# Composition and inverse [74 marks]

**1.** [Maximum mark: 5] SPM.1.SL.TZ0.5

The functions $f$ and $g$ are defined such that $f\left(x\right)=\frac{x+3}{4}$ and $g\left(x\right)=8x+5$.

(a) Show that $\left(g\circ f\right)\left(x\right)=2x+11$. [2]

Markscheme

attempt to form composition ***M1***

correct substitution $g\left(\frac{x+3}{4}\right)=8\left(\frac{x+3}{4}\right)+5$ ***A1***

$\left(g\circ f\right)\left(x\right)=2x+11$ ***AG***

***[2 marks]***

(b) Given that $\left(g\circ f\right)^{-1}\left(a\right)=4$, find the value of $a$. [3]

Markscheme

attempt to substitute 4 (seen anywhere) ***(M1)***

correct equation $a=2\times4+11$ ***(A1)***

$a$ = 19 ***A1***

***[3 marks]***

**2.** [Maximum mark: 5] 23N.1.AHL.TZ1.1

Consider the functions $f(x)=x-3$ and $g(x)=x^2+k^2$, where $k$ is a real constant.

(a) Write down an expression for $(g \circ f)(x)$. [2]

Markscheme

attempt to form $(g \circ f)(x)$ ***(M1)***

$((g \circ f)(x)) = (x - 3)^2 + k^2 \ \left(= x^2 - 6x + 9 + k^2\right)$ ***A1***

***[2 marks]***

(b) Given that $(g \circ f)(2) = 10$, find the possible values of $k$. [3]

Markscheme

substituting $x = 2$ into their $(g \circ f)(x)$ and setting their expression $= 10$ ***(M1)***

$(2 - 3)^2 + k^2 = 10$ OR $2^2 - 6(2) + 9 + k^2 = 10$

$k^2 = 9$ ***(A1)***

$k = \pm 3$ ***A1***

***[3 marks]***

**3.** [Maximum mark: 6] EXN.1.SL.TZ0.5

The functions $f$ and $g$ are defined for $x \in \mathbb{R}$ by $f(x) = x - 2$ and $g(x) = ax + b$, where $a,\ b \in \mathbb{R}$.

Given that $(f \circ g)(2) = -3$ and $(g \circ f)(1) = 5$, find the value of $a$ and the value of $b$. [6]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$(f \circ g)(x) = ax + b - 2$ **(M1)**

$(f \circ g)(2) = -3 \Rightarrow 2a + b - 2 = -3\ (2a + b = -1)$ **A1**

$(g \circ f)(x) = a(x - 2) + b$ **(M1)**

$(g \circ f)(1) = 5 \Rightarrow -a + b = 5$ **A1**

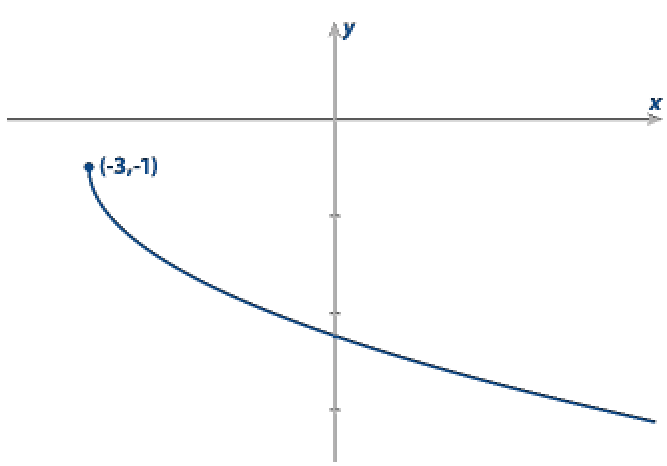
a valid attempt to solve their two linear equations for $a$ and $b$ **M1**

so $a = -2$ and $b = 3$ **A1**

**[6 marks]**

**4.** [Maximum mark: 14] EXN.1.SL.TZ0.8

The following diagram shows the graph of $y = -1 - \sqrt{x + 3}$ for $x \geq -3$.



(a) Describe a sequence of transformations that transforms the graph of $y = \sqrt{x}$ for $x \geq 0$ to the graph of $y = -1 - \sqrt{x+3}$ for $x \geq -3$. [3]

Markscheme

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for example,

a reflection in the $x$-axis (in the line $y = 0$) **A1**

a horizontal translation (shift) $3$ units to the left **A1**

a vertical translation (shift) down by $1$ unit **A1**

**Note:** Award **A1** for each correct transformation applied in a correct position in the sequence. Do not accept use of the "move" for a translation.

**Note:** Award **A1A1A1** for a correct alternative sequence of transformations. For example,

a vertical translation (shift) down by $1$ unit, followed by a horizontal translation (shift) $3$ units to the left and then a reflection in the line $y = -1$.

**[3 marks]**

A function $f$ is defined by $f(x) = -1 - \sqrt{x+3}$ for $x \geq -3$.

(b) State the range of $f$. [1]

Markscheme

range is $f(x) \leq -1$ **A1**

**Note:** Correct alternative notations include $]-\infty, -1]$, $(-\infty, -1]$ or $y \leq -1$.

**[1 mark]**

(c) Find an expression for $f^{-1}(x)$, stating its domain. [5]

Markscheme

$-1 - \sqrt{y+3} = x$ **M1**

**Note:** Award **M1** for interchanging $x$ and $y$ (can be done at a later stage).

$\sqrt{y+3} = -x - 1(= -(x+1))$ **A1**

$y + 3 = (x+1)^2$ **A1**

so $f^{-1}(x) = (x+1)^2 - 3 \ \left(f^{-1}(x) = x^2 + 2x - 2\right)$ **A1**

domain is $x \leq -1$ **A1**

**Note:** Correct alternative notations include $]-\infty, -1]$ or $(-\infty, -1]$.

**[5 marks]**

(d) Find the coordinates of the point(s) where the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect. [5]

Markscheme

the point of intersection lies on the line $y = x$

**EITHER**

$(x+1)^2 - 3 = x$ **M1**

attempts to solve their quadratic equation **M1**

for example, $(x+2)(x-1) = 0$ or $x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2}$ $\left(x = \frac{-1 \pm 3}{2}\right)$

**OR**

$-1 - \sqrt{x+3} = x$ **M1**

$\left(-1 - \sqrt{x+3}\right)^2 = x^2 \Rightarrow 2\sqrt{x+3} + x + 4 = x^2$

substitutes $2\sqrt{x+3} = -2(x+1)$ to obtain $-2(x+1) + x + 4 = x^2$

attempts to solve their quadratic equation **M1**

for example, $(x+2)(x-1) = 0$ or $x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2}$ $\left(x = \frac{-1 \pm 3}{2}\right)$

**THEN**

$x = -2, 1$ **A1**

as $x \leq -1$, the only solution is $x = -2$ **R1**

so the coordinates of the point of intersection are $(-2,\ -2)$ **A1**

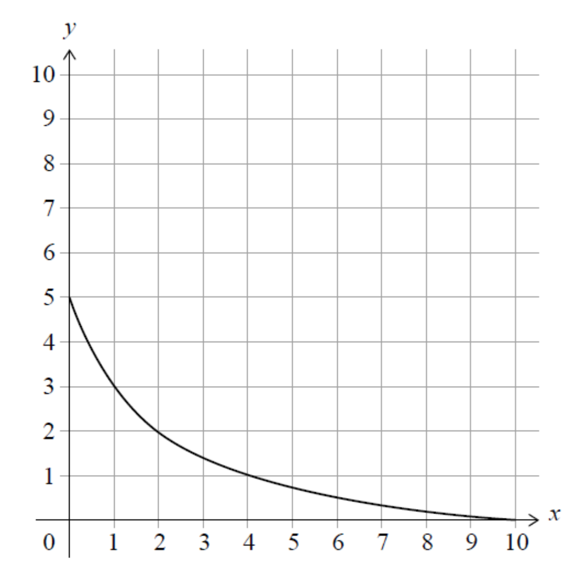
**Note:** Award **R0A1** if $(-2,\ -2)$ is stated without a valid reason given for rejecting $(1,\ 1)$.

**[5 marks]**

**5.** [Maximum mark: 5] 24M.1.SL.TZ2.1

The graph of $y = f(x)$ for $0 \leq x \leq 10$ is shown in the following diagram.

The graph intercepts the axes at $(10,\ 0)$ and $(0,\ 5)$.



(a) Write down the value of

(a.i) $f(4)$; [1]

Markscheme

$f(4) = 1$ ***A1***

***[1 mark]***

(a.ii) $f \circ f(4)$; [1]

Markscheme

$f \circ f(4) = 3$ ***A1***

***[1 mark]***

(a.iii) $f^{-1}(3)$. [1]

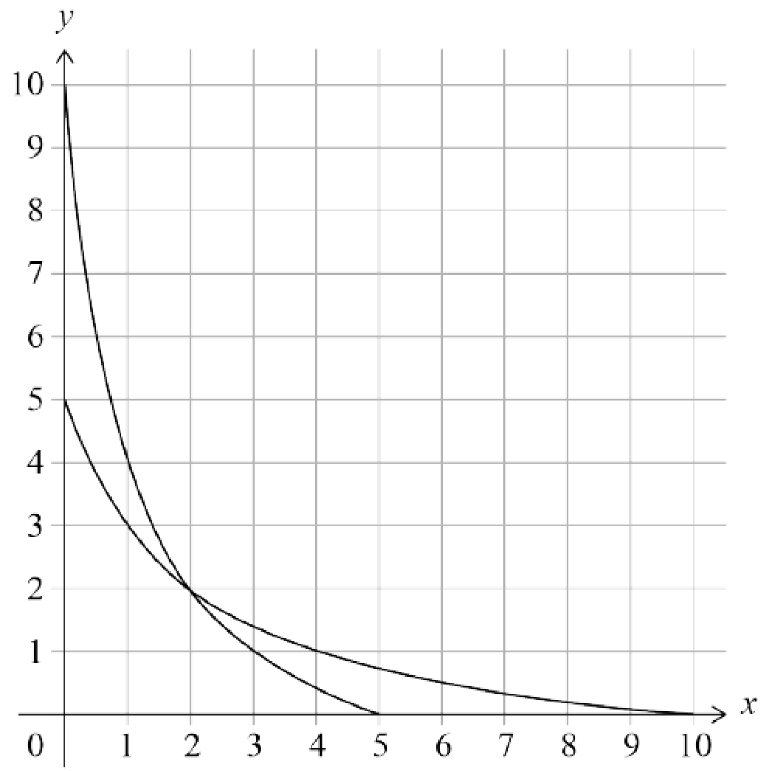
Markscheme

$f^{-1}(3) = 1$ ***A1***

***[1 mark]***

(b) On the axes above, sketch the graph of $y = f^{-1}(x)$. Show clearly where the graph intercepts the axes. [2]

Markscheme




concave up curve with $y$ intercept at $(0,\ 10)$ and $x$ intercept at $(5,\ 0)$ ***A1***

curve passes through $(2,\ 2)$ OR through $(1,\ 4)$ and $(3,\ 1)$ ***A1***

**Note:** Do not award the second mark unless the first mark has been awarded. (Do not award ***A0A1***).

***[2 marks]***

**6.** [Maximum mark: 7] 23M.1.AHL.TZ1.1

The function $f$ is defined by $f(x) = \frac{7x+7}{2x-4}$ for $x \in \mathbb{R}, x \neq 2$.

(a) Find the zero of $f(x)$. [2]

Markscheme

recognizing $f(x) = 0$ *(M1)*

$x = -1$ ***A1***

***[2 marks]***

(b) For the graph of $y = f(x)$, write down the equation of

(b.i) the vertical asymptote; [1]

Markscheme

$x = 2$ (must be an equation with $x$) ***A1***

***[1 mark]***

(b.ii) the horizontal asymptote. [1]

Markscheme

$y = \frac{7}{2}$ (must be an equation with $y$) ***A1***

***[1 mark]***

(c) Find $f^{-1}(x)$, the inverse function of $f(x)$. [3]

Markscheme

**EITHER**

interchanging $x$ and $y$ ***(M1)***

$2xy - 4x = 7y + 7$

correct working with $y$ terms on the same side: $2xy - 7y = 4x + 7$
***(A1)***

**OR**

$2yx - 4y = 7x + 7$

correct working with $x$ terms on the same side: $2yx - 7x = 4y + 7$
***(A1)***

interchanging $x$ and $y$ OR making $x$ the subject $x = \frac{4y+7}{2y-7}$ ***(M1)***

**THEN**

$f^{-1}(x) = \frac{4x+7}{2x-7}$ (or equivalent) $\left(x \neq \frac{7}{2}\right)$ ***A1***

***[3 marks]***

**7.** [Maximum mark: 7] 23M.1.AHL.TZ2.5

The functions $f$ and $g$ are defined for $x \in \mathbb{R}$ by

$f(x) = ax + b$, where $a, b \in \mathbb{Z}$

$g(x) = x^2 + x + 3$.

Find the two possible functions $f$ such that
$\left(g \circ f\right)\left(x\right) = 4x^2 - 14x + 15$.

[7]

Markscheme

attempts to form $(g \circ f)(x)$ ***(M1)***

$[f(x)]^2 + f\left(x\right) + 3$ OR $(ax + b)^2 + ax + b + 3$

$a^2x^2 + 2abx + b^2 + ax + b + 3\left(= 4x^2 - 14x + 15\right)$ ***(A1)***

equates their corresponding terms to form at least one equation ***(M1)***

$a^2x^2 = 4x^2$ OR $a^2 = 4$ OR $2abx + ax = -14x$ OR
$2ab + a = -14$ OR $b^2 + b + 3 = 15$

$a = \pm 2$ (seen anywhere) ***A1***

attempt to use $2ab + a = -14$ to pair the correct values (seen anywhere) ***(M1)***

$f(x) = 2x - 4$ (accept $a = 2$ with $b = -4$), $f(x) = -2x + 3$ (accept $a = -2$ with $b = 3$) ***A1A1***

***[7 marks]***

**8.** [Maximum mark: 5] 22M.1.SL.TZ2.1

The following table shows values of $f(x)$ and $g(x)$ for different values of $x$.

Both $f$ and $g$ are one-to-one functions.

| $x$ | −2 | 0 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 8 | 4 | 0 | −3 |
| $g(x)$ | −5 | −2 | 4 | 0 |

(a) Find $g(0)$. [1]

Markscheme

$g(0) = -2$ ***A1***

***[1 mark]***

(b) Find $(f \circ g)(0)$. [2]

Markscheme

evidence of using composite function ***(M1)***

$f(g(0))$ OR $f(-2)$

$(f \circ g)(0) = 8$ ***A1***

***[2 marks]***

(c) Find the value of $x$ such that $f(x) = 0$. [2]

Markscheme

$x = 3$ ***A2***

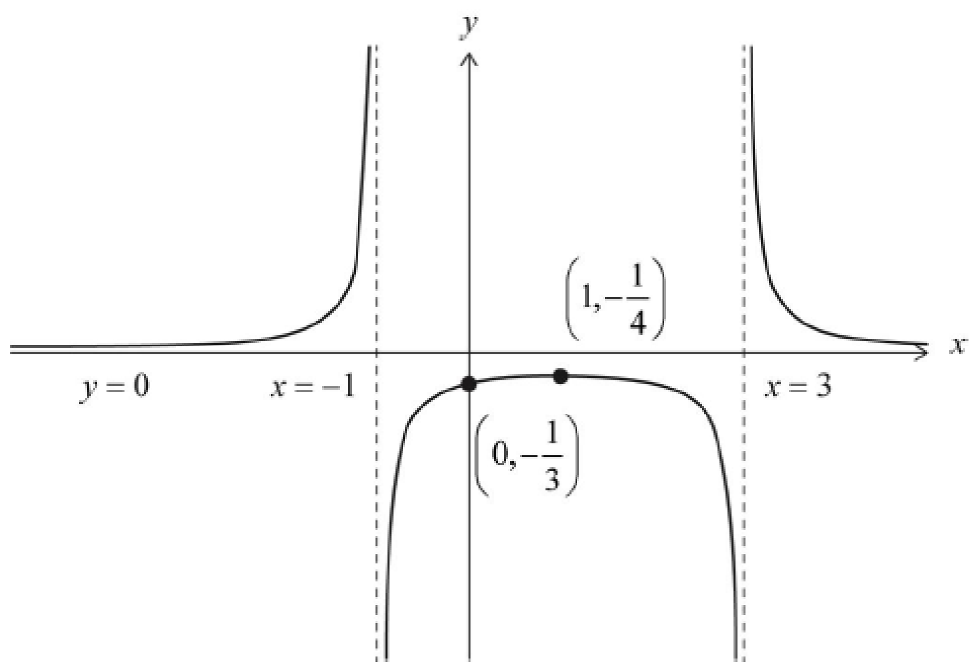
***[2 marks]***

**9.** [Maximum mark: 20] 22M.1.AHL.TZ2.11

A function $f$ is defined by $f(x) = \frac{1}{x^2-2x-3}$, where $x \in \mathbb{R},\ x \neq -1,\ x \neq 3$.

(a) Sketch the curve $y = f(x)$, clearly indicating any asymptotes with their equations. State the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [6]

Markscheme



$y$-intercept $\left(0, -\frac{1}{3}\right)$ ***A1***

**Note:** Accept an indication of $-\frac{1}{3}$ on the $y$-axis.

vertical asymptotes $x = -1$ and $x = 3$ ***A1***

horizontal asymptote $y = 0$ ***A1***

uses a valid method to find the $x$-coordinate of the local maximum point ***(M1)***

**Note:** For example, uses the axis of symmetry or attempts to solve $f\prime(x) = 0$.

local maximum point $\left(1, -\frac{1}{4}\right)$ ***A1***

**Note:** Award ***(M1)A0*** for a local maximum point at $x = 1$ and coordinates not given.

three correct branches with correct asymptotic behaviour and the key features in approximately correct relative positions to each other ***A1***

***[6 marks]***

A function $g$ is defined by $g(x) = \frac{1}{x^2-2x-3}$, where $x \in \mathbb{R},\ x > 3$.

The inverse of $g$ is $g^{-1}$.

(b.i) Show that $g^{-1}(x) = 1 + \frac{\sqrt{4x^2+x}}{x}$. [6]

Markscheme

$x = \frac{1}{y^2-2y-3}$ ***M1***

**Note:** Award ***M1*** for interchanging $x$ and $y$ (this can be done at a later stage).

**EITHER**

attempts to complete the square ***M1***

$y^2 - 2y - 3 = (y-1)^2 - 4$ ***A1***

$x = \frac{1}{(y-1)^2-4}$

$(y-1)^2 - 4 = \frac{1}{x}\left((y-1)^2 = 4 + \frac{1}{x}\right)$ ***A1***

$y - 1 = \pm\sqrt{4+\frac{1}{x}}\left(= \pm\sqrt{\frac{4x+1}{x}}\right)$

**OR**

attempts to solve $xy^2 - 2xy - 3x - 1 = 0$ for $y$ ***M1***

$y = \frac{-(-2x)\pm\sqrt{(-2x)^2+4x(3x+1)}}{2x}$ ***A1***

**Note:** Award ***A1*** even if $-$ (in $\pm$) is missing

$= \frac{2x \pm \sqrt{16x^2+4x}}{2x}$ ***A1***

**THEN**

$= 1 \pm \frac{\sqrt{4x^2+x}}{x}$ ***A1***

$y > 3$ and hence $y = 1 - \frac{\sqrt{4x^2+x}}{x}$ is rejected ***R1***

**Note:** Award ***R1*** for concluding that the expression for $y$ must have the '$+$' sign.
The ***R1*** may be awarded earlier for using the condition $x > 3$.

$y = 1 + \frac{\sqrt{4x^2+x}}{x}$

$g^{-1}(x) = 1 + \frac{\sqrt{4x^2+x}}{x}$ ***AG***

***[6 marks]***

(b.ii) State the domain of $g^{-1}$. [1]

Markscheme

domain of $g^{-1}$ is $x > 0$ ***A1***

***[1 mark]***

A function $h$ is defined by $h(x) = \arctan \frac{x}{2}$, where $x \in \mathbb{R}$.

(c) Given that $(h \circ g)(a) = \frac{\pi}{4}$, find the value of $a$.

Give your answer in the form $p + \frac{q}{2}\sqrt{r}$, where $p,\ q,\ r \in \mathbb{Z}^+$. [7]

Markscheme

attempts to find $(h \circ g)(a)$ ***(M1)***

$(h \circ g)(a) = \arctan\left(\frac{g(a)}{2}\right)$ $\left((h \circ g)(a) = \arctan\left(\frac{1}{2(a^2-2a-3)}\right)\right)$

***(A1)***

$\arctan\left(\frac{g(a)}{2}\right) = \frac{\pi}{4}$ $\left(\arctan\left(\frac{1}{2(a^2-2a-3)}\right) = \frac{\pi}{4}\right)$

attempts to solve for $g(a)$ ***M1***

$\Rightarrow g(a) = 2$ $\left(\frac{1}{(a^2-2a-3)} = 2\right)$

**EITHER**

$\Rightarrow a = g^{-1}(2)$ ***A1***

attempts to find their $g^{-1}(2)$ ***M1***

$a = 1 + \frac{\sqrt{4(2)^2+2}}{2}$ ***A1***

**Note:** Award all available marks to this stage if $x$ is used instead of $a$.

**OR**

$\Rightarrow 2a^2 - 4a - 7 = 0$ ***A1***

attempts to solve their quadratic equation ***M1***

$$a = \frac{-(-4) \pm \sqrt{(-4)^2 + 4(2)(7)}}{4} \quad \left(= \frac{4 \pm \sqrt{72}}{4}\right)$$ ***A1***

**Note:** Award all available marks to this stage if $x$ is used instead of $a$.

**THEN**

$a = 1 + \frac{3}{2}\sqrt{2}$ (as $a > 3$) ***A1***

$(p = 1,\ q = 3,\ r = 2)$

**Note:** Award ***A1*** for $a = 1 + \frac{1}{2}\sqrt{18}$ $(p = 1,\ q = 1,\ r = 18)$

***[7 marks]***