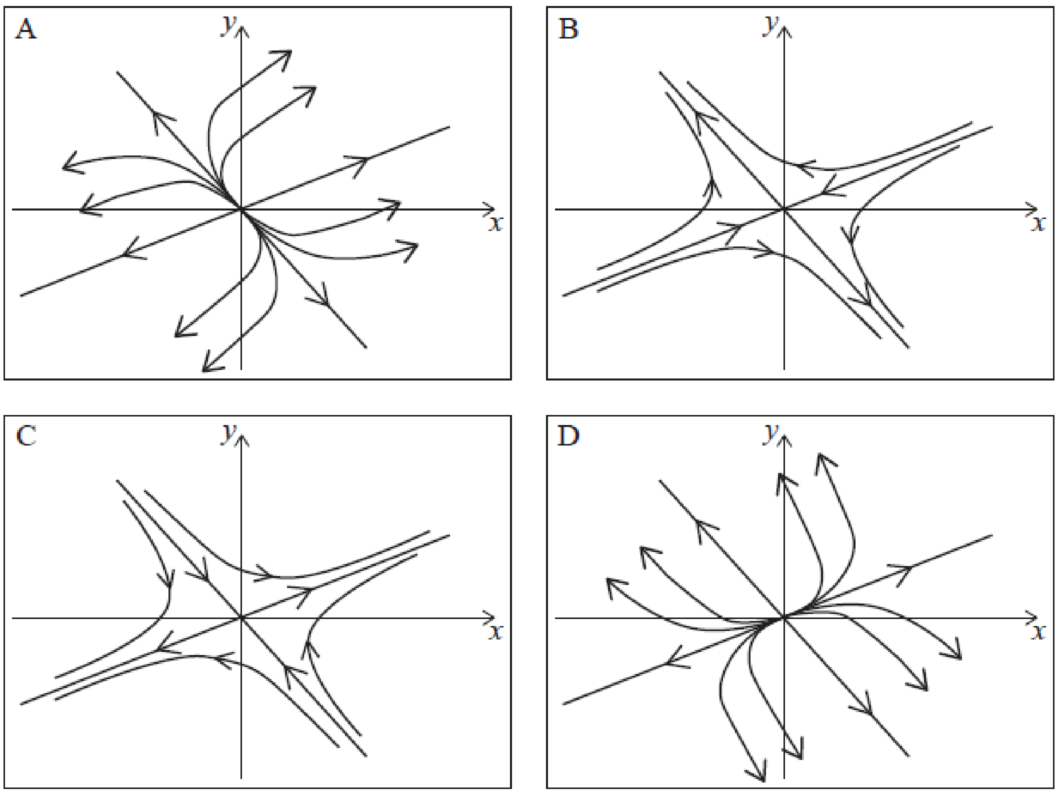
Coupled differential equations [86 marks]

1. [Maximum mark: 5] 22N.1.AHL.TZ0.12

Four possible phase portraits for the coupled differential equations $\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$ are shown, labelled A, B, C and D.



The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has eigenvalues λ_1 and λ_2.

(a) Complete the following table by writing down the letter of the phase portrait that best matches the description.

Description	Phase portrait
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = -3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	
$\lambda_1 = -2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	

[3]

Markscheme

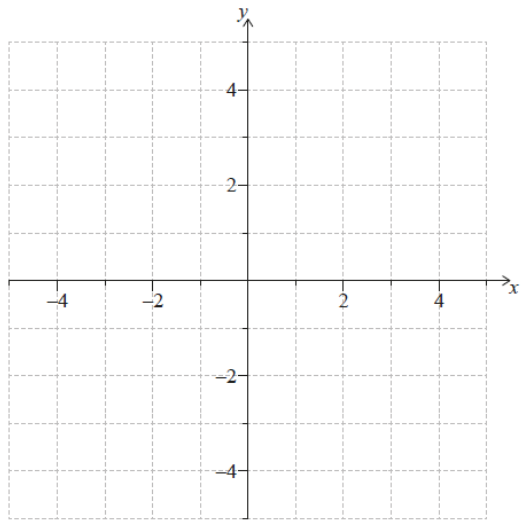
Description	Phase portrait
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	**D**
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = -3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	**C**
$\lambda_1 = -2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	**B**

A1A1A1

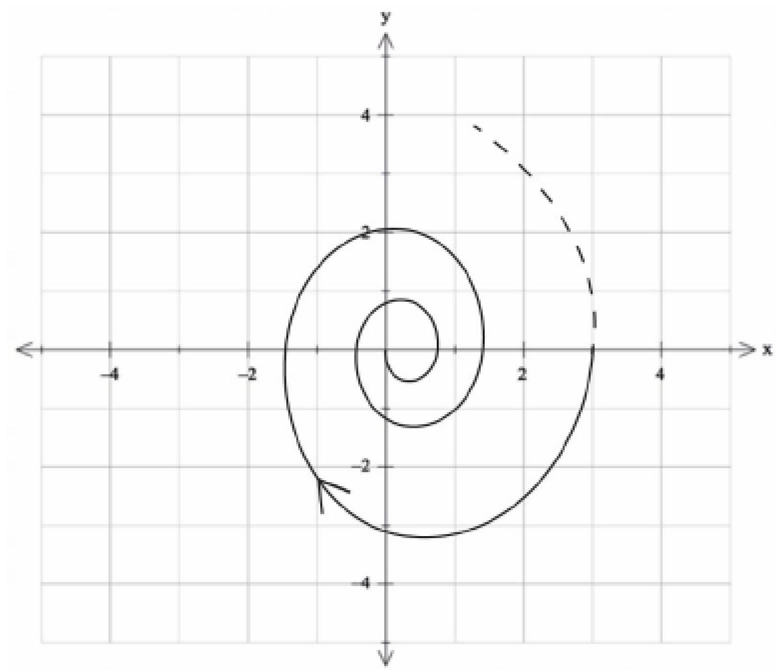
[3 marks]

(b) On the following axes, sketch the phase portrait that corresponds to $\lambda_1 = -2 + 3\mathrm{i}$ and $\lambda_2 = -2 - 3\mathrm{i}$, given that $\frac{\mathrm{d}y}{\mathrm{d}t} = -12$ at $(3,\ 0)$.

[2]



Markscheme



spiral (crossing x-axis at least twice), centre at origin ***A1***

arrow indicating clockwise, passing through or starting from $(3,\ 0)$ ***A1***

[2 marks]

2. [Maximum mark: 6] 24M.1.AHL.TZ1.15

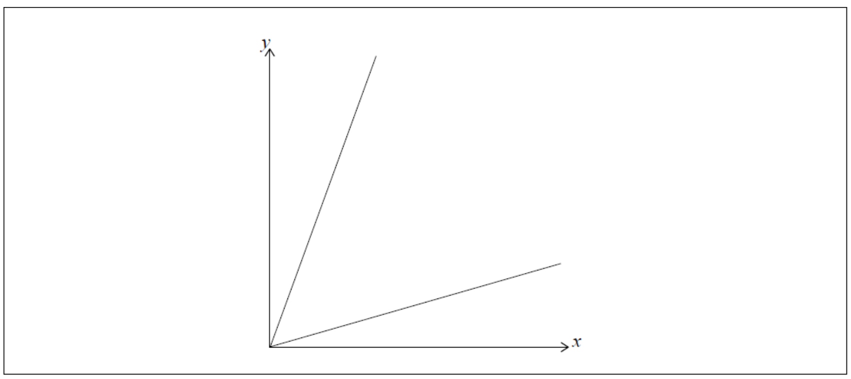
A system of differential equations of the form

$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by,\ \frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$ has eigenvalues $\lambda = -1$ and $\lambda = 2$ with corresponding eigenvectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

The following incomplete phase portrait for this system, with $x,\ y \geq 0$, shows lines through $(0,\ 0)$ parallel to the eigenvectors.

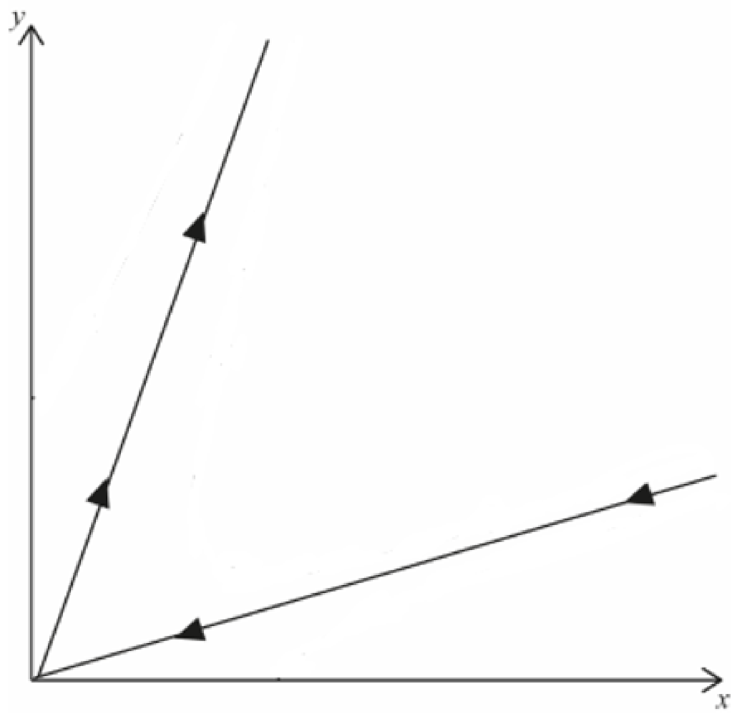
(a) On the phase portrait

(a.i) show the direction of motion along the eigenvectors.



[1]

Markscheme

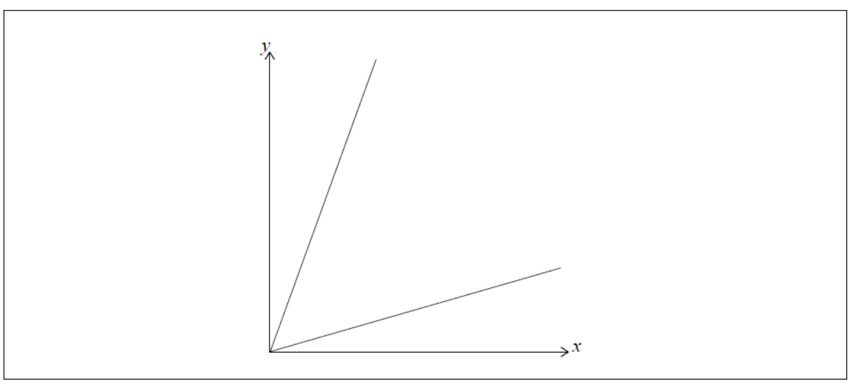


A1

Note: Award ***A1*** for correct directions on eigenvectors

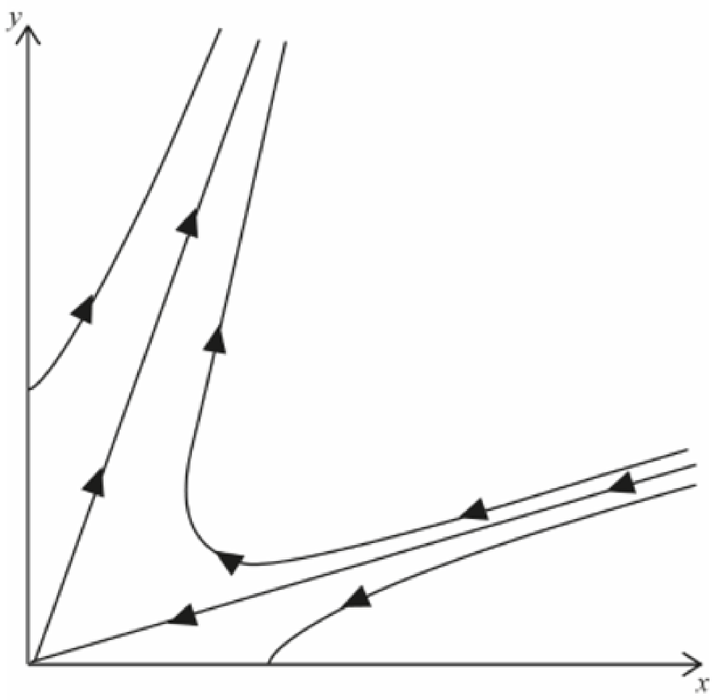
[1 mark]

(a.ii) sketch one trajectory in each of the three regions.



[2]

Markscheme



A1A1

Note: ***A1*** for correct trajectories, ***A1*** for correct arrows on trajectories.

[2 marks]

In the system described above, x and y are the population sizes of two species, X and Y. The population of Y is vulnerable, so it will be increased by adding more animals from a different area. Currently, $x = 252$ and $y = 60$.

(b) Find the minimum number of new animals from species Y that need to be added for the population not to reduce to 0 over time. [3]

Markscheme

for Y not to die out $y > \frac{1}{3}x$ ***(R1)***

as $x = 252,\ y > 84$ ***(M1)***

(minimum number of new animals is) 25 ***A1***

Note: Award ***(R1)(M1)A0*** for an unsupported 24.

[3 marks]

3. [Maximum mark: 15] 23M.2.AHL.TZ1.6

A model speedboat has its position, at time t seconds $t \geq 0$, defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5y - 0.05x, \ \frac{\mathrm{d}y}{\mathrm{d}t} = -5x - 0.05y,$$

where x metres is the distance east and y metres is the distance north of a fixed point O.

(a) Find the eigenvalues of $\boldsymbol{A} = \begin{pmatrix} -0.05 & 5 \\ -5 & -0.05 \end{pmatrix}$, giving your answers in the form $a + b\mathrm{i}$, where $a \neq 0, b \neq 0$. [4]

Markscheme

attempt to solve $\det(A - \lambda I) = 0$ ***(M1)***

$(-0.05 - \lambda)^2 + 25 = 0$ ***(A1)***

$-0.05 - \lambda = \pm 5\mathrm{i}$ ***(A1)***

$\lambda = -0.05 \pm 5\mathrm{i}$ ***A1***

[4 marks]

(b.i) State what $a \neq 0$ indicates about the path of the speedboat. [1]

Markscheme

spiral ***A1***

[1 mark]

(b.ii) State what the sign of a indicates about the path of the speedboat. [1]

Markscheme

inwards / towards O ***A1***

[1 mark]

At time $t = 0$, the speedboat has position $(20,\ 0)$.

(c) At time $t = 0$, find the value of

(c.i) $\frac{\mathrm{d}y}{\mathrm{d}t}$. [2]

Markscheme

attempt to substitute $(20,\ 0)$ into expression for $\frac{\mathrm{d}y}{\mathrm{d}t}$ ***(M1)***

$-5(20) - 0.05(0)$

$\frac{\mathrm{d}y}{\mathrm{d}t} = -100\left(\mathrm{m\,s^{-1}}\right)$ ***A1***

[2 marks]

(c.ii) $\frac{\mathrm{d}y}{\mathrm{d}x}$. [3]

Markscheme

$\frac{\mathrm{d}x}{\mathrm{d}t} = -1$ ***(A1)***

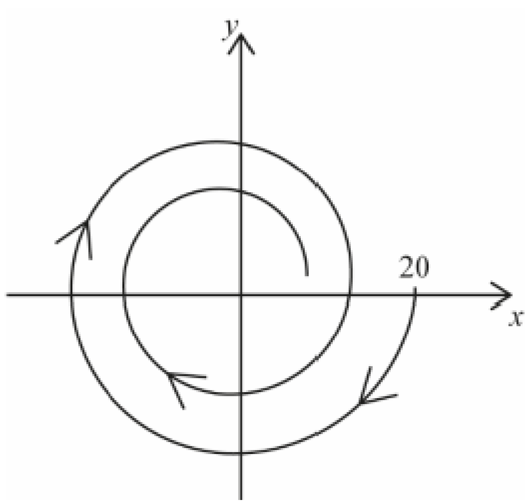
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$ **OR** $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$ ***(M1)***

$(= -100 \div -1 =)\ 100$ ***A1***

[3 marks]

(d) Use your answers to parts (b) and (c) to sketch the path of the model speedboat. [4]

Markscheme



A4

Note: Award ***A1*** for starting at $(20,\ 0)$, ***A1*** for spiral inwards, ***A1*** for clockwise, ***A1*** for non-negative gradient at (20, 0).

[4 marks]

4. [Maximum mark: 6] SPM.1.AHL.TZ0.13

The rates of change of the area covered by two types of fungi, X and Y, on a particular tree are given by the following equations, where x is the area covered by X and y is the area covered by Y.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x - 2y$$

The matrix $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ has eigenvalues of 2 and −1 with corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Initially x = 8 cm^2 and y = 10 cm^2.

(a) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $t = 0$. [2]

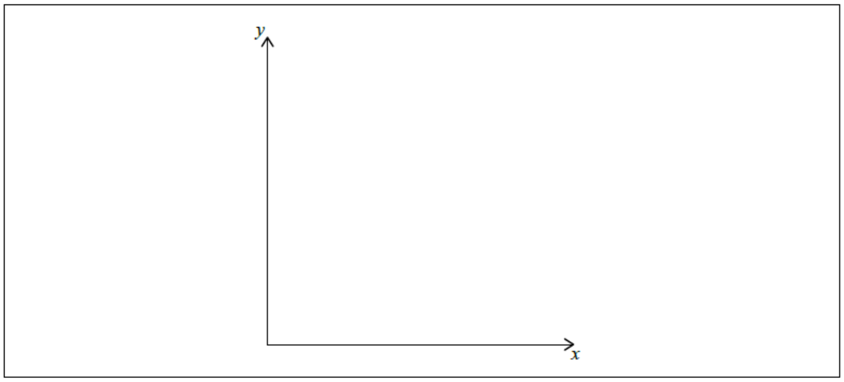
Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{16-20}{24-20}$ ***M1***

= −1 ***A1***

[2 marks]

(b) On the following axes, sketch a possible trajectory for the growth of the two fungi, making clear any asymptotic behaviour.



[4]

Markscheme

asymptote of trajectory along $\boldsymbol{r} = k\begin{pmatrix}2\\1\end{pmatrix}$ ***M1A1***

Note: Award ***M1A0*** if asymptote along $\begin{pmatrix}1\\2\end{pmatrix}$.

trajectory begins at (8, 10) with negative gradient ***A1A1***

[4 marks]

5. [Maximum mark: 19] EXN.2.AHL.TZ0.5

A change in grazing habits has resulted in two species of herbivore, X and Y, competing for food on the same grasslands. At time $t = 0$ environmentalists begin to record the sizes of both populations. Let the size of the population of X be x, and the size of the population Y be y. The following model is proposed for predicting the change in the sizes of the two populations:

$$\dot{x} = 0.3x - 0.1y$$

$$\dot{y} = -0.2x + 0.4y$$

for $x,\ y > 0$

For this system of coupled differential equations find

(a.i) the eigenvalues. [3]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$\begin{vmatrix} 0.3-\lambda & -0.1 \\ -0.2 & 0.4-\lambda \end{vmatrix} = 0$ **(M1)(A1)**

$\lambda = 0.5$ and 0.2 **A1**

[3 marks]

(a.ii) the eigenvectors. [3]

Markscheme

Attempt to solve either

$\begin{pmatrix} -0.2 & -0.1 \\ -0.2 & -0.1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ **or**

$\begin{pmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or equivalent **(M1)**

$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **A1A1**

Note: accept equivalent forms

[3 marks]

(b) Hence write down the general solution of the system of equations. [1]

Markscheme

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{0.5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + B\mathrm{e}^{0.2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ **A1**

[1 mark]

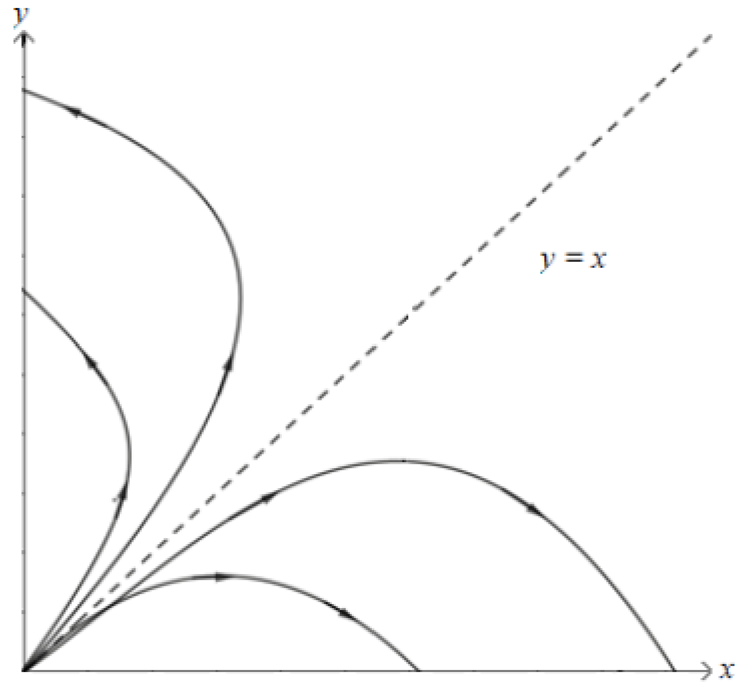
(c) Sketch the phase portrait for this system, for $x,\ y > 0$.

On your sketch show

- the equation of the line defined by the eigenvector in the first quadrant
- at least two trajectories either side of this line using arrows on those trajectories to represent the change in populations as *t* increases

[3]

Markscheme



A1A1A1

Note: **A1** for $y = x$ correctly labelled, **A1** for at least two trajectories above $y = x$ and **A1** for at least two trajectories below $y = x$, including arrows.

[3 marks]

When $t = 0$ X has a population of 2000.

(d) Write down a condition on the size of the initial population of Y if it is to avoid its population reducing to zero. [1]

Markscheme

$y > 2000$ **A1**

[1 mark]

It is known that Y has an initial population of 2900.

(e.i) Find the value of t at which $x = 0$. [6]

Markscheme

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{0.5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + Be^{0.2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

At $t = 0\ 2000 = A + B,\ 2900 = -2A + B$ **M1A1**

Note: Award **M1** for the substitution of 2000 and 2900

Hence $A = -300,\ B = 2300$ **A1A1**

$0 = -300\mathrm{e}^{0.5t} + 2300\mathrm{e}^{0.2t}$ **M1**

$t = 6.79\ (6.7896\ldots)$ (years) **A1**

[6 marks]

(e.ii) Find the population of Y at this value of t. Give your answer to the nearest 10 herbivores. [2]

Markscheme

$y = 600\mathrm{e}^{0.5\times 6.79} + 2300\mathrm{e}^{0.2\times 6.79}$ **(M1)**

$= 26827.9\ldots$

$= 26830$ (to the nearest 10 animals) **A1**

[2 marks]

6. [Maximum mark: 18] EXM.2.AHL.TZ0.1

Consider the system of paired differential equations

$$\dot{x} = 3x + 2y$$

$$\dot{y} = 2x + 3y.$$

This represents the populations of two species of symbiotic toadstools in a large wood.

Time t is measured in decades.

(a) Use the eigenvalue method to find the general solution to this system of equations. [10]

Markscheme

The characteristic equation is given by

$$\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 6\lambda + 5 = 0 \Rightarrow \lambda = 1 \text{ or } 5$$

M1A1A1A1

$$\lambda = 1 \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives an eigenvector of form } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

M1A1

$\lambda = 5 \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ gives an eigenvector of form $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

M1A1

General solution is $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ***A1A1***

[10 marks]

(b.i) Given the initial conditions that when $t = 0, x = 150$, $y = 50$, find the particular solution. [3]

Markscheme

Require $A + B = 150, \ - A + B = 50 \Rightarrow A = 50, \ \mathrm{B} = 100$ ***M1A1***

Particular solution is $\begin{pmatrix} x \\ y \end{pmatrix} = 50e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 100e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ***A1***

[3 marks]

(b.ii) Hence find the solution when $t = 1$. [1]

Markscheme

$t = 1 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15000 \\ 14700 \end{pmatrix} \ (3sf)$ ***A1***

[1 mark]

(c) As $t \to \infty$, find an asymptote to the trajectory of the particular solution found in (b)(i) and state if this trajectory will be moving towards or away from the origin. [4]

Markscheme

The dominant term is $100e^{5t}\begin{pmatrix}1\\1\end{pmatrix}$ so as $t\to\infty$,

$\begin{pmatrix}x\\y\end{pmatrix}\simeq 100e^{5t}\begin{pmatrix}1\\1\end{pmatrix}$ ***M1A1***

Giving the asymptote as $y=x$ ***A1***

The trajectory is moving away from the origin. ***A1***

[4 marks]

7. [Maximum mark: 17] 21M.2.AHL.TZ2.7

Consider the following system of coupled differential equations.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-4x$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=3x-2y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix}-4 & 0\\ 3 & -2\end{pmatrix}$. [6]

Markscheme

$\begin{vmatrix}-4-\lambda & 0\\ 3 & -2-\lambda\end{vmatrix}=0$ ***(M1)***

$(-4-\lambda)(-2-\lambda)=0$ ***(A1)***

$\lambda=-4$ **OR** $\lambda=-2$ ***A1***

$\lambda=-4$

$$\begin{pmatrix}-4 & 0\\ 3 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-4x\\ -4y\end{pmatrix}$$ ***(M1)***

Note: This ***M1*** can be awarded for attempting to find either eigenvector.

$3x-2y=-4y$

$3x=-2y$

possible eigenvector is $\begin{pmatrix}-2\\ 3\end{pmatrix}$ (or any real multiple) ***A1***

$\lambda=-2$

$$\begin{pmatrix}-4 & 0\\ 3 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-2x\\ -2y\end{pmatrix}$$

$x=0,\ y=1$

possible eigenvector is $\begin{pmatrix}0\\ 1\end{pmatrix}$ (or any real multiple) ***A1***

[6 marks]

(b) Hence, write down the general solution of the system. [2]

Markscheme

$$\begin{pmatrix}x\\ y\end{pmatrix}=A\mathrm{e}^{-4t}\begin{pmatrix}-2\\ 3\end{pmatrix}+B\mathrm{e}^{-2t}\begin{pmatrix}0\\ 1\end{pmatrix}$$ ***(M1)A1***

Note: Award ***M1A1*** for $x=-2A\mathrm{e}^{-4t},\ y=3A\mathrm{e}^{-4t}+B\mathrm{e}^{-2t}$, ***M1A0*** if LHS is missing or incorrect.

[2 marks]

(c) Determine, with justification, whether the equilibrium point $(0,\ 0)$ is stable or unstable. [2]

Markscheme

two (distinct) real negative eigenvalues ***R1***

(or equivalent (*eg* both $\mathrm{e}^{-4t} \to 0,\ \mathrm{e}^{-2t} \to 0$ as $t \to \infty$))

⇒ stable equilibrium point ***A1***

Note: Do not award ***R0A1***.

[2 marks]

Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$

(d) (i) at $(4,\ 0)$.

(ii) at $(-4,\ 0)$. [3]

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x-2y}{-4x}$ ***(M1)***

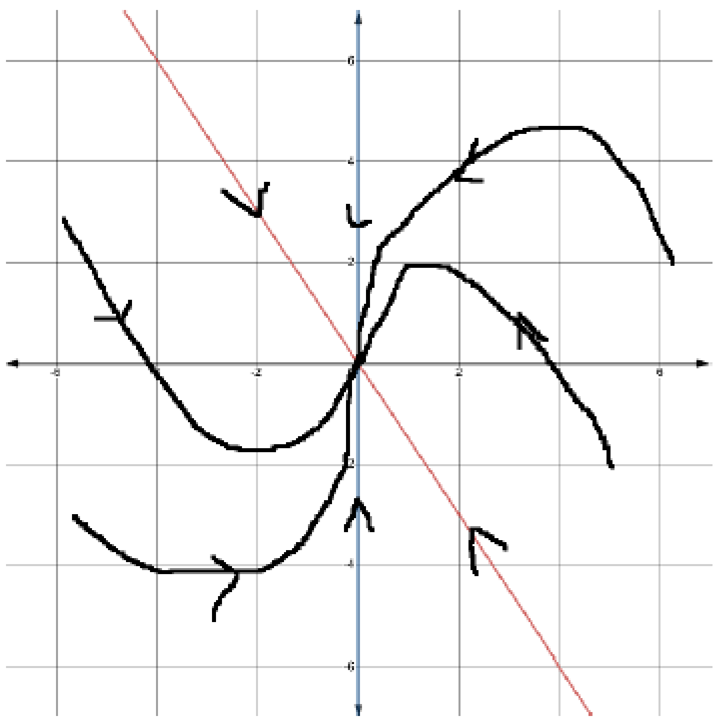
(i) $\left(4,\ 0\right) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3}{4}$ ***A1***

(ii) $\left(-4,\ 0\right) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3}{4}$ ***A1***

[3 marks]

(e) Sketch a phase portrait for the general solution to the system of coupled differential equations for $-6 \leq x \leq 6$, $-6 \leq y \leq 6$. [4]

Markscheme



A1A1A1A1

Note: Award ***A1*** for a phase plane, with correct axes (condone omission of labels) and at least three non-overlapping trajectories. Award ***A1*** for all trajectories leading to a stable node at $(0,\ 0)$. Award ***A1*** for showing gradient is negative at $x = 4$ and -4. Award ***A1*** for both eigenvectors on diagram.

[4 marks]