Sum/difference - to - product formulae

When we solve an equation like:

$$x^2 - 4x - 12 = 0 (1)$$

what we want to do is to turn it (if possible) into:

$$(x-6)(x+2) = 0 (2)$$

because this immediately gives us the solutions x = 6 or x = -2.

We turned equation (1) where we add terms to an equation (2) where we multiply terms. This is often useful when solving equations.

Tomasz Lechowski DP1 AA HL October 16, 2024 2 / 14

When we solve an equation like:

$$x^2 - 4x - 12 = 0 (1)$$

what we want to do is to turn it (if possible) into:

$$(x-6)(x+2) = 0 (2)$$

because this immediately gives us the solutions x = 6 or x = -2.

We turned equation (1) where we add terms to an equation (2) where we multiply terms. This is often useful when solving equations.

Tomasz Lechowski DP1 AA HL October 16, 2024 2 / 14

We want to do something similar to trigonometric functions. Suppose we have an equation:

$$\sin A + \cos B = 0$$

can we somehow turn this sum into a product?

The answer is of course yes and we will see how to do this on the next few slides.

Tomasz Lechowski DP1 AA HL October 16, 2024 3 / 14

We want to do something similar to trigonometric functions. Suppose we have an equation:

$$\sin A + \cos B = 0$$

can we somehow turn this sum into a product?

The answer is of course yes and we will see how to do this on the next few slides.

Recall that we already have the following four formulae:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha$$
$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha$$
$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$
$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

Tomasz Lechowski DP1 AA HL October 16, 2024 4 / 14

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$$

Now let  $A = \alpha + \beta$  and  $B = \alpha - \beta$ . This gives  $\alpha = \frac{\alpha + \beta}{2}$  and

A - B

 $\beta = \frac{1}{2}$  . So we get the following formula:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Similarly if we subtracted the second equation from the first one we would get:

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

Tomasz Lechowski DP1 AA HL October 16, 2024 5 / 14

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$$

Now let 
$$A = \alpha + \beta$$
 and  $B = \alpha - \beta$ .

Tomasz Lechowski DP1 AA HL October 16, 2024 5 / 14

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$$

Now let 
$$A = \alpha + \beta$$
 and  $B = \alpha - \beta$ . This gives  $\alpha = \frac{A + B}{2}$  and

$$\beta = \frac{A - B}{2}.$$

$$\sin A + \sin B = 2 \sin \left( -\frac{1}{2} \right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A}{a}\right)$$

$$\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

Tomasz Lechowski DP1 AA HL October 16, 2024 5/14

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$$

Now let  $A = \alpha + \beta$  and  $B = \alpha - \beta$ . This gives  $\alpha = \frac{A + B}{2}$  and  $\beta = \frac{A - B}{2}$ . So we get the following formula:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Tomasz Lechowski DP1 AA HL October 16, 2024 5 / 14

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$$

Now let  $A = \alpha + \beta$  and  $B = \alpha - \beta$ . This gives  $\alpha = \frac{A + B}{2}$  and A - B

 $\beta = \frac{A - B}{2}$ . So we get the following formula:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Similarly if we subtracted the second equation from the first one we would get:

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

We can also add the third and fourth equations to get:

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$$

and again using the substitutions  $A = \alpha + \beta$  and  $B = \alpha - \beta$  we get:

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Subtracting the fourth equation from the third one girl

We can also add the third and fourth equations to get:

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$$

and again using the substitutions  $A = \alpha + \beta$  and  $B = \alpha - \beta$  we get:

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Subtracting the fourth equation from the third one gives:

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Tomasz Lechowski DP1 AA HL October 16, 2024 6 / 14

In the end we get the following four formulae:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

The IB does not require you to know these, but they are very helpful. I would highly recommend to learn them by heart (and since they're not required by the IB, they're not in the formula booklet).

Tomasz Lechowski DP1 AA HL October 16, 2024 7 / 14

In the end we get the following four formulae:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

The IB does not require you to know these, but they are very helpful. I would highly recommend to learn them by heart (and since they're not required by the IB, they're not in the formula booklet).

Tomasz Lechowski DP1 AA HL October 16, 2024 7 / 14

Evaluate  $\sin 105^{\circ} - \sin 15^{\circ}$ .

$$\sin 105^\circ - \sin 15^\circ = 2 \sin 45^\circ \cos 60^\circ = 2 \times \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{2}}{2}$$

8/14

Tomasz Lechowski DP1 AA HL October 16, 2024

Evaluate  $\sin 105^{\circ} - \sin 15^{\circ}$ .

$$\sin 105^{\circ} - \sin 15^{\circ} = 2 \sin 45^{\circ} \cos 60^{\circ} = 2 \times \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{2}}{2}$$

Tomasz Lechowski DP1 AA HL October 16, 2024 8 / 14

Evaluate  $\cos 105^{\circ} - \sin 75^{\circ}$ .

We first need to change sin into cos. We can do so using the reduction formulae  $\sin(90^\circ - \alpha) = \cos \alpha$ :

 $\cos 105^\circ - \sin 75^\circ = \cos 105^\circ - \cos 15^\circ = -2 \sin 60^\circ \sin 45^\circ = -rac{\sqrt{2}}{2}$ 

Tomasz Lechowski DP1 AA HL October 16, 2024 9 / 14

Evaluate  $\cos 105^{\circ} - \sin 75^{\circ}$ .

We first need to change sin into cos. We can do so using the reduction formulae  $\sin(90^{\circ} - \alpha) = \cos \alpha$ :

Tomasz Lechowski DP1 AA HL October 16, 2024 9 / 14

Evaluate  $\cos 105^{\circ} - \sin 75^{\circ}$ .

We first need to change sin into cos. We can do so using the reduction formulae  $\sin(90^{\circ} - \alpha) = \cos \alpha$ :

$$\cos 105^{\circ} - \sin 75^{\circ} = \cos 105^{\circ} - \cos 15^{\circ} = -2\sin 60^{\circ} \sin 45^{\circ} = -\frac{\sqrt{6}}{2}$$

Tomasz Lechowski DP1 AA HL October 16, 2024 9 / 14

Express  $\sin \theta + \sin 3\theta$  as a product of two trigonometric functions. Hence solve  $\sin \theta + \sin 2\theta + \sin 3\theta = 0$  for  $0 \le \theta \le \pi$ .

10 / 14

Tomasz Lechowski DP1 AA HL October 16, 2024

Express  $\sin \theta + \sin 3\theta$  as a product of two trigonometric functions. Hence solve  $\sin \theta + \sin 2\theta + \sin 3\theta = 0$  for  $0 \le \theta \le \pi$ .

We have:

$$\sin \theta + \sin 3\theta = 2\sin\left(\frac{\theta + 3\theta}{2}\right)\cos\left(\frac{\theta - 3\theta}{2}\right) =$$

$$= 2\sin 2\theta\cos(-\theta) = 2\sin 2\theta\cos\theta$$

October 16, 2024

10 / 14

Tomasz Lechowski DP1 AA HL

Express  $\sin\theta + \sin 3\theta$  as a product of two trigonometric functions. Hence solve  $\sin\theta + \sin 2\theta + \sin 3\theta = 0$  for  $0 \le \theta \le \pi$ .

We have:

$$\sin \theta + \sin 3\theta = 2\sin\left(\frac{\theta + 3\theta}{2}\right)\cos\left(\frac{\theta - 3\theta}{2}\right) =$$

$$= 2\sin 2\theta\cos(-\theta) = 2\sin 2\theta\cos\theta$$

So we get:

$$\sin \theta + \sin 2\theta + \sin 3\theta = 0$$
$$2\sin 2\theta \cos \theta + \sin 2\theta = 0$$
$$\sin 2\theta (2\cos \theta + 1) = 0$$

Which gives  $\sin 2\theta = 0$  or  $2\cos\theta + 1 = 0$ 

4 ロ ト 4 個 ト 4 差 ト 4 差 ト 9 へ ()

10 / 14

Tomasz Lechowski DP1 AA HL October 16, 2024

We want to solve  $\sin 2\theta = 0$  or  $2\cos \theta + 1 = 0$  for  $0 \le \theta \le \pi$ .

The first equation gives us  $\theta \in \{0, \frac{\pi}{2}, \pi\}$ , the second gives us  $\theta = \frac{2\pi}{3}$ , so in the end we have four solutions:  $0, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$ .

11 / 14

Tomasz Lechowski DP1 AA HL October 16, 2024

We want to solve  $\sin 2\theta = 0$  or  $2\cos \theta + 1 = 0$  for  $0 \le \theta \le \pi$ .

The first equation gives us  $\theta \in \{0, \frac{\pi}{2}, \pi\}$ , the second gives us  $\theta = \frac{2\pi}{3}$ , so in the end we have four solutions:  $0, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$ .

<□ > <□ > <□ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > <

Tomasz Lechowski DP1 AA HL October 16, 2024 11/14

Solve  $\cos 5\theta - \cos \theta = \sin 2\theta$  for  $0 \le \theta \le 2\pi$ .

We start by combining  $\cos 5 heta - \cos heta$  . We get

$$\cos 5\theta - \cos \theta = -2\sin\left(\frac{5\theta + \theta}{2}\right)\sin\left(\frac{5\theta - \theta}{2}\right) = -2\sin 2\theta\sin 2\theta$$

 $= -2 \sin 3\theta \sin 2\theta$ 

Hence we get:

$$\cos 5\theta - \cos \theta = \sin 2\theta \quad \Rightarrow \quad -2\sin 3\theta \sin 2\theta = \sin 2\theta$$

This gives

 $\sin 2\theta + 2\sin 3\theta \sin 2\theta = 0$ 

4 D > 4 B > 4 E > 4 E > E 990

12 / 14

Solve  $\cos 5\theta - \cos \theta = \sin 2\theta$  for  $0 \le \theta \le 2\pi$ .

We start by combining  $\cos 5\theta - \cos \theta$ .

$$\cos 5\theta - \cos \theta = -2\sin\left(\frac{5\theta + \theta}{2}\right)\sin\left(\frac{5\theta - \theta}{2}\right) = -2\sin 3\theta \sin 2\theta$$

Hence we get:

$$\cos 5\theta - \cos \theta = \sin 2\theta \quad \Rightarrow \quad -2\sin 3\theta \sin 2\theta = \sin 2\theta$$

This gives

 $\sin 2\theta + 2\sin 3\theta \sin 2\theta = 0$ 

4 D > 4 B > 4 E > 4 E > E 9 Q C

12 / 14

Solve  $\cos 5\theta - \cos \theta = \sin 2\theta$  for  $0 \le \theta \le 2\pi$ .

We start by combining  $\cos 5\theta - \cos \theta$ . We get:

$$\cos 5\theta - \cos \theta = -2\sin\left(\frac{5\theta + \theta}{2}\right)\sin\left(\frac{5\theta - \theta}{2}\right) =$$
$$= -2\sin 3\theta \sin 2\theta$$

Hence we get:

 $\cos 5\theta - \cos \theta = \sin 2\theta \quad \Rightarrow \quad -2\sin 3\theta \sin 2\theta = \sin 2\theta$ 

This gives

 $\sin 2\theta + 2 \sin 3\theta \sin 2\theta = 0$ 

Tomasz Lechowski DP1 AA HL October 16, 2024 12 / 14

Solve  $\cos 5\theta - \cos \theta = \sin 2\theta$  for  $0 \le \theta \le 2\pi$ .

We start by combining  $\cos 5\theta - \cos \theta$ . We get:

$$\cos 5\theta - \cos \theta = -2\sin\left(\frac{5\theta + \theta}{2}\right)\sin\left(\frac{5\theta - \theta}{2}\right) =$$
$$= -2\sin 3\theta \sin 2\theta$$

Hence we get:

$$\cos 5\theta - \cos \theta = \sin 2\theta \quad \Rightarrow \quad -2\sin 3\theta \sin 2\theta = \sin 2\theta$$

This gives

Solve  $\cos 5\theta - \cos \theta = \sin 2\theta$  for  $0 \le \theta \le 2\pi$ .

We start by combining  $\cos 5\theta - \cos \theta$ . We get:

$$\cos 5\theta - \cos \theta = -2\sin\left(\frac{5\theta + \theta}{2}\right)\sin\left(\frac{5\theta - \theta}{2}\right) =$$
$$= -2\sin 3\theta \sin 2\theta$$

Hence we get:

$$\cos 5\theta - \cos \theta = \sin 2\theta \quad \Rightarrow \quad -2\sin 3\theta \sin 2\theta = \sin 2\theta$$

This gives:

$$\sin 2\theta + 2\sin 3\theta \sin 2\theta = 0$$

Tomasz Lechowski DP1 AA HL October 16, 2024 12/14

Factoring  $\sin 2\theta$  gives:

$$\sin 2\theta (1+2\sin 3\theta)=0$$

So

$$\sin 2\theta = 0$$
 or  $\sin 3\theta = -\frac{1}{2}$ 

From the first equation we get  $\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ . The second equation gives  $\theta \in \{\frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}\}$ 

So in the end we have 11 solutions:

 $\theta \in \{0, \frac{7\pi}{18}, \frac{\pi}{2}, \frac{11\pi}{18}, \pi, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{3\pi}{2}, \frac{31\pi}{18}, \frac{35\pi}{18}, 2\pi\}$ 

13 / 14

Tomasz Lechowski DP1 AA HL October 16, 2024

Factoring  $\sin 2\theta$  gives:

$$\sin 2\theta (1+2\sin 3\theta)=0$$

So

$$\sin 2\theta = 0$$
 or  $\sin 3\theta = -\frac{1}{2}$ 

From the first equation we get  $\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ . The second equation gives  $\theta \in \{\frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}\}$ 

So in the end we have 11 solutions:

 $\theta \in \{0, \frac{7\pi}{18}, \frac{\pi}{9}, \frac{11\pi}{18}, \pi, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{3\pi}{9}, \frac{31\pi}{18}, \frac{35\pi}{18}, 2\pi\}$ 

Factoring  $\sin 2\theta$  gives:

$$\sin 2\theta (1+2\sin 3\theta)=0$$

So

$$\sin 2\theta = 0$$
 or  $\sin 3\theta = -\frac{1}{2}$ 

From the first equation we get  $\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ . The second equation gives  $\theta \in \{\frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}\}$ 

So in the end we have 11 solutions:

13 / 14

Factoring  $\sin 2\theta$  gives:

$$\sin 2\theta (1+2\sin 3\theta)=0$$

So

$$\sin 2\theta = 0$$
 or  $\sin 3\theta = -\frac{1}{2}$ 

From the first equation we get  $\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ . The second equation gives  $\theta \in \{\frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}\}$ 

So in the end we have 11 solutions:

$$\theta \in \{0, \tfrac{7\pi}{18}, \tfrac{\pi}{2}, \tfrac{11\pi}{18}, \pi, \tfrac{19\pi}{18}, \tfrac{23\pi}{18}, \tfrac{3\pi}{2}, \tfrac{31\pi}{18}, \tfrac{35\pi}{18}, 2\pi\}$$

- 4 ロ ト 4 昼 ト 4 種 ト - 種 - り Q (C)

13 / 14

In case of any questions you can message me via Librus or MS Teams.