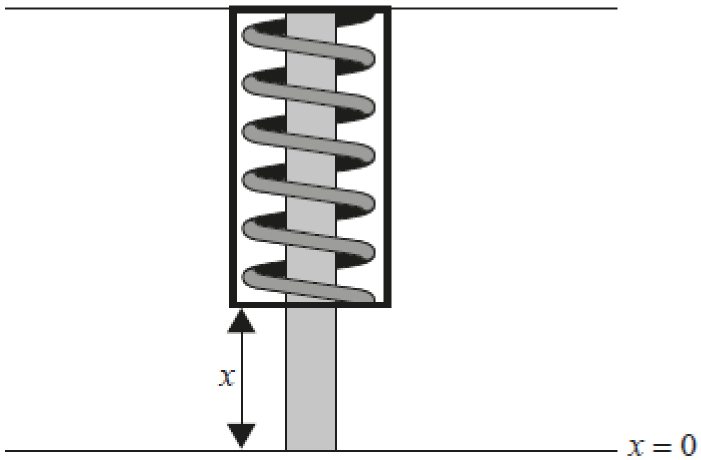
Coupled differential equations 2 [61 marks]

1. [Maximum mark: 15] 21N.2.AHL.TZ0.6

A shock absorber on a car contains a spring surrounded by a fluid. When the car travels over uneven ground the spring is compressed and then returns to an equilibrium position.



The displacement, x, of the spring is measured, in centimetres, from the equilibrium position of $x = 0$. The value of x can be modelled by the following second order differential equation, where t is the time, measured in seconds, after the initial displacement.

$$\ddot{x} + 3\dot{x} + 1.25x = 0$$

(a) Given that $y = \dot{x}$, show that $\dot{y} = -1.25x - 3y$. [2]

Markscheme

$y = \dot{x} \Rightarrow \dot{y} = \ddot{x}$ ***A1***

$\dot{y} + 3(y) + 1.25x = 0$ ***R1***

Note: If no explicit reference is made to $\dot{y} = \ddot{x}$, or equivalent, award ***A0R1*** if second line is seen. If $\frac{\mathrm{d}y}{\mathrm{d}x}$ used instead of $\frac{\mathrm{d}y}{\mathrm{d}t}$, award ***A0R0***.

$\dot{y} = -3y - 1.25x$ ***AG***

[2 marks]

The differential equation can be expressed in the form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} x \\ y \end{pmatrix}$, where $\boldsymbol{A}$ is a 2×2 matrix.

(b) Write down the matrix $\boldsymbol{A}$. [1]

Markscheme

$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -1.25 & -3 \end{pmatrix}$ ***A1***

[1 mark]

(c.i) Find the eigenvalues of matrix $\boldsymbol{A}$. [3]

Markscheme

$\begin{vmatrix} -\lambda & 1 \\ -1.25 & -3 - \lambda \end{vmatrix} = 0$ ***(M1)***

$\lambda(\lambda + 3) + 1.25 = 0$ ***(A1)***

$\lambda = -2.5\ ;\ \lambda = -0.5$ ***A1***

[3 marks]

(c.ii) Find the eigenvectors of matrix $\boldsymbol{A}$. [3]

Markscheme

$$\begin{pmatrix} 2.5 & 1 \\ -1.25 & -0.5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ ***(M1)***

$$2.5a + b = 0$$

$$v_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$ ***A1***

$$\begin{pmatrix} 0.5 & 1 \\ -1.25 & -2.5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0.5a + b = 0$$

$$v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$ ***A1***

Note: Award ***M1*** for a valid attempt to find either eigenvector. Accept equivalent forms of the eigenvectors.
Do not award ***FT*** for eigenvectors that do not satisfy both rows of the matrix.

[3 marks]

(d) Given that when $t = 0$ the shock absorber is displaced 8 cm and its velocity is zero, find an expression for x in terms of t. [6]

Markscheme

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{-2.5t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} + B\mathrm{e}^{-0.5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$ ***M1A1***

$t = 0 \Rightarrow x = 8,\ \dot{x} = y = 0$ ***(M1)***

$-2A - 2B = 8$

$5A + B = 0$ ***(M1)***

$A = 1\ ;\ B = -5$ ***A1***

$x = -2\mathrm{e}^{-2.5t} + 10\mathrm{e}^{-0.5t}$ ***A1***

Note: Do not award the final ***A1*** if the answer is given in the form $\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{-2.5t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} + B\mathrm{e}^{-0.5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

[6 marks]

2. [Maximum mark: 6] EXN.1.AHL.TZ0.13

Consider the second order differential equation

$$\ddot{x} + 4(\dot{x})^2 - 2t = 0$$

where x is the displacement of a particle for $t \geq 0$.

(a) Write the differential equation as a system of coupled first order differential equations. [2]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$\dot{x} = y$ **M1**

$\dot{y} = 2t - 4y^2$ **A1**

[2 marks]

(b) When $t = 0, x = \dot{x} = 0$

Use Euler's method with a step length of 0.1 to find an estimate for the value of the displacement and velocity of the particle when $t = 1$. [4]

Markscheme

$t_{n+1} = t_n + 0.1$

$x_{n+1} = x_n + 0.1y_n$

$y_{n+1} = y_n + 0.1\left(2t_n - 4y_n^2\right)$ **(M1)(A1)**

Note: Award **M1** for a correct attempt to substitute the functions in part (a) into the formula for Euler's method for coupled systems.

When $t = 1$

$x = 0.202\ (0.20201\ldots)$ **A1**

$\dot{x} = 0.598\ (0.59822\ldots)$ **A1**

Note: Accept $y = 0.598$.

[4 marks]

3. [Maximum mark: 6] 23M.1.AHL.TZ1.13

The displacement, x (cm), of the end of a spring, at time t (seconds), is given by

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0.$$

At $t = 0$, $x = 0.75$ and $\frac{dx}{dt} = 0$.

Use Euler's method, with a step length 0.1 seconds, to estimate the value of x when $t = 0.5$. [6]

Markscheme

$\frac{dx}{dt} = y$ ***(A1)***

$\frac{dy}{dt} = -10x - 2y$ ***(A1)***

Note: Writing $\frac{d^2x}{dt^2} = -10x - 2\frac{dx}{dt}$ is a valid approach and should be awarded ***A1A1***.

attempt to use the Euler equations shown by finding either a correct x_{n+1} or y_{n+1} ***(M1)*** correct equations for both x_{n+1} and y_{n+1}

$$x_{n+1} = x_n + 0.1(y_n), \quad y_{n+1} = y_n + 0.1(-10x_n - 2y_n)$$

(accept equivalent notation)

$$(t_{n+1} = t_n + 0.1)$$

Note: All of the above marks can be implied by a correct second row in a table **OR** by a correct f_1 and f_2 clearly identified for use in Euler's method formula.

T	x	y
0	0.75	0
$\mathbf{0.1}$	$\mathbf{0.75}$	$\mathbf{-0.75}$
0.2	0.675	-1.35
0.3	0.54	-1.755
0.4	0.3645	-1.944
0.5	0.1701	

so estimate is 0.170 ***A2***

Note: Accept 0.17 rounded to 2 sf.

[6 marks]

4. [Maximum mark: 5] 22N.1.AHL.TZ0.15

An electrical circuit contains a capacitor. The charge on the capacitor, q Coulombs, at time t seconds, satisfies the differential equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + 5\frac{\mathrm{d}q}{\mathrm{d}t} + 20q = 200.$$

Initially $q = 1$ and $\frac{\mathrm{d}q}{\mathrm{d}t} = 8$.

Use Euler's method with $h = 0.1$ to estimate the maximum charge on the capacitor during the first second. [5]

Markscheme

EITHER

$$q_{n+1} = q_n + 0.1\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)_n$$

$$\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)_{n+1} = \left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)_n + 0.1\left(\frac{\mathrm{d}^2q}{\mathrm{d}t^2}\right)_n$$ ***(M1)***

OR

let $\frac{\mathrm{d}q}{\mathrm{d}t} = y$

$$q_{n+1} = q_n + 0.1y_n$$

$$y_{n+1} = y_n + 0.1\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)_n$$ ***(M1)***

THEN

EITHER

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 200 - 5y - 20q$$ ***(A1)***

OR

$$\frac{d^2q}{dt^2} = 200 - 5\frac{dq}{dt} - 20q$$ ***(A1)***

THEN

evidence of using Euler's method (e.g.) ***(M1)***

0	1	8	140
0.1	1.8	22	54

maximum charge $= 12.7$ (Coulombs, at $t = 0.7$) ***A2***

Note: Award ***A0A1*** for a final answer of 10.8, from reading the value at $t = 1$.

[5 marks]

5. [Maximum mark: 13] 22M.2.AHL.TZ1.4

A particle moves such that its displacement, x metres, from a point O at time t seconds is given by the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

(a.i) Use the substitution $y = \frac{dx}{dt}$ to show that this equation can be written as

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$ [1]

Markscheme

$y = \frac{dx}{dt} \Rightarrow \frac{dy}{dt} + 5\frac{dx}{dt} + 6x = 0$ **OR** $\frac{dy}{dt} + 5y + 6x = 0$ ***M1***

Note: Award **M1** for substituting $\frac{\mathrm{d}y}{\mathrm{d}t}$ for $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$.

$$\begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$ ***AG***

[1 mark]

(a.ii) Find the eigenvalues for the matrix $\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$. [3]

Markscheme

$$\det\begin{pmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{pmatrix} = 0$$ ***(M1)***

Note: Award ***M1*** for an attempt to find eigenvalues. Any indication that $\det(\boldsymbol{M} - \lambda\boldsymbol{I}) = 0$ has been used is sufficient for the ***(M1)***.

$-\lambda(-5-\lambda) + 6 = 0$ **OR** $\lambda^2 + 5\lambda + 6 = 0$ ***(A1)***

$\lambda = -2, \ -3$ ***A1***

[3 marks]

(a.iii) Hence state the long-term velocity of the particle. [1]

Markscheme

(on a phase portrait the particle approaches $(0,\ 0)$ as t increases so long term velocity (y) is)

0 ***A1***

Note: Only award ***A1*** for 0 if both eigenvalues in part (a)(ii) are negative. If at least one is positive accept an answer of '*no limit*' or '*infinity*', or in the case of one positive and one negative also accept '*no limit or 0 (depending on initial conditions)*'.

[1 mark]

The equation for the motion of the particle is amended to

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 3t + 4.$$

(b.i) Use the substitution $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ to write the differential equation as a system of coupled, first order differential equations. [2]

Markscheme

$y = \frac{\mathrm{d}x}{\mathrm{d}t}$

$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}y}{\mathrm{d}t}$ ***(A1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} + 5y + 6x = 3t + 4$ ***A1***

[2 marks]

When $t = 0$ the particle is stationary at O.

(b.ii) Use Euler's method with a step length of 0.1 to find the displacement of the particle when $t = 1$. [5]

Markscheme

recognition that $h = 0.1$ in any recurrence formula ***(M1)***

$(t_{n+1} = t_n + 0.1)$

$x_{n+1} = x_n + 0.1y_n$ ***(A1)***

$y_{n+1} = y_n + 0.1(3t_n + 4 - 5y_n - 6x_n)$ ***(A1)***

(when $t = 1$,) $x = 0.64402\ldots \approx 0.644\,\mathrm{m}$ ***A2***

[5 marks]

(b.iii) Find the long-term velocity of the particle. [1]

Markscheme

recognizing that y is the velocity

$0.5\,\mathrm{m\,s^{-1}}$ ***A1***

[2 marks]

6. [Maximum mark: 16] 22M.2.AHL.TZ2.7

An environmental scientist is asked by a river authority to model the effect of a leak from a power plant on the mercury levels in a local river. The variable x measures the concentration of mercury in micrograms per litre.

The situation is modelled using the second order differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$$

where $t \geq 0$ is the time measured in days since the leak started. It is known that when $t = 0,\ x = 0$ and $\frac{dx}{dt} = 1$.

(a) Show that the system of coupled first order equations:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -2x - 3y$$

can be written as the given second order differential equation. [2]

Markscheme

differentiating first equation. ***M1***

$$\frac{d^2x}{dt^2} = \frac{dy}{dt}$$

substituting in for $\frac{dy}{dt}$ ***M1***

$$= -2x - 3y = -2x - 3\frac{dx}{dt}$$

therefore $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$ ***AG***

Note: The **AG** line must be seen to award the final ***M1*** mark.

[2 marks]

(b) Find the eigenvalues of the system of coupled first order equations given in part (a). [3]

Markscheme

the relevant matrix is $\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ ***(M1)***

Note: $\begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$ is also possible.

(this has characteristic equation) $-\lambda(-3-\lambda)+2=0$ ***(A1)***

$\lambda=-1,\ -2$ ***A1***

[3 marks]

(c) Hence find the exact solution of the second order differential equation. [5]

Markscheme

EITHER

the general solution is $x = A\mathrm{e}^{-t} + B\mathrm{e}^{-2t}$ ***M1***

Note: Must have constants, but condone sign error for the ***M1***.

so $\frac{\mathrm{d}x}{\mathrm{d}t} = -A\mathrm{e}^{-t} - 2B\mathrm{e}^{-2t}$ ***M1A1***

OR

attempt to find eigenvectors ***(M1)***

respective eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (or any multiple)

$\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + B\mathrm{e}^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ ***(M1)A1***

THEN

the initial conditions become:

$0 = A + B$

$1 = -A - 2B$ ***M1***

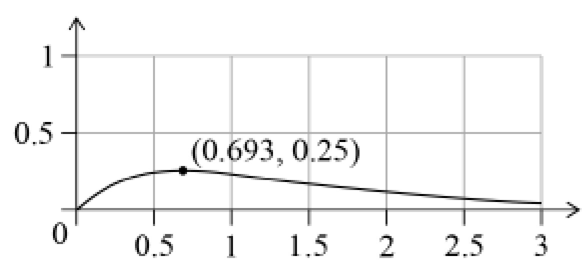
this is solved by $A = 1,\ B = -1$

so the solution is $x = \mathrm{e}^{-t} - \mathrm{e}^{-2t}$ ***A1***

[5 marks]

(d) Sketch the graph of x against t, labelling the maximum point of the graph with its coordinates. [2]

Markscheme



A1A1

Note: Award ***A1*** for correct shape (needs to go through origin, have asymptote at $y = 0$ and a single maximum; condone $x < 0$). Award ***A1*** for correct coordinates of maximum.

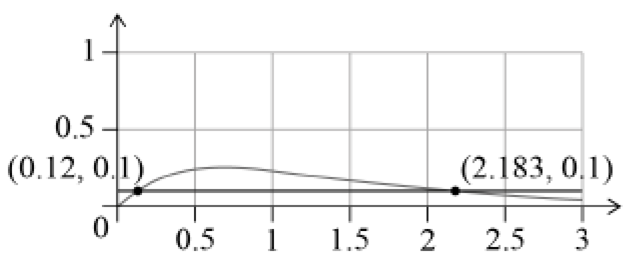
[2 marks]

If the mercury levels are greater than 0.1 micrograms per litre, fishing in the river is considered unsafe and is stopped.

(e) Use the model to calculate the total amount of time when fishing should be stopped. [3]

Markscheme

intersecting graph with $y = 0.1$ ***(M1)***



so the time fishing is stopped between $2.1830\ldots$ and $0.11957\ldots$ ***(A1)***

$= 2.06\ (343\ldots)$ days ***A1***

[3 marks]

The river authority decides to stop people from fishing in the river for 10% longer than the time found from the model.

(f) Write down one reason, with reference to the context, to support this decision. [1]

Markscheme

Any reasonable answer. For example:

There are greater downsides to allowing fishing when the levels may be dangerous than preventing fishing when the levels are safe.

The concentration of mercury may not be uniform across the river due to natural variation / randomness.

The situation at the power plant might get worse.

Mercury levels are low in water but still may be high in fish. ***R1***

Note: Award ***R1*** for a reasonable answer that refers to this specific context (and not a generic response that could apply to *any* model).

[1 mark]