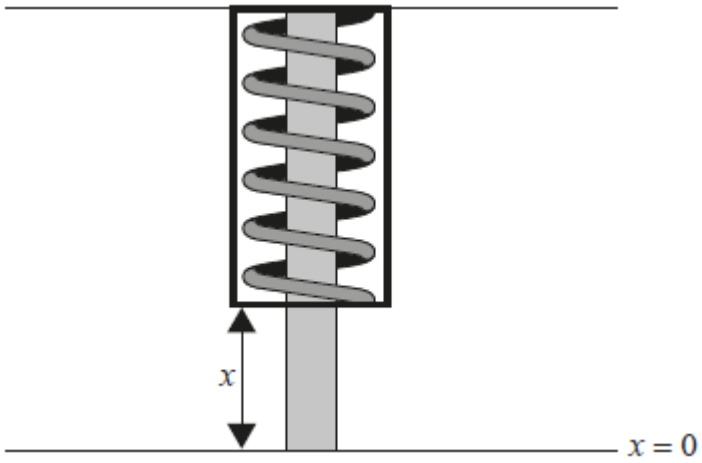
Coupled differential equations 2 [61 marks]

1. [Maximum mark: 15] 21N.2.AHL.TZ0.6

A shock absorber on a car contains a spring surrounded by a fluid. When the car travels over uneven ground the spring is compressed and then returns to an equilibrium position.



The displacement, x, of the spring is measured, in centimetres, from the equilibrium position of $x = 0$. The value of x can be modelled by the following second order differential equation, where t is the time, measured in seconds, after the initial displacement.

$$\ddot{x} + 3\dot{x} + 1.25x = 0$$

(a) Given that $y = \dot{x}$, show that $\dot{y} = -1.25x - 3y$. [2]

The differential equation can be expressed in the form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} x \\ y \end{pmatrix}$, where $\boldsymbol{A}$ is a 2×2 matrix.

(b) Write down the matrix $\boldsymbol{A}$. [1]

(c.i) Find the eigenvalues of matrix $\boldsymbol{A}$. [3]

(c.ii) Find the eigenvectors of matrix $\boldsymbol{A}$. [3]

(d) Given that when $t = 0$ the shock absorber is displaced $8\,\text{cm}$ and its velocity is zero, find an expression for x in terms of t. [6]

2. [Maximum mark: 6] EXN.1.AHL.TZ0.13

Consider the second order differential equation

$$\ddot{x} + 4(\dot{x})^2 - 2t = 0$$

where x is the displacement of a particle for $t \geq 0$.

(a) Write the differential equation as a system of coupled first order differential equations. [2]

(b) When $t = 0, x = \dot{x} = 0$

Use Euler's method with a step length of 0.1 to find an estimate for the value of the displacement and velocity of the particle when $t = 1$. [4]

3. [Maximum mark: 6] 23M.1.AHL.TZ1.13

The displacement, x (cm), of the end of a spring, at time t (seconds), is given by

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 10x = 0.$$

At $t = 0, x = 0.75$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$.

Use Euler's method, with a step length 0.1 seconds, to estimate the value of x when $t = 0.5$. [6]

4. [Maximum mark: 5] 22N.1.AHL.TZ0.15

An electrical circuit contains a capacitor. The charge on the capacitor, q Coulombs, at time t seconds, satisfies the differential equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + 5\frac{\mathrm{d}q}{\mathrm{d}t} + 20q = 200.$$

Initially $q = 1$ and $\frac{\mathrm{d}q}{\mathrm{d}t} = 8$.

Use Euler's method with $h = 0.1$ to estimate the maximum charge on the capacitor during the first second. [5]

5. [Maximum mark: 13] 22M.2.AHL.TZ1.4

A particle moves such that its displacement, x metres, from a point O at time t seconds is given by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 0$$

(a.i) Use the substitution $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ to show that this equation can be written as

$$\begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$ [1]

(a.ii) Find the eigenvalues for the matrix $\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$. [3]

(a.iii) Hence state the long-term velocity of the particle. [1]

The equation for the motion of the particle is amended to

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 3t + 4.$$

(b.i) Use the substitution $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ to write the differential equation as a system of coupled, first order differential equations. [2]

When $t = 0$ the particle is stationary at O.

(b.ii) Use Euler's method with a step length of 0.1 to find the displacement of the particle when $t = 1$. [5]

(b.iii) Find the long-term velocity of the particle. [1]

6. [Maximum mark: 16] 22M.2.AHL.TZ2.7

An environmental scientist is asked by a river authority to model the effect of a leak from a power plant on the mercury levels in a local river. The variable x measures the concentration of mercury in micrograms per litre.

The situation is modelled using the second order differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = 0$$

where $t \geq 0$ is the time measured in days since the leak started. It is known that when $t = 0,\ x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$.

(a) Show that the system of coupled first order equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2x - 3y$$

can be written as the given second order differential equation. [2]

(b) Find the eigenvalues of the system of coupled first order equations given in part (a). [3]

(c) Hence find the exact solution of the second order differential equation. [5]

(d) Sketch the graph of x against t, labelling the maximum point of the graph with its coordinates. [2]

If the mercury levels are greater than 0.1 micrograms per litre, fishing in the river is considered unsafe and is stopped.

(e) Use the model to calculate the total amount of time when fishing should be stopped. [3]

The river authority decides to stop people from fishing in the river for 10% longer than the time found from the model.

(f) Write down one reason, with reference to the context, to support this decision. [1]