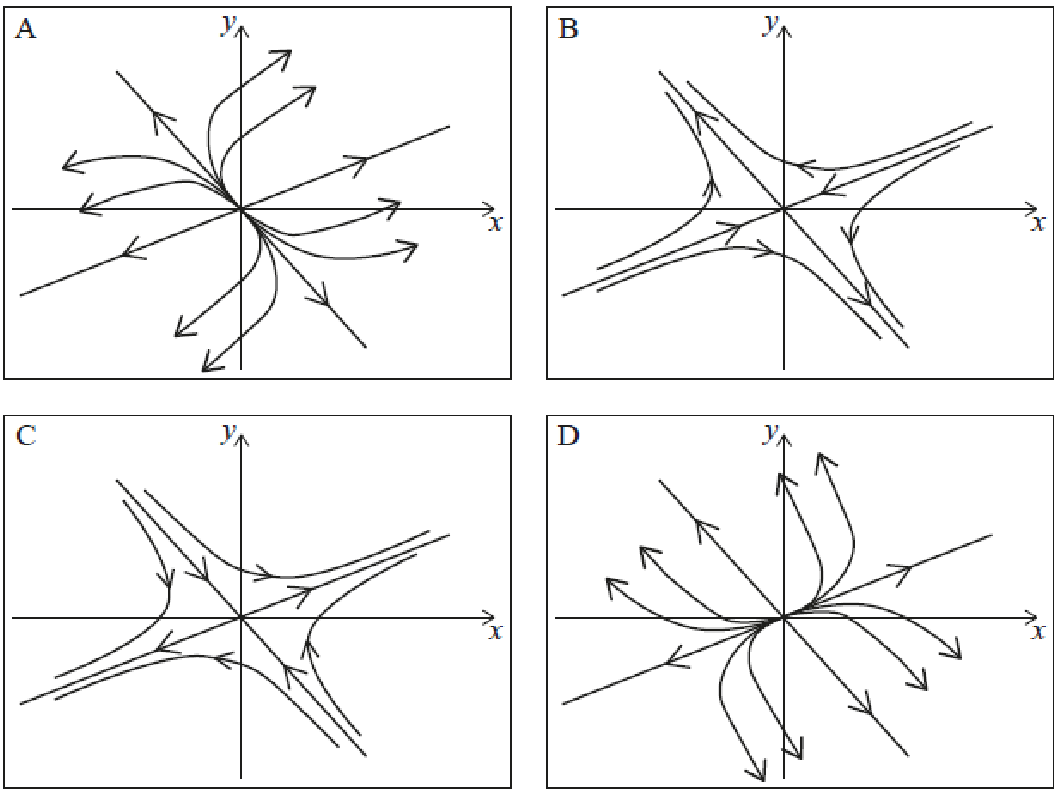
Coupled differential equations [86 marks]

1. [Maximum mark: 5] 22N.1.AHL.TZ0.12

Four possible phase portraits for the coupled differential equations $\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$ are shown, labelled A, B, C and D.



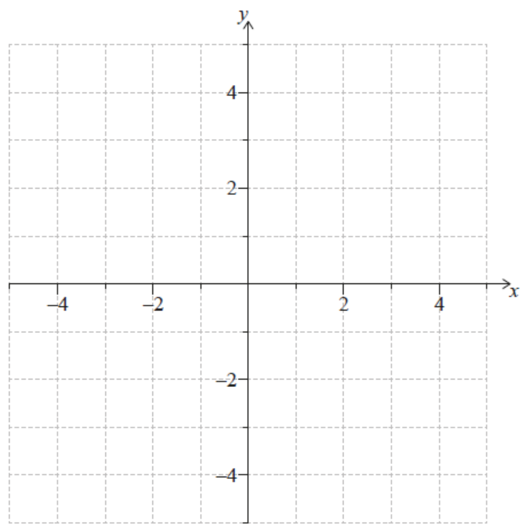
The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has eigenvalues λ_1 and λ_2.

(a) Complete the following table by writing down the letter of the phase portrait that best matches the description.

Description	Phase portrait
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix}2\\1\end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix}1\\-1\end{pmatrix}$	
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix}2\\1\end{pmatrix}$ and $\lambda_2 = -3$ with eigenvector $\begin{pmatrix}1\\-1\end{pmatrix}$	
$\lambda_1 = -2$ with eigenvector $\begin{pmatrix}2\\1\end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix}1\\-1\end{pmatrix}$	

[3]

(b) On the following axes, sketch the phase portrait that corresponds to $\lambda_1 = -2 + 3\text{i}$ and $\lambda_2 = -2 - 3\text{i}$, given that $\frac{\mathrm{d}y}{\mathrm{d}t} = -12$ at $(3,\ 0)$.



[2]

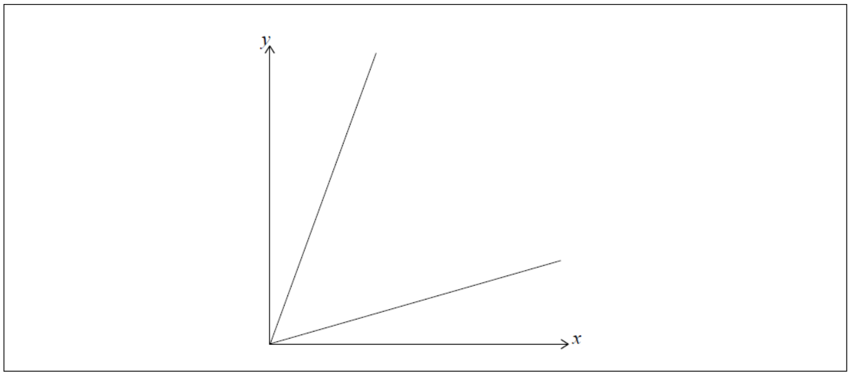
2. [Maximum mark: 6] 24M.1.AHL.TZ1.15

A system of differential equations of the form $\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by,\ \frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$ has eigenvalues $\lambda = -1$ and $\lambda = 2$ with corresponding eigenvectors $\begin{pmatrix}3\\1\end{pmatrix}$ and $\begin{pmatrix}1\\3\end{pmatrix}$.

The following incomplete phase portrait for this system, with $x,\ y \geq 0$, shows lines through $(0,\ 0)$ parallel to the eigenvectors.

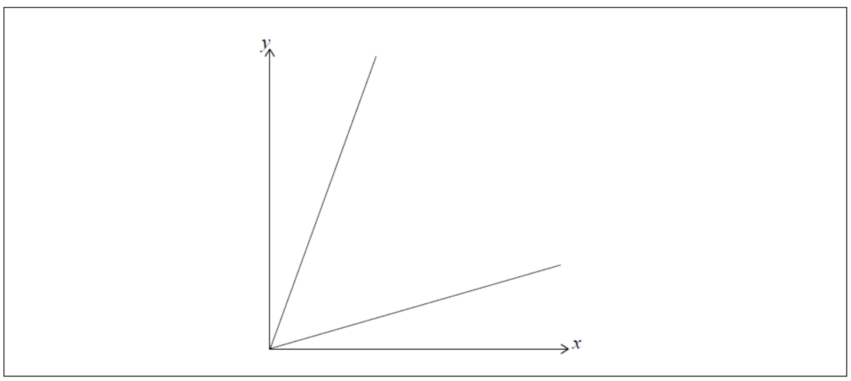
(a) On the phase portrait

(a.i) show the direction of motion along the eigenvectors.



[1]

(a.ii) sketch one trajectory in each of the three regions.



[2]

In the system described above, x and y are the population sizes of two species, X and Y. The population of Y is vulnerable, so it will be increased by adding more animals from a different area. Currently, $x = 252$ and $y = 60$.

(b) Find the minimum number of new animals from species Y that need to be added for the population not to reduce to 0 over time. [3]

3. [Maximum mark: 15] 23M.2.AHL.TZ1.6

A model speedboat has its position, at time t seconds $t \geq 0$, defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5y - 0.05x,\ \frac{\mathrm{d}y}{\mathrm{d}t} = -5x - 0.05y,$$

where x metres is the distance east and y metres is the distance north of a fixed point O.

(a) Find the eigenvalues of $\boldsymbol{A} = \begin{pmatrix} -0.05 & 5 \\ -5 & -0.05 \end{pmatrix}$, giving your answers in the form $a + b\mathrm{i}$, where $a \neq 0, b \neq 0$. [4]

(b.i) State what $a \neq 0$ indicates about the path of the speedboat. [1]

(b.ii) State what the sign of a indicates about the path of the speedboat. [1]

At time $t = 0$, the speedboat has position $(20,\ 0)$.

(c) At time $t = 0$, find the value of

(c.i) $\frac{\mathrm{d}y}{\mathrm{d}t}$. [2]

(c.ii) $\frac{\mathrm{d}y}{\mathrm{d}x}$. [3]

(d) Use your answers to parts (b) and (c) to sketch the path of the model speedboat. [4]

4. [Maximum mark: 6] SPM.1.AHL.TZ0.13

The rates of change of the area covered by two types of fungi, X and Y, on a particular tree are given by the following equations, where x is the area covered by X and y is the area covered by Y.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 2y$$

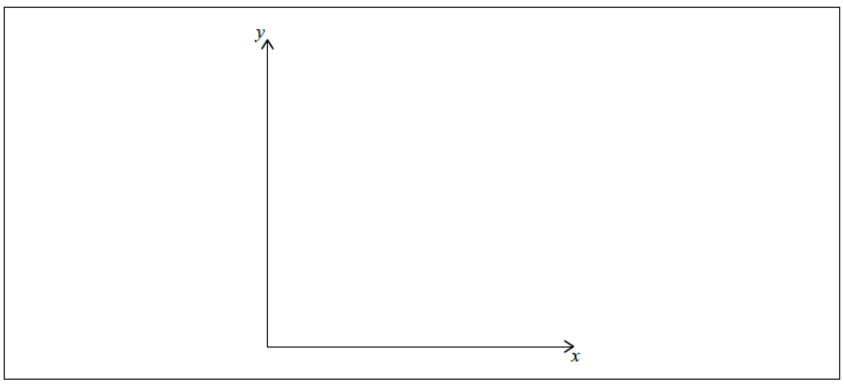
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x - 2y$$

The matrix $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ has eigenvalues of 2 and −1 with corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Initially $x = 8\ \text{cm}^2$ and $y = 10\ \text{cm}^2$.

(a) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $t = 0$. [2]

(b) On the following axes, sketch a possible trajectory for the growth of the two fungi, making clear any asymptotic behaviour.



[4]

5. [Maximum mark: 19] EXN.2.AHL.TZ0.5

A change in grazing habits has resulted in two species of herbivore, X and Y, competing for food on the same grasslands. At time $t = 0$ environmentalists begin to record the sizes of both populations. Let the size of the population of X be x, and the size of the population Y be y. The following model is proposed for predicting the change in the sizes of the two populations:

$$\dot{x} = 0.3x - 0.1y$$

$$\dot{y} = -0.2x + 0.4y$$

for $x,\ y > 0$

For this system of coupled differential equations find (a.i) the eigenvalues. [3]

(a.ii) the eigenvectors. [3]

(b) Hence write down the general solution of the system of equations. [1]

(c) Sketch the phase portrait for this system, for $x,\ y > 0$.

On your sketch show

- the equation of the line defined by the eigenvector in the first quadrant
- at least two trajectories either side of this line using arrows on those trajectories to represent the change in populations as *t* increases [3]

When $t = 0$ X has a population of 2000.

(d) Write down a condition on the size of the initial population of Y if it is to avoid its population reducing to zero. [1]

It is known that Y has an initial population of 2900.

(e.i) Find the value of t at which $x = 0$. [6]

(e.ii) Find the population of Y at this value of t. Give your answer to the nearest 10 herbivores. [2]

6. [Maximum mark: 18] EXM.2.AHL.TZ0.1

Consider the system of paired differential equations

$$\dot{x} = 3x + 2y$$

$$\dot{y} = 2x + 3y.$$

This represents the populations of two species of symbiotic toadstools in a large wood.

Time t is measured in decades.

(a) Use the eigenvalue method to find the general solution to this system of equations. [10]

(b.i) Given the initial conditions that when $t = 0, x = 150$, $y = 50$, find the particular solution. [3]

(b.ii) Hence find the solution when $t = 1$. [1]

(c) As $t \to \infty$, find an asymptote to the trajectory of the particular solution found in (b)(i) and state if this trajectory will be moving towards or away from the origin. [4]

7. [Maximum mark: 17] 21M.2.AHL.TZ2.7

Consider the following system of coupled differential equations.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3x - 2y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}$. [6]

(b) Hence, write down the general solution of the system. [2]

(c) Determine, with justification, whether the equilibrium point $(0,\ 0)$ is stable or unstable. [2]

Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$

(d) (i) at $(4,\ 0)$.

(ii) at $(-4,\ 0)$. [3]

(e) Sketch a phase portrait for the general solution to the system of coupled differential equations for $-6 \le x \le 6$, $-6 \le y \le 6$. [4]