Differential equations [133 marks]

1. [Maximum mark: 4] 24M.1.AHL.TZ1.4

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \log_{10}(x+y)$, where $x \geq 0$ and $y > 0$.

Given that $y = 1$ when $x = 0$, use Euler's method with a step length of 0.1 to find an approximate value for y when $x = 2$. [4]

Markscheme

attempt to use Euler

$y_{n+1} = y_n + 0.1 \log(x_n + y_n)$ ***(A1)***

$y_1 (= 1 + 0.1 \times \log_{10}(1)) = 1$ ***(A1)***

$y_2 = 1.004139\ldots$ ***(A1)***

THEN

when $x = 2$ $y(2) \approx 1.61\ (1.60536\ldots)$ ***A1***

[4 marks]

2. [Maximum mark: 4] 24M.1.AHL.TZ2.7

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = xy - 1$, given that $y = 2$ when $x = 1$.

Use Euler's method with step size 0.1 to find the approximate value of y when $x = 1.5$. [4]

Markscheme

attempt at Euler ***(M1)***

$y_{n+1} = (y_n x_n - 1)\ 0.1 \times y_n$ ***(A1)***

x	1	1.1	1.2	1.3	1.4	1.5
y	2	2.1	2.231	2.39872	2.61055…	2.87603…

(A1)

Note: Any correct value either as a separate calculation or as part of a table, (including the result).

$y = 2.88\ (2.87603\ldots)$ ***A1***

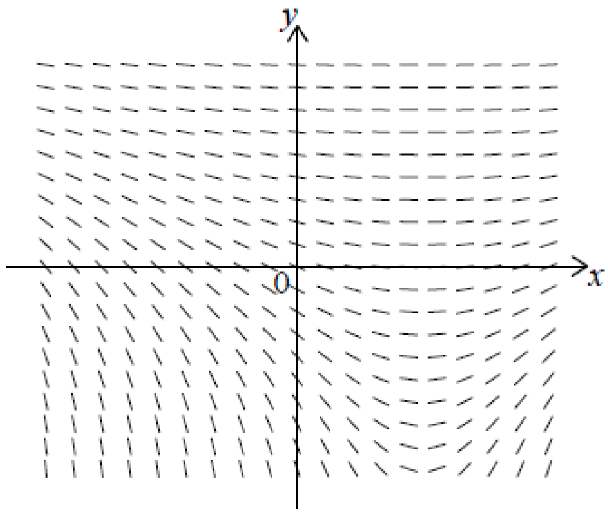
[4 marks]

3. [Maximum mark: 13] 24M.2.AHL.TZ1.5

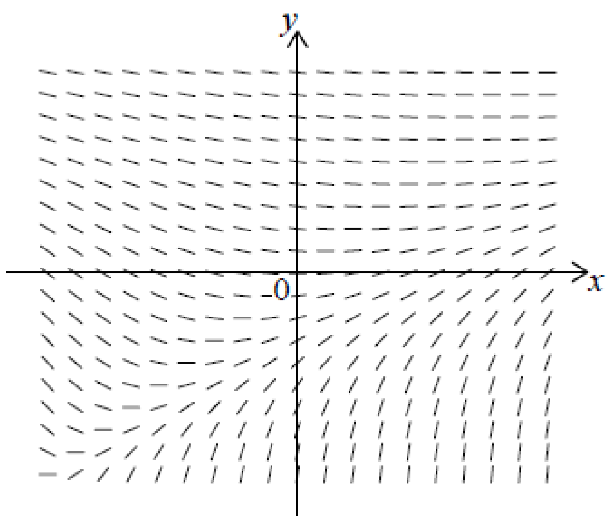
Consider the differential equation $\frac{dy}{dx} = \frac{x}{e^{2y}}$.

(a) Identify which of the following diagrams, **A**, **B** or **C**, represents the slope field for the differential equation. Give a reason for your answer.

A.

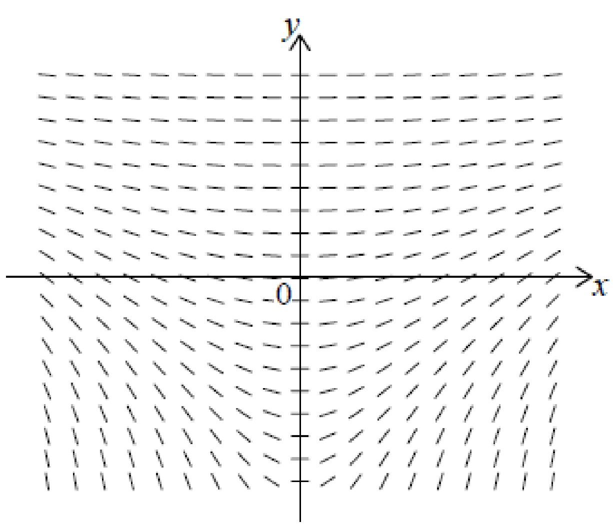


B.



[2]

C.



Markscheme

C. ***A1***

Any valid reason for accepting C. or rejecting A. and B. ***R1***

for example:

- when $x = 0$ slopes have (or appear to have) zero gradient

- (slope field is) always positive for $x > 0$

Note: Allow ***A1R0***.

[2 marks]

It is given that, for a particular solution, $x = 0$ and $y = 0$.

(b) Find an expression for y, in terms of x, for this solution. [7]

Markscheme

$\int \mathrm{e}^{2y} dy = \int x \mathrm{d}x$ ***(M1)***

$\frac{1}{2}\mathrm{e}^{2y} = \frac{1}{2}x^2\ (+c)$ ***(A1)(A1)***

Note: ***A1*** for left hand side, ***A1*** for right hand side.

substituting in $x = 0, y = 0$ ***(M1)***

$\frac{1}{2} = c$ ***(A1)***

Note: The substitution may be seen and credited later, however at that point the constant term may be 1.

$e^{2y} = x^2 + 1$

$y = \frac{1}{2} \ln \left(x^2 + 1\right)$ ***M1A1***

Note: Award ***M1*** for use of log law.

[7 marks]

(c) Find $\frac{dy}{dx}$, in terms of x, by differentiating your answer from part (b). [2]

Markscheme

$\frac{dy}{dx} = \frac{1}{2} \times 2x \times \frac{1}{x^2+1} \left(= \frac{x}{x^2+1}\right)$ ***M1A1***

Note: Award ***M1*** for use of chain rule, or use of implicit differentiation of the penultimate line of the answer to (b).

[2 marks]

(d) Hence verify that your answer to part (b) is a solution to $\frac{dy}{dx} = \frac{x}{e^{2y}}$. [2]

Markscheme

substitution of $e^{2y} = x^2 + 1$ from part (b) into part(c)(i) or original differential equation ***M1***

$\frac{dy}{dx} = \frac{x}{x^2+1} = \frac{x}{e^{2y}}$ ***A1***

and hence $y = \frac{1}{2} \ln \left(x^2 + 1\right)$ is a solution for the differential equation ***AG***

Note: Only award the ***A1*** as follow-through if their $\frac{dy}{dx}$ is of the form $\frac{x}{x^2+c}$.

[2 marks]

4. [Maximum mark: 4] 23N.1.AHL.TZ0.12

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}\chi} = 3x - y + 1$.

(a) Find the equation of the tangent to the solution curve at the point $(-1,\ -1)$ in the form $ax + by + c = 0$. [2]

Markscheme

gradient $(= \ -3\ +\ 1\ +\ 1)\ =\ -1$ ***A1***

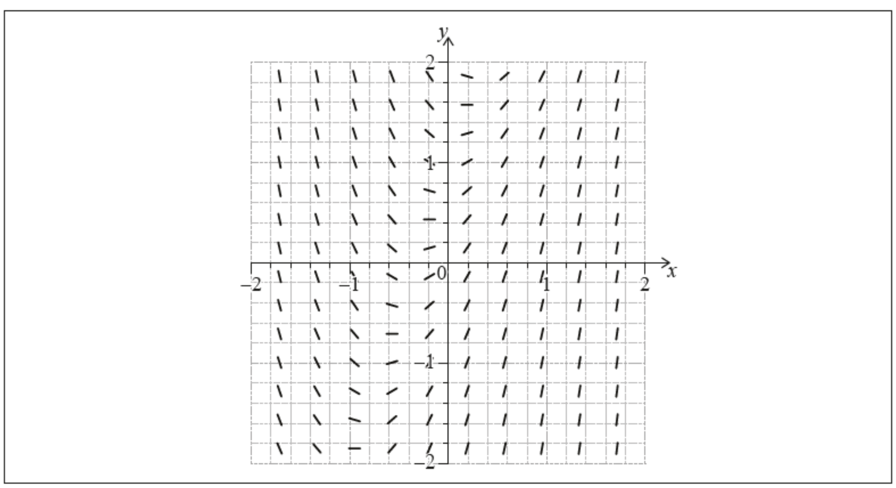
$y\ +\ 1\ =\ -1(x\ +\ 1)$

$x\ +\ y\ +\ 2\ =\ 0$ ***A1***

[2 marks]

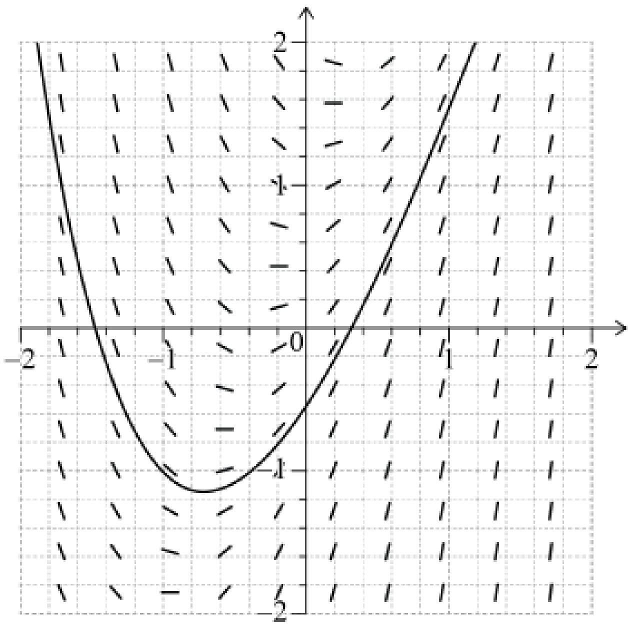
The slope field for this differential equation is shown in the following diagram.

(b) Sketch the solution curve that passes through the point $(-1,\ -1)$.



[2]

Markscheme



A1A1

Note: Award ***A1*** for (approximately) intersecting $(-1,\ -1)$ and with correct gradient, ***A1*** for generally plausible shape (e.g. not crossing over LOTS of isoclines).

[2 marks]

5. [Maximum mark: 6] 23M.1.AHL.TZ2.17

Consider the differential equation

$\left(x^2+1\right)\frac{dy}{dx} = \frac{x}{2y-2}$, for $x \geq 0, y \geq 1$,

where $y = 1$ when $x = 0$.

(a) Explain why Euler's method cannot be used to find an approximate value for y when $x = 0.1$. [1]

Markscheme

$\frac{dy}{dx}$ is undetermined at $(0,\ 1)$ ***R1***

$\left(\text{so cannot use } y_n = y_{n-1} + h\left(\frac{x}{(x^2+1)(2y-2)}\right)\right)$

Note: Accept "undefined", "indeterminate" or "division by zero" in place of "undetermined".

[1 mark]

(b) By solving the differential equation, show that $y = 1 + \sqrt{\frac{\ln(x^2+1)}{2}}$. [4]

Markscheme

$\int \left(2y - 2\right) dy = \int \frac{x}{x^2+1} dx$ ***M1***

$y^2 - 2y = \frac{1}{2} \ln\left(x^2 + 1\right) + c$ ***A1***

substituting $x = 0, y = 1$ ***M1***

$c = -1$

$y^2 - 2y + 1 = \frac{1}{2} \ln\left(x^2 + 1\right)$

$\left(y - 1\right)^2 = \frac{1}{2} \ln\left(x^2 + 1\right)$ ***A1***

$y - 1 = \sqrt{\frac{1}{2} \ln\left(x^2 + 1\right)}$ (where positive root required as $y \geq 1$)

$y = 1 + \sqrt{\frac{\ln(x^2+1)}{2}}$ ***AG***

[4 marks]

(c) Hence deduce the value of y when $x = 0.1$. [1]

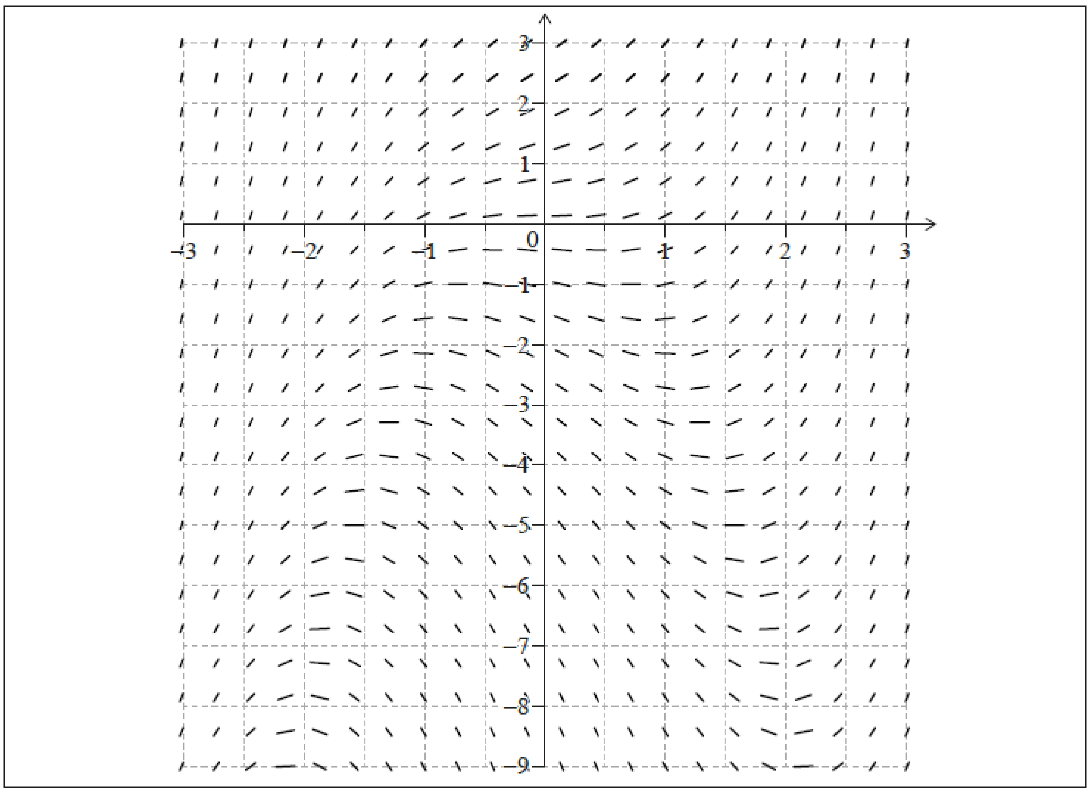
Markscheme

(when $x = 0.1$) $y = 1.07\ (1.07053\ldots)$ ***A1***

[1 mark]

6. [Maximum mark: 4] 22M.1.AHL.TZ1.7

A slope field for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + \frac{y}{2}$ is shown.



Some of the solutions to the differential equation have a local maximum point and a local minimum point.

(a.i) Write down the equation of the curve on which all these maximum and minimum points lie. [1]

Markscheme

$x^2 + \frac{y}{2} = 0\ \left(y = -2x^2\right)$ ***A1***

[1 mark]

(a.ii) Sketch this curve on the slope field. [1]

Markscheme

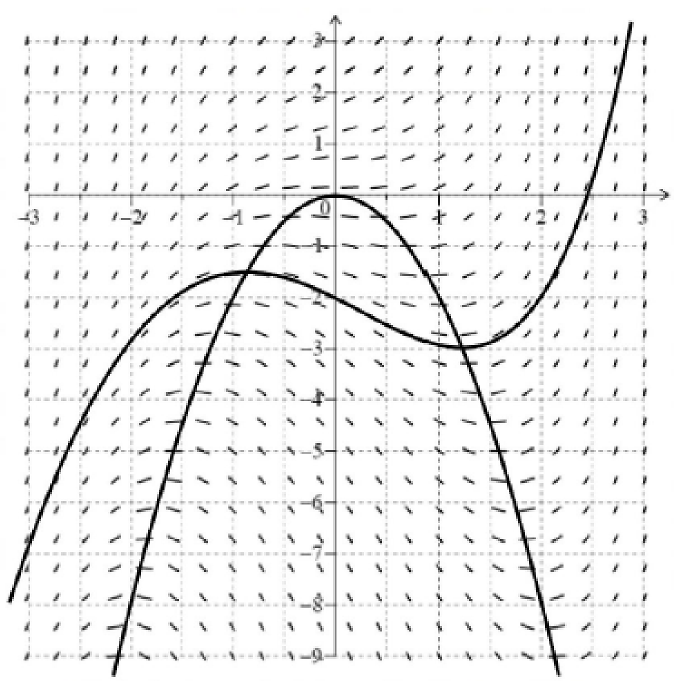
$y = -2x^2$ drawn on diagram (correct shape with a maximum at $(0, 0)$) ***A1***

[1 mark]

(b) The solution to the differential equation that passes through the point $(0,\ -2)$ has both a local maximum point and a local minimum point.

On the slope field, sketch the solution to the differential equation that passes through $(0,\ -2)$. [2]

Markscheme



correct shape with a local maximum and minimum, passing through $(0,\ -2)$ ***A1***

local maximum and minimum on the graph of $y = -2x^2$ ***A1***

[2 marks]

7. [Maximum mark: 7] 21N.1.AHL.TZ0.13

The slope field for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-x^2} - y$ is shown in the following two graphs.

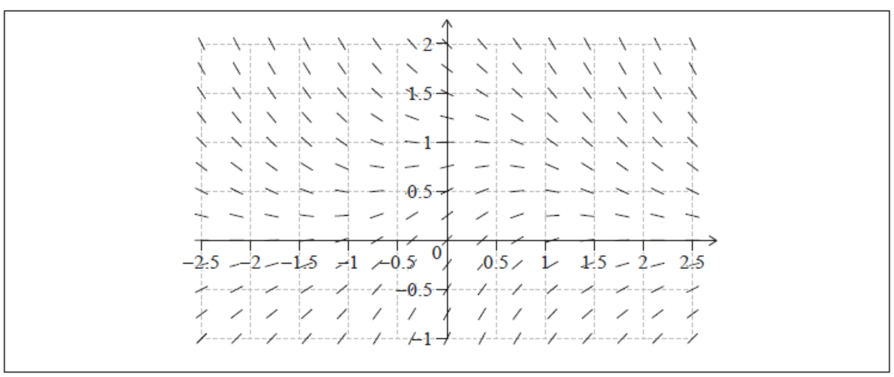
(a) Calculate the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(0,\ 1)$. [1]

Markscheme

$\left(\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^0 - 1\right) = 0$ ***A1***

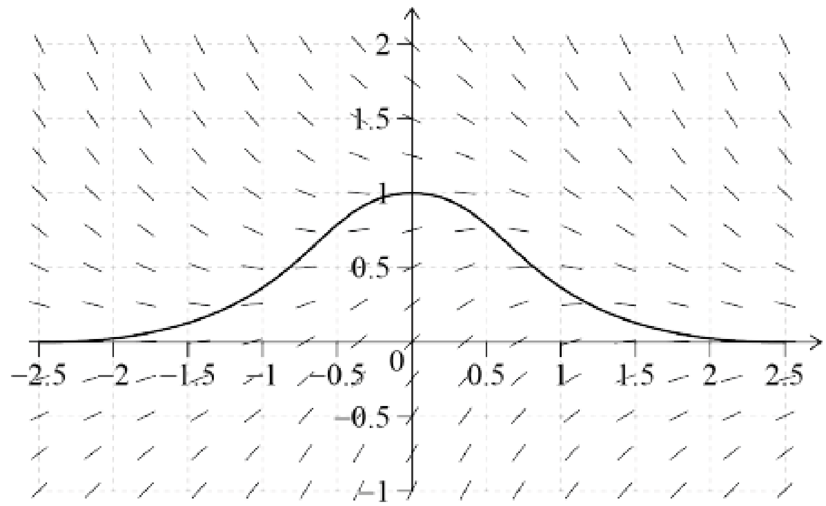
[1 mark]

(b) Sketch, on the first graph, a curve that represents the points where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.



[2]

Markscheme



gradient $= 0$ at $(0,\ 1)$ ***A1***

correct shape ***A1***

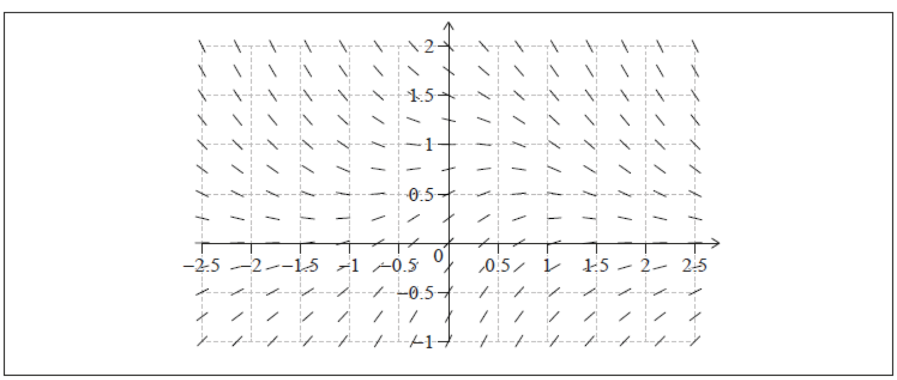
Note: Award second ***A1*** for horizontal asymptote of $y = 0$, and general symmetry about the y-axis.

[2 marks]

On the second graph,

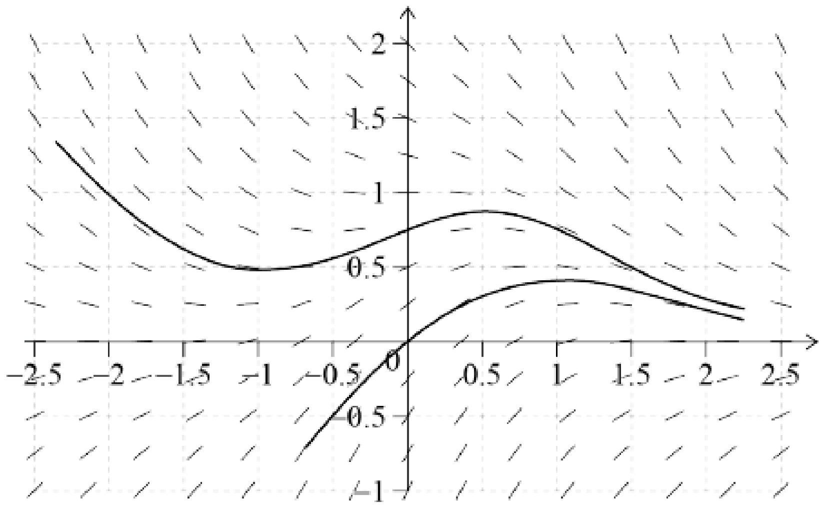
(c) (i) sketch the solution curve that passes through the point $(0,\ 0)$.

(ii) sketch the solution curve that passes through the point $(0,\ 0.75)$.



[4]

Markscheme



(i) positive gradient at origin ***A1***

correct shape ***A1***

Note: Award second ***A1*** for a single maximum in 1st quadrant and tending toward an asymptote.

(ii) positive gradient at $(0,\ 0.75)$ ***A1***

correct shape ***A1***

Note: Award second ***A1*** for a single minimum in 2nd quadrant, single maximum in 1st quadrant and tending toward an asymptote.

[4 marks]

8. [Maximum mark: 8] 21M.1.AHL.TZ1.12

A tank of water initially contains 400 litres. Water is leaking from the tank such that after 10 minutes there are 324 litres remaining in the tank.

The volume of water, V litres, remaining in the tank after t minutes, can be modelled by the differential equation

$\frac{\mathrm{d}V}{\mathrm{d}t} = -k\sqrt{V}$, where k is a constant.

(a) Show that $V = \left(20 - \frac{t}{5}\right)^2$. [6]

Markscheme

$\frac{\mathrm{d}V}{\mathrm{d}t} = -kV^{\frac{1}{2}}$

use of separation of variables ***(M1)***

$\Rightarrow \int V^{-\frac{1}{2}}\mathrm{d}V = \int -k\mathrm{d}t$ ***A1***

$2V^{\frac{1}{2}} = -kt\ (+c)$ ***A1***

considering initial conditions $40 = c$ ***A1***

$2\sqrt{324} = -10k + 40$

$\Rightarrow k = 0.4$ ***A1***

$2\sqrt{V} = -0.4t + 40$

$\Rightarrow \sqrt{V} = 20 - 0.2t$ ***A1***

Note: Award ***A1*** for any correct intermediate step that leads to the ***AG***.

$\Rightarrow V = \left(20 - \frac{t}{5}\right)^2$ ***AG***

Note: Do not award the final ***A1*** if the ***AG*** line is not stated.

[6 marks]

(b) Find the time taken for the tank to empty. [2]

Markscheme

$0 = \left(20 - \frac{t}{5}\right)^2 \Rightarrow t = 100$ minutes ***(M1)A1***

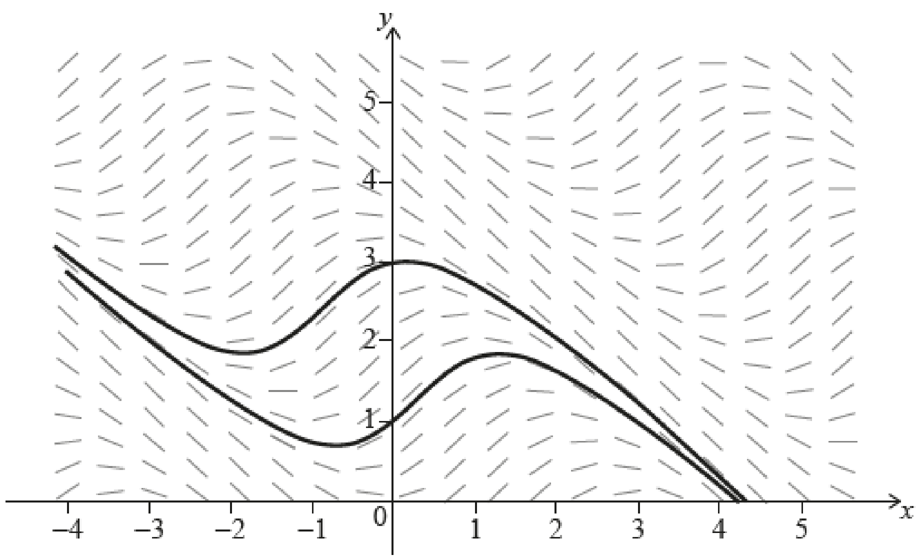
[2 marks]

9. [Maximum mark: 5] 21M.1.AHL.TZ1.15

The diagram shows the slope field for the differential equation

$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(x+y), \ -4 \le x \le 5, \ 0 \le y \le 5.$

The graphs of the two solutions to the differential equation that pass through points $(0,\ 1)$ and $(0,\ 3)$ are shown.



For the two solutions given, the local minimum points lie on the straight line L_1.

(a) Find the equation of L_1, giving your answer in the form $y = mx + c$. [3]

Markscheme

$\sin(x+y) = 0$ ***A1***

$\Rightarrow x + y = 0$ ***(M1)***

(the equation of L_1 is) $y = -x$ ***A1***

[3 marks]

(b) For the two solutions given, the local maximum points lie on the straight line L_2.

Find the equation of L_2. [2]

Markscheme

$x + y = \pi$ **OR** $y = -x + \pi$ ***(M1)A1***

[2 marks]

10. [Maximum mark: 20] SPM.2.AHL.TZ0.7

An object is placed into the top of a long vertical tube, filled with a thick viscous fluid, at time $t = 0$ seconds.

Initially it is thought that the resistance of the fluid would be proportional to the velocity of the object. The following model was proposed, where the object's displacement, x, from the top of the tube, measured in metres, is given by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 9.81 - 0.9\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right).$$

(a) By substituting $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ into the equation, find an expression for the velocity of the particle at time t. Give your answer in the form $v = f(t)$. [7]

Markscheme

$\frac{\mathrm{d}v}{\mathrm{d}t} = 9.81 - 0.9v$ ***M1***

$\int \frac{1}{9.81-0.9v}\,\mathrm{d}v = \int 1\,\mathrm{d}t$ ***M1***

$-\frac{1}{0.9}\ln(9.81-0.9v)=t+c$ ***A1***

$9.81-0.9v=A\mathrm{e}^{-0.9t}$ ***A1***

$v=\frac{9.81-A\mathrm{e}^{-0.9t}}{0.9}$ ***A1***

when $t=0, v=0$ hence $A=9.81$ ***A1***

$v=\frac{9.81\left(1-\mathrm{e}^{-0.9t}\right)}{0.9}$

$v=10.9\left(1-\mathrm{e}^{-0.9t}\right)$ ***A1***

[7 marks]

The maximum velocity approached by the object as it falls is known as the terminal velocity.

(b) From your solution to part (a), or otherwise, find the terminal velocity of the object predicted by this model. [2]

Markscheme

either let t tend to infinity, or $\frac{\mathrm{d}v}{\mathrm{d}t}=0$ ***(M1)***

$v=10.9$ ***A1***

[2 marks]

An experiment is performed in which the object is placed in the fluid on a number of occasions and its terminal velocity recorded. It is found that the terminal velocity was consistently smaller than that predicted by the model used. It was suggested that the resistance to motion is actually proportional to the velocity squared and so the following model was set up.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=9.81-0.9\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2$$

(c) Write down the differential equation as a system of first order differential equations. [2]

Markscheme

$\frac{\mathrm{d}x}{\mathrm{d}t} = y$ ***M1***

$\frac{\mathrm{dy}}{\mathrm{d}t} = 9.81 - 0.9y^2$ ***A1***

[2 marks]

(d) Use Euler's method, with a step length of 0.2, to find the displacement and velocity of the object when $t = 0.6$. [4]

Markscheme

$x_{n+1} = x_n + 0.2y_n, y_{n+1} = y_n + 0.2\left(9.81 - 0.9(y_n)^2\right)$ ***(M1)(A1)***

$x = 1.04, \frac{\mathrm{d}x}{\mathrm{d}t} = 3.31$ ***(M1)A1***

[4 marks]

(e) By repeated application of Euler's method, find an approximation for the terminal velocity, to five significant figures. [1]

Markscheme

3.3015 ***A1***

[1 mark]

At terminal velocity the acceleration of an object is equal to zero.

(f) Use the differential equation to find the terminal velocity for the object. [2]

Markscheme

$0 = 9.81 - 0.9(v)^2$ ***M1***

$\Rightarrow v = \sqrt{\frac{9.81}{0.9}} = 3.301511\ldots\ (= 3.30)$ ***A1***

[2 marks]

(g) Use your answers to parts (d), (e) and (f) to comment on the accuracy of the Euler approximation to this model. [2]

Markscheme

the model found the terminal velocity very accurately, so good approximation ***R1***

intermediate values had object exceeding terminal velocity so not good approximation ***R1***

[2 marks]

11. [Maximum mark: 28] 23N.3.AHL.TZ0.1

This question uses differential equations to model the maximum velocity of a skydiver in free fall.

In 2012, Felix Baumgartner jumped from a height of $40\,000\,\mathrm{m}$. He was attempting to travel at the speed of sound, $330\ \mathrm{m\,s^{-1}}$, whilst free-falling to the Earth.

Before making his attempt, Felix used mathematical models to check how realistic his attempt would be. The simplest model he used suggests that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g$$

where $v\,\mathrm{m\,s^{-1}}$ is Felix's velocity and $g\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity. The time since he began to free-fall is t seconds and the displacement from his initial position is s metres.

Throughout this question, the direction towards the centre of the Earth is taken to be positive and v is a positive quantity.

When $s = 0$, it is given that Felix jumps with an initial velocity $v = 10$.

(a.i) Use the chain rule to show that $\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s}$. [1]

Markscheme

$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}t} \times \frac{\mathrm{d}v}{\mathrm{d}s}$ ***A1***

$\left(v = \frac{\mathrm{d}s}{\mathrm{d}t}\right)$

$\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s}$ ***AG***

[1 mark]

(a.ii) Assuming that g is a constant, solve the differential equation $v\frac{\mathrm{d}v}{\mathrm{d}s} = g$ to find v as a function of s. [4]

Markscheme

$v\frac{\mathrm{d}v}{\mathrm{d}s} = g$

attempt to separate variables ***M1***

$\int v \ \mathrm{d} \ v = \int g \ \mathrm{d} \ s$

$\frac{v^2}{2} = gs(+c)$ ***A1***

using initial conditions (can be done at any point) ***M1***

$50 = c$

so $v = \sqrt{2gs + 100}$ ***A1***

Note: Marks are intentionally unimplied to ensure on-syllabus techniques are used.

[4 marks]

(a.iii) Using $g = 9.8$, determine whether the model predicts that Felix will succeed in travelling at the speed of sound at some point before $s = 40\,000$. Justify your answer. [3]

Markscheme

EITHER

attempt to use their part (a)(ii) to find a value of s when $v = 330$ ***(M1)***

$330 = \sqrt{2gs + 100}$

therefore $s = 5551.02\ldots$ ***A1***

$(5551.02 < 40000)$

so (the model does predict) he will reach the speed of sound ***A1***

OR

attempt to use their part (a)(ii) to find a value of v when $s = 40000$ ***(M1)***

$v = \sqrt{2g(40000) + 100}$

$= 885 \ \ (885.49\ldots)$ ***A1***

$(885 > 330)$

so (the model does predict) he will reach the speed of sound (before $s = 40000$) ***A1***

Note: For the **OR** method, accept any large s that leads to $v = 330$.

FT from $\sqrt{2gs}$ gives $885 \ (885.437\ldots)$ for v and $5560 \ (5556.12\ldots)$ for s

FT from their v or their s for the final ***A1***, provided ***M1*** is awarded

[3 marks]

(b) To test the model

$\frac{dv}{dt} = g$,

Felix conducted a trial jump from a lower height, and data for v against t was found.

(b.i) If the model is correct, describe the shape of the graph of v against t. [2]

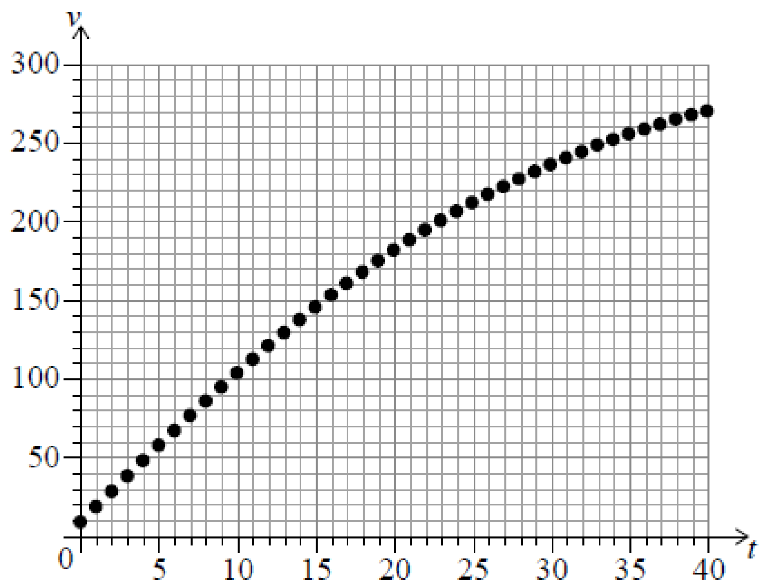
Markscheme

$v = gt + (c)$ **OR** gradient is a constant *(M1)*

so the graph should be a straight line *A1*

[2 marks]

Felix's data are plotted on the following graph.



(b.ii) Use the plot to comment on the validity of the model in part (a). [1]

Markscheme

the graph is not a straight line / only (approx.) straight for small *t*, so the model does not appear to be valid *R1*

Note: Award ***R1*** for recognising that the graph is non-linear **AND** stating that the model does not appear to be valid

[1 mark]

(c) An improved model considers air resistance, using

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - kv^2$$

where k is a positive constant. You are reminded that initially $s = 0$ and $v = 10$.

(c.i) By using $\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s}$, solve the differential equation to find v in terms of s, g and k. You may assume that $g - kv^2 > 0$. [5]

Markscheme

$v\frac{\mathrm{d}v}{\mathrm{d}s} = g - kv^2$

separating variables *(M1)*

$\int \frac{v}{g-kv^2}\ \mathrm{d}\ v = \int \mathrm{d}\ s$

$-\frac{1}{2k}\ \ln\left(g - kv^2\right) = s(+c)$ **OR** $-\frac{1}{2k}\ \ln\left|g - kv^2\right| = s(+c)$ *(A1)*

rearranging to make v the subject *(M1)*

Note: Award *(M1)* for making v the subject of their equation and not just an attempt, or an erroneous equation with v also on the RHS.

$g - kv^2 = A\mathrm{e}^{-2ks}$

$v = \sqrt{\frac{g - A\mathrm{e}^{-2ks}}{k}}$

applying initial conditions (here or elsewhere) *(M1)*

$100 = \frac{g-A}{k}$

$A = g - 100k$

so

$v = \sqrt{\frac{g-(g-100k)\mathrm{e}^{-2ks}}{k}}$ *A1*

[5 marks]

Felix uses the graph of v against t shown in part (b) to estimate the value of k.

(c.ii) The gradient is estimated to be 9.672 when $v = 40$. Taking g to be 9.8, use this information to show that Felix found that $k = 8 \times 10^{-5}$.

[2]

Markscheme

$9.672 = 9.8 - 1600k$ ***A1A1***

Note: Award ***A1*** for correct left-hand side and ***A1*** for correct right-hand side.

$k = \frac{9.8-9.672}{1600}$

$k = 8 \times 10^{-5}$ ***AG***

Note: Award ***A1A0*** for $k = 8 \times 10^{-5}$ substituted into the right-hand side of the expression, leading to 9.672.

[2 marks]

(c.iii) Hence, find the value of v predicted by this model, as s tends to infinity. [2]

Markscheme

$s \to \infty,\ \mathrm{e}^{-2ks} \to 0$ **OR** $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$ **OR** graph/table ***(M1)***

$\left(v_{\max} = \sqrt{\frac{g}{k}} =\right) 350\left(ms^{-1}\right)$ ***A1***

[2 marks]

(c.iv) Find the upper bound for the velocity according to this model, given that $0 < s \leq 40\,000$. Give your answer to four significant figures. [2]

Markscheme

upper limit occurs when $s = 40000$ ***(M1)***

Note: The ***M1*** can be implied by 40000 substituted into their part (c)(i).

$349.7\left(\mathrm{ms}^{-1}\right)$ ***A1***

Note: Answer must be to 4 sf.

[2 marks]

The assumption that the value of g is constant is not correct. It can be shown that

$$g = \frac{3.98 \times 10^{14}}{(6.41 \times 10^{6} - s)^{2}}.$$

Hence, the new model is given by

$$v\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{3.98 \times 10^{14}}{(6.41 \times 10^{6} - s)^{2}} - (8 \times 10^{-5})v^{2}.$$

When $s = 0$, it is known that $v = 10$.

(d) Use Euler's method with a step length of 4000 to estimate the value of v when $s = 40\,000$. [4]

Markscheme

$s_{n+1} = s_n + 4000$ ***(A1)***

$$v_{n+1} = v_n + 4000 \times \left(\frac{3.98 \times 10^{14}}{v_n(6.41 \times 10^{6} - s_n)^{2}} - (8 \times 10^{-5})v_n\right)$$ ***(M1)(A1)***

Note: Award ***(M1)*** for attempt to use Euler method formula **AND** dividing through by v.

if $v_0 = 10$, then $v_{10} = 361 \ \ (360.658\ldots)$ ***A1***

[4 marks]

(e) After Felix completed his record-breaking jump, he found that the answer from part (d) was not supported by data collected during the jump.

(e.i) Suggest **one** improvement to the use of Euler's method which might increase the accuracy of the prediction of the model. [1]

Markscheme

Use a smaller step length ***R1***

OR

Use a better method such as Runge-Kutta ***R1***

OR

(Try to) solve the equation exactly ***R1***

[1 mark]

(e.ii) Suggest **one** factor **not** explicitly considered by the model in part (d) which might lead to a difference between the model's prediction and the data collected. [1]

Markscheme

Any reasonable response: ***R1***

For example:

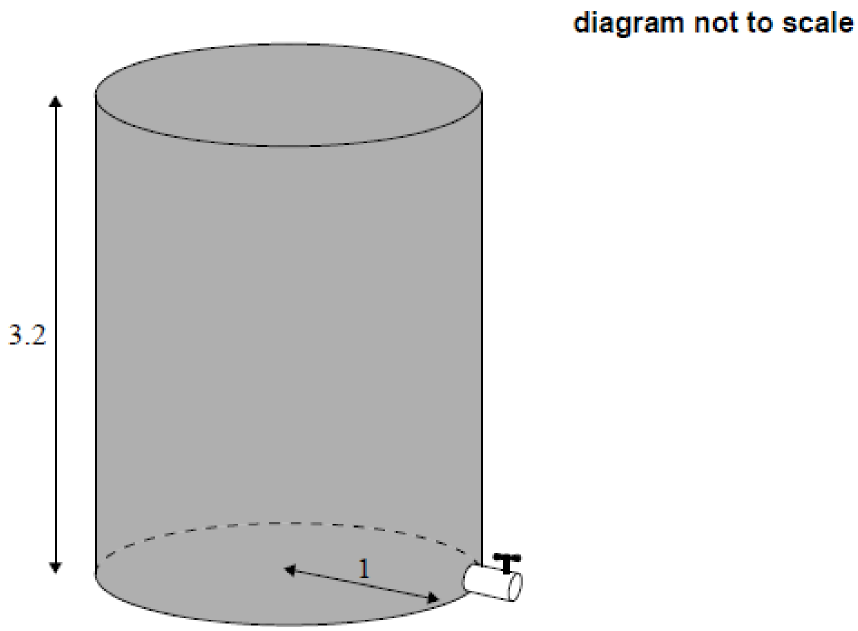
Ignoring parachute / end point of motion / only valid for certain domain.
Treating Felix as a point object.
Ignoring weather / wind / air currents.
Assuming path is directly downwards.
Assuming perfect measurement of initial speed.

[1 mark]

12. [Maximum mark: 30] 21N.3.AHL.TZ0.2

This question explores models for the height of water in a cylindrical container as water drains out.

The diagram shows a cylindrical water container of height 3.2 metres and base radius 1 metre. At the base of the container is a small circular valve, which enables water to drain out.



Eva closes the valve and fills the container with water.

At time $t = 0$, Eva opens the valve. She records the height, h metres, of water remaining in the container every 5 minutes.

Time, t (minutes)	Height, h (metres)
0	3.2
5	2.4
10	1.6
15	1.1
20	0.5

Eva first tries to model the height using a linear function, $h(t) = at + b$, where $a,\ b \in \mathbb{R}$.

(a.i) Find the equation of the regression line of h on t. [2]

Markscheme

$h(t) = -0.134t + 3.1$ ***A1A1***

Note: Award ***A1*** for an equation in h and t and ***A1*** for the coefficient -0.134 and constant 3.1.

[2 marks]

(a.ii) Interpret the meaning of parameter a in the context of the model. [1]

Markscheme

EITHER

the rate of change of height (of water in metres per minute) ***A1***

Note: Accept “rate of decrease” or “rate of increase” in place of “rate of change”.

OR

the (average) amount that the height (of the water) decreases each minute ***A1***

[1 mark]

Eva uses the equation of the regression line of h on t, to predict the time it will take for all the water to drain out of the container.

(a.iii) Suggest why Eva’s use of the linear regression equation in this way could be unreliable. [1]

Markscheme

EITHER

unreliable to use h on t equation to estimate t ***A1***

OR

unreliable to extrapolate from original data ***A1***

OR

rate of change (of height) might not remain constant (as the water drains out) ***A1***

[1 mark]

Eva thinks she can improve her model by using a quadratic function, $h(t) = pt^2 + qt + r$, where $p,\ q,\ r \in \mathbb{R}$.

(b.i) Find the equation of the least squares quadratic regression curve. [1]

Markscheme

$h(t) = 0.002t^2 - 0.174t + 3.2$ ***A1***

[1 mark]

Eva uses this equation to predict the time it will take for all the water to drain out of the container and obtains an answer of k minutes.

(b.ii) Find the value of k. [2]

Markscheme

$0.002t^2 - 0.174t + 3.2 = 0$ ***(M1)***

$26.4\ \ (26.4046\ldots)$ ***A1***

[2 marks]

(b.iii) Hence, write down a suitable domain for Eva's function $h(t) = pt^2 + qt + r$. [1]

Markscheme

EITHER

$(0 \leq) t \leq 26.4 \quad (t \leq 26.4046\ldots)$ ***A1***

OR

$(0 \leq) t \leq 20$ (due to range of original data / interpolation) ***A1***

[1 mark]

Let V be the volume, in cubic metres, of water in the container at time t minutes.
Let R be the radius, in metres, of the circular valve.

Eva does some research and discovers a formula for the rate of change of V.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\pi R^2 \sqrt{70\,560h}$$

(c) Show that $\frac{\mathrm{d}h}{\mathrm{d}t} = -R^2 \sqrt{70\,560h}$. [3]

Markscheme

$V = \pi(1)^2 h$ ***(A1)***

EITHER

$\frac{\mathrm{d}V}{\mathrm{d}t} = \pi \frac{\mathrm{d}h}{\mathrm{d}t}$ ***M1***

OR

attempt to use chain rule ***M1***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$

THEN

$$\frac{dh}{dt} = \frac{1}{\pi} \times -\pi R^2 \sqrt{70\,560h} \quad \textit{A1}$$

$$\frac{dh}{dt} = -R^2 \sqrt{70\,560h} \quad \textit{AG}$$

[3 marks]

(d) By solving the differential equation $\frac{dh}{dt} = -R^2\sqrt{70\,560h}$, show that the general solution is given by $h = 17\,640\left(c - R^2t\right)^2$, where $c \in \mathbb{R}$. [5]

Markscheme

attempt to separate variables ***M1***

$$\int \frac{1}{\sqrt{70\,560h}}\, \mathrm{d}\, h = \int -R^2\, \mathrm{d}\, t \quad \textit{A1}$$

$$\frac{2\sqrt{h}}{\sqrt{70\,560}} = -R^2t + c \quad \textit{A1A1}$$

Note: Award ***A1*** for each correct side of the equation.

$$\sqrt{h} = \frac{\sqrt{70\,560}}{2}\left(c - R^2t\right) \quad \textit{A1}$$

Note: Award the final ***A1*** for any correct intermediate step that clearly leads to the given equation.

$$h = 17\,640\left(c - R^2t\right)^2 \quad \textit{AG}$$

[5 marks]

Eva measures the radius of the valve to be 0.023 metres. Let T be the time, in minutes, it takes for all the water to drain out of the container.

(e) Use the general solution from part (d) and the initial condition $h(0) = 3.2$ to predict the value of T. [4]

Markscheme

$t = 0 \Rightarrow 3.2 = 17640c^2$ ***(M1)***

$c = 0.0134687\ldots$ ***(A1)***

substituting $h = 0$ and their non-zero value of c ***(M1)***

$T = \frac{c}{R^2} = \frac{0.0134687\ldots}{0.023^2}$

$= 25.5$ (minutes) $(25.4606\ldots)$ ***A1***

[4 marks]

Eva wants to use the container as a timer. She adjusts the initial height of water in the container so that all the water will drain out of the container in 15 minutes.

(f) Find this new height. [3]

Markscheme

$h = 0 \Rightarrow c = R^2t$

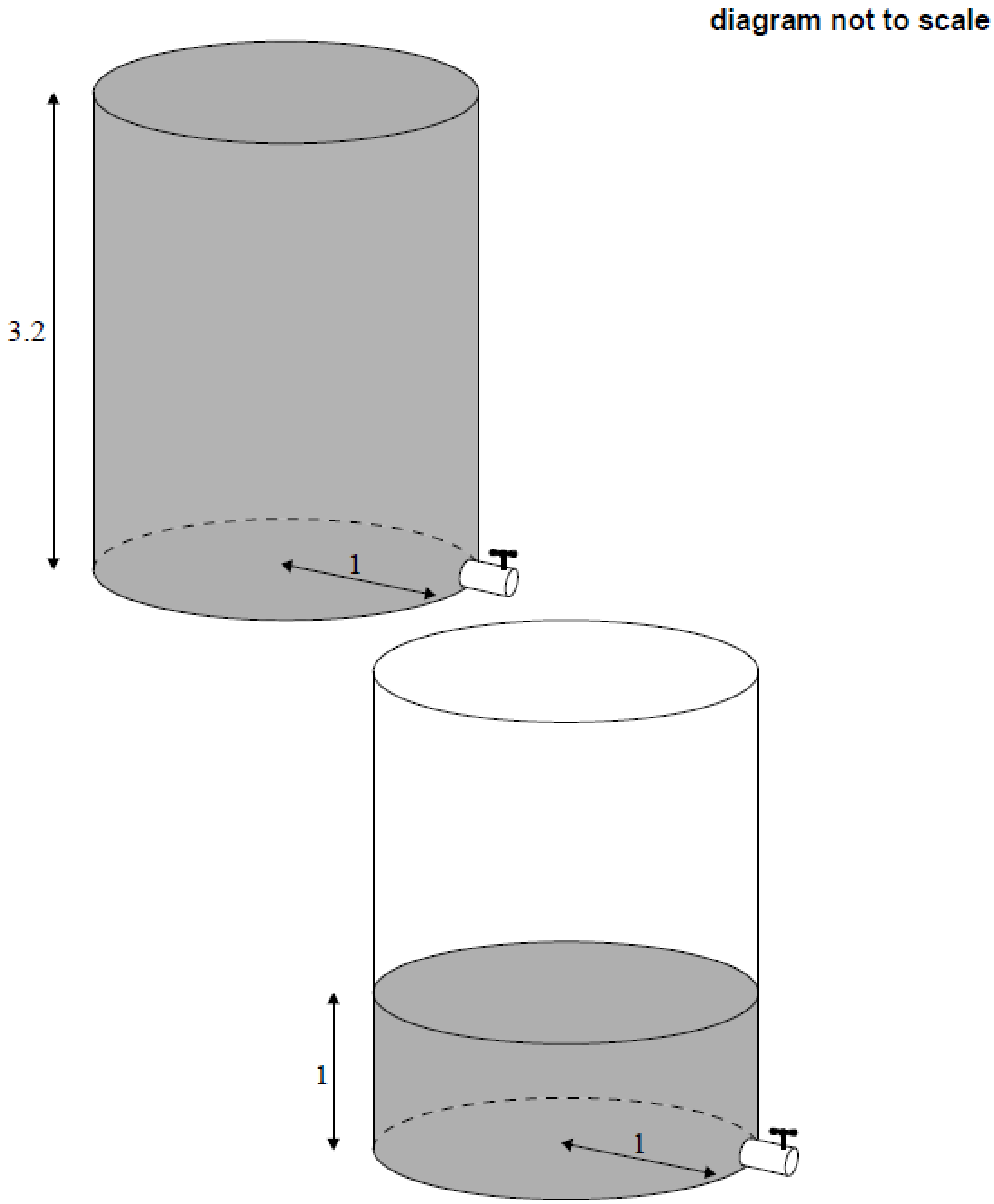
$c = 0.023^2 \times 15 \ (= 0.007935)$ ***(A1)***

$t = 0 \Rightarrow h = 17640\left(0.023^2 \times 15\right)^2$ ***(M1)***

$h = 1.11$ (metres) $(1.11068\ldots)$ ***A1***

[3 marks]

Eva has another water container that is identical to the first one. She places one water container above the other one, so that all the water from the highest container will drain into the lowest container. Eva completely fills the highest container, but only fills the lowest container to a height of 1 metre, as shown in the diagram.



At time $t = 0$ Eva opens both valves. Let H be the height of water, in metres, in the lowest container at time t.

(g.i) Show that $\frac{\mathrm{d}H}{\mathrm{d}t} \approx 0.2514 - 0.009873t - 0.1405\sqrt{H}$, where $0 \leq t \leq T$. [4]

Markscheme

let h be the height of water in the highest container from parts (d) and (e) we get

$\frac{dh}{dt} = -35280R^2(0.0134687\ldots - R^2t)$ ***(M1)(A1)***

so $\frac{dH}{dt} = 35280R^2(0.0135 - R^2t) - R^2\sqrt{70560H}$ ***M1A1***

$\left(\frac{dH}{dt} = 18.6631\ldots(0.0134687\ldots - 0.000529t) - 0.000529\sqrt{70560H}\right)$

$\left(\frac{dH}{dt} = 0.251367\ldots - 0.0987279\ldots - 0.140518\ldots\sqrt{H}\right)$

$\frac{dH}{dt} \approx 0.2514 - 0.009873t - 0.1405\sqrt{H}$ ***AG***

[4 marks]

(g.ii) Use Euler's method with a step length of 0.5 minutes to estimate the maximum value of H. [3]

Markscheme

evidence of using Euler's method correctly

e.g. $y_1 = 1.05545\ldots$ ***(A1)***

maximum value of $H = 1.45$ (metres) (at 8.5 minutes) ***A2***

($1.44678\ldots$ metres)

[3 marks]