\frac{2}{3}\sin x + \frac{\sqrt{5}}{3}\cos x\right) =$$

$$= 3\left(\cos\theta\sin x + \sin\theta\cos x\right)$$

$$= 3\sin(x+\theta)$$

where 
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.

So  $f(x) = 3\sin(x + \theta)$ , which means that the range of f(x) is [-3, 3].

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We get:

$$2\sin x + \sqrt{5}\cos x = 3\left(\frac{2}{3}\sin x + \frac{\sqrt{5}}{3}\cos x\right) =$$

$$= 3\left(\cos\theta\sin x + \sin\theta\cos x\right)$$

$$= 3\sin(x+\theta)$$

where 
$$\theta = \arctan\left(\frac{\sqrt{5}}{2}\right)$$
.

So  $f(x) = 3\sin(x + \theta)$ , which means that the range of f(x) is [-3, 3].

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In case of any questions you can message me via Librus or MS Teams.