Sum and difference of angles

The goal is to learn the identities for the sine and cosine of a sum or a difference of two angles.

Identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Easy consequences of the above identities:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$
$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

If you don't think that the above are easy consequences, let me know and I will explain in class.

Identities

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \sin\beta\cos\alpha \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
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are called double angle identities.

They're easy consequences of the first four identities. For instance if we set $\alpha = \beta$ into the first identity, we get:

$$\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \sin\alpha\cos\alpha$$

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Identities

We will prove:

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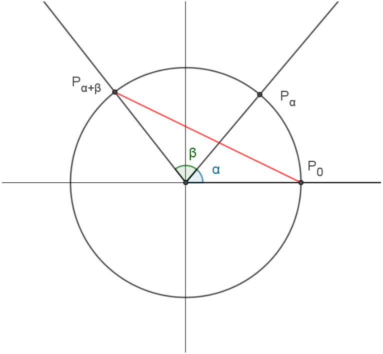
Proof

Let's put $\alpha + \beta$ on the unit circle.



We then have from the definition of cosine that $\cos(\alpha + \beta)$ is the x-coordinate of point $P_{\alpha+\beta}$. We will calculate the length of $P_0P_{\alpha+\beta}$

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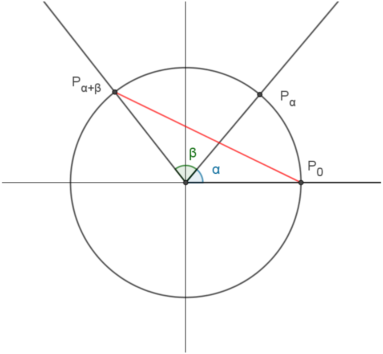

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Proof

Using Pythagorean Theorem:

$$|P_0P_{\alpha+\beta}|^2 = (1 - \cos(\alpha+\beta))^2 + \sin^2(\alpha+\beta)$$

We get:

$$|P_0P_{\alpha+\beta}|^2 = 1 - 2\cos(\alpha+\beta) + \cos^2(\alpha+\beta) + \sin^2(\alpha+\beta) = 2 - 2\cos(\alpha+\beta)$$

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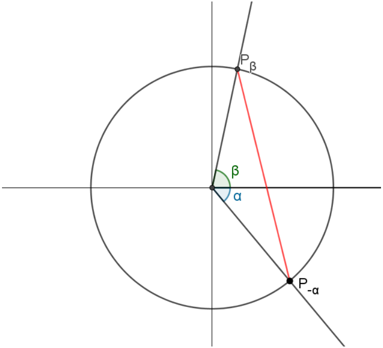
Now if we rotate our triangle we get:



Of course the triangle did not change, so the length of the red segment is the same:

$$|P_0 P_{\alpha+\beta}| = |P_{-\alpha} P_{\beta}|$$

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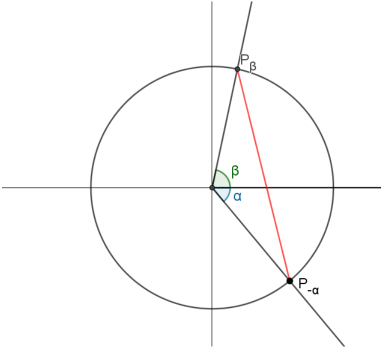

Of course the triangle did not change, so the length of the red segment is the same:

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Proof

We will calculate $|P_{-\alpha}P_{\beta}|^2$.

$$|P_{-\alpha}P_{\beta}|^2 = (\cos\beta - \cos(-\alpha))^2 + (\sin(-\alpha) - \sin\beta)^2$$

We get:

$$\begin{aligned}|P_{-\alpha}P_{\beta}|^2 &= \cos^2\beta - 2\cos\beta\cos(-\alpha) + \cos^2(-\alpha) + \\ &+ \sin^2(-\alpha) - 2\sin(-\alpha)\sin\beta + \sin^2\beta = \\ &= 2 - 2\cos\beta\cos\alpha + 2\sin\alpha\sin\beta\end{aligned}$$

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Proof

So finally we got:

$$2 - 2\cos(\alpha + \beta) = 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$$

So:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

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Summary

Please try and understand this proof. If you think some details are missing or unclear, let me know. It is important that you understand every step of this proof.

Applications

Let's calculate $\sin 105^\circ$.

$$\begin{aligned}\sin(105^\circ) &= \sin(45^\circ + 60^\circ) = \\ &= \sin 45^\circ \cos 60^\circ + \sin 60^\circ \cos 45^\circ = \\ &= \frac{\sqrt{2}}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \\ &= \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

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Applications

How about $\cos\frac{\pi}{12}$?

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) =$$

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} =$$

$$= \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} =$$

$$= \frac{\sqrt{2} + \sqrt{6}}{4}$$

We could've predicted the result, since:

$$\sin 105^\circ = \sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{2} + \frac{\pi}{12}\right) = \cos\frac{\pi}{12}$$

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In case of any questions you can message me via Librus or MS Teams.