Differential equations [133 marks]

1. [Maximum mark: 4] 24M.1.AHL.TZ1.4

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \log_{10}(x+y)$, where $x \geq 0$ and $y > 0$.

Given that $y = 1$ when $x = 0$, use Euler's method with a step length of 0.1 to find an approximate value for y when $x = 2$. [4]

2. [Maximum mark: 4] 24M.1.AHL.TZ2.7

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = xy - 1$, given that $y = 2$ when $x = 1$.

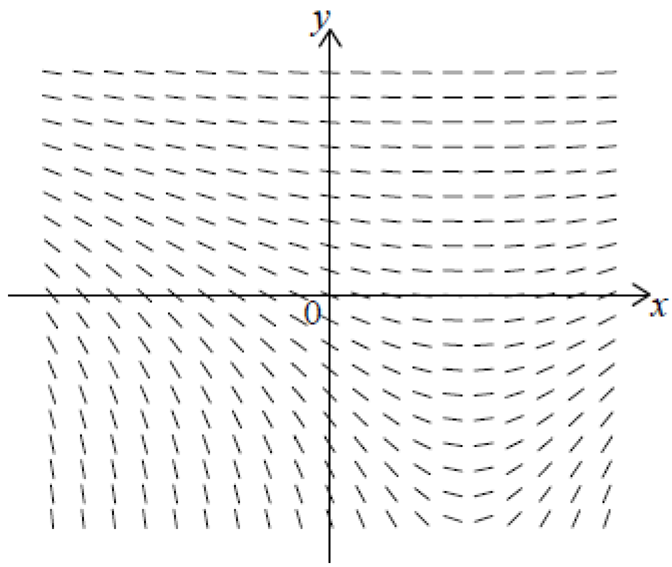
Use Euler's method with step size 0.1 to find the approximate value of y when $x = 1.5$. [4]

3. [Maximum mark: 13] 24M.2.AHL.TZ1.5

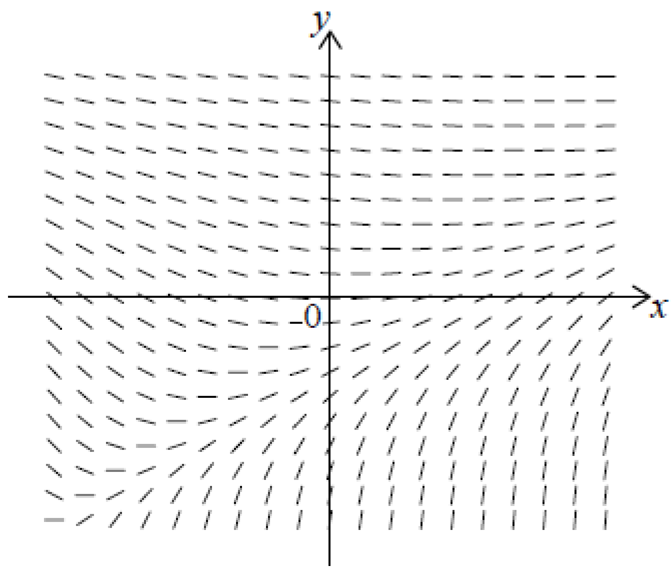
Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\mathrm{e}^{2y}}$.

(a) Identify which of the following diagrams, **A**, **B** or **C**, represents the slope field for the differential equation. Give a reason for your answer.

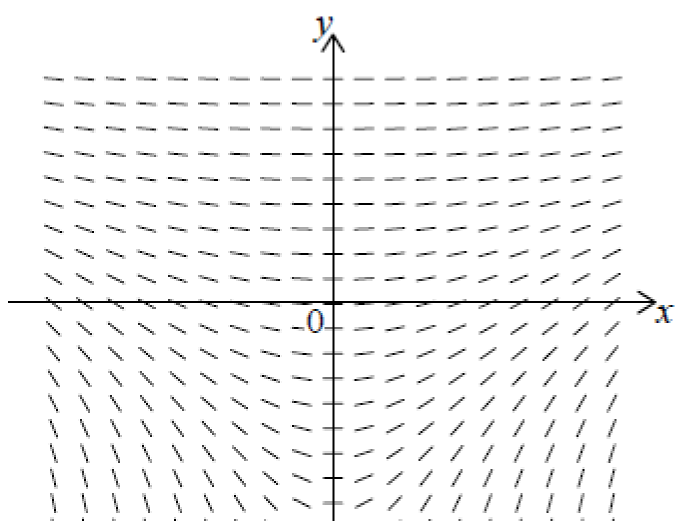
A.



B.



C.



[2]

It is given that, for a particular solution, $x = 0$ and $y = 0$.

(b) Find an expression for y, in terms of x, for this solution. [7]

(c) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, in terms of x, by differentiating your answer from part (b). [2]

(d) Hence verify that your answer to part (b) is a solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\mathrm{e}^{2y}}$. [2]

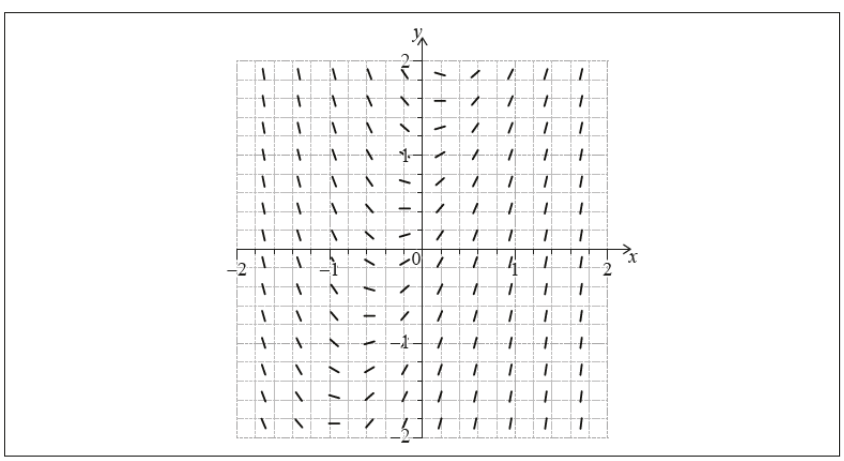
4. [Maximum mark: 4] 23N.1.AHL.TZ0.12

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}\chi} = 3x - y + 1$.

(a) Find the equation of the tangent to the solution curve at the point $(-1, \ -1)$ in the form $ax + by + c = 0$. [2]

The slope field for this differential equation is shown in the following diagram.

(b) Sketch the solution curve that passes through the point $(-1, \ -1)$.



[2]

5. [Maximum mark: 6] 23M.1.AHL.TZ2.17

Consider the differential equation

$$\left(x^2+1\right)\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x}{2y-2}\text{, for } x\geq 0, y\geq 1,$$

where $y=1$ when $x=0$.

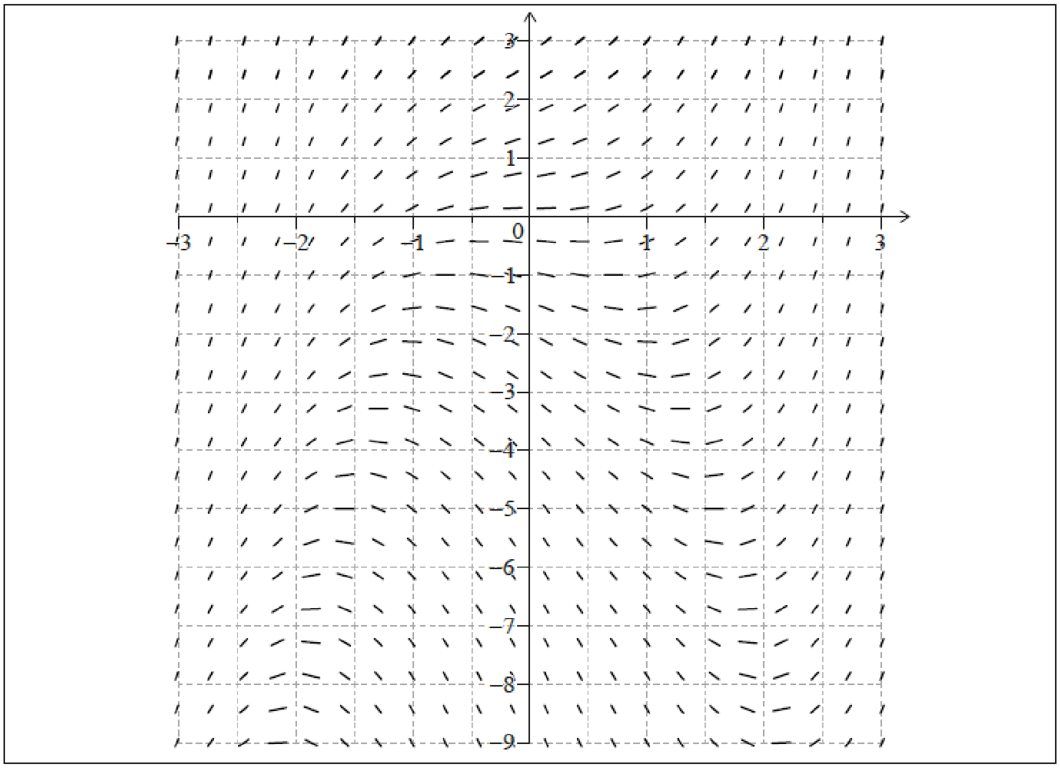
(a) Explain why Euler's method cannot be used to find an approximate value for y when $x=0.1$. [1]

(b) By solving the differential equation, show that $y=1+\sqrt{\frac{\ln\left(x^2+1\right)}{2}}$. [4]

(c) Hence deduce the value of y when $x=0.1$. [1]

6. [Maximum mark: 4] 22M.1.AHL.TZ1.7

A slope field for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=x^2+\frac{y}{2}$ is shown.



Some of the solutions to the differential equation have a local maximum point and a local minimum point.

(a.i) Write down the equation of the curve on which all these maximum and minimum points lie. [1]

(a.ii) Sketch this curve on the slope field. [1]

(b) The solution to the differential equation that passes through the point $(0,\ -2)$ has both a local maximum point and a local minimum point.

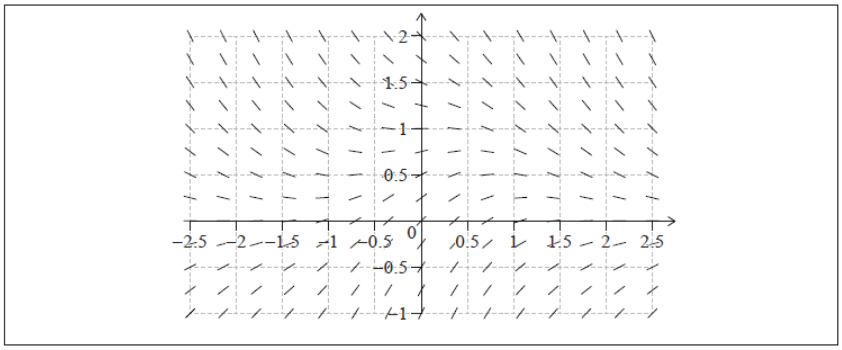
On the slope field, sketch the solution to the differential equation that passes through $(0,\ -2)$. [2]

7. [Maximum mark: 7] 21N.1.AHL.TZ0.13

The slope field for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-x^2} - y$ is shown in the following two graphs.

(a) Calculate the value of $\frac{dy}{dx}$ at the point $(0,\ 1)$. [1]

(b) Sketch, on the first graph, a curve that represents the points where $\frac{dy}{dx} = 0$.

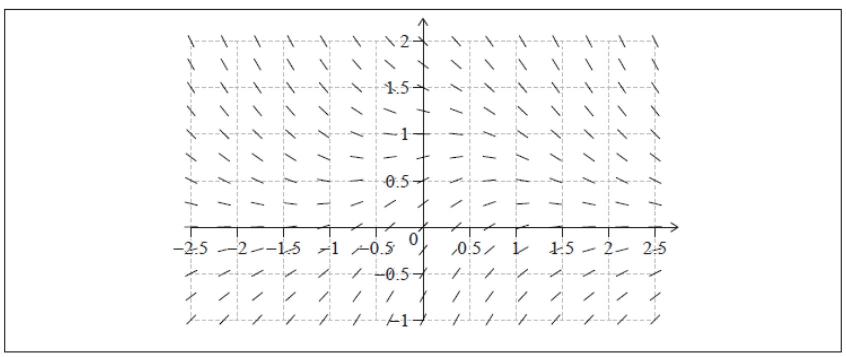


[2]

On the second graph,

(c) (i) sketch the solution curve that passes through the point $(0,\ 0)$.

(ii) sketch the solution curve that passes through the point $(0,\ 0.75)$.



[4]

8. [Maximum mark: 8] 21M.1.AHL.TZ1.12

A tank of water initially contains 400 litres. Water is leaking from the tank such that after 10 minutes there are 324 litres remaining in the tank.

The volume of water, V litres, remaining in the tank after t minutes, can be modelled by the differential equation

$\frac{\mathrm{d}V}{\mathrm{d}t} = -k\sqrt{V}$, where k is a constant.

(a) Show that $V = \left(20 - \frac{t}{5}\right)^2$. [6]

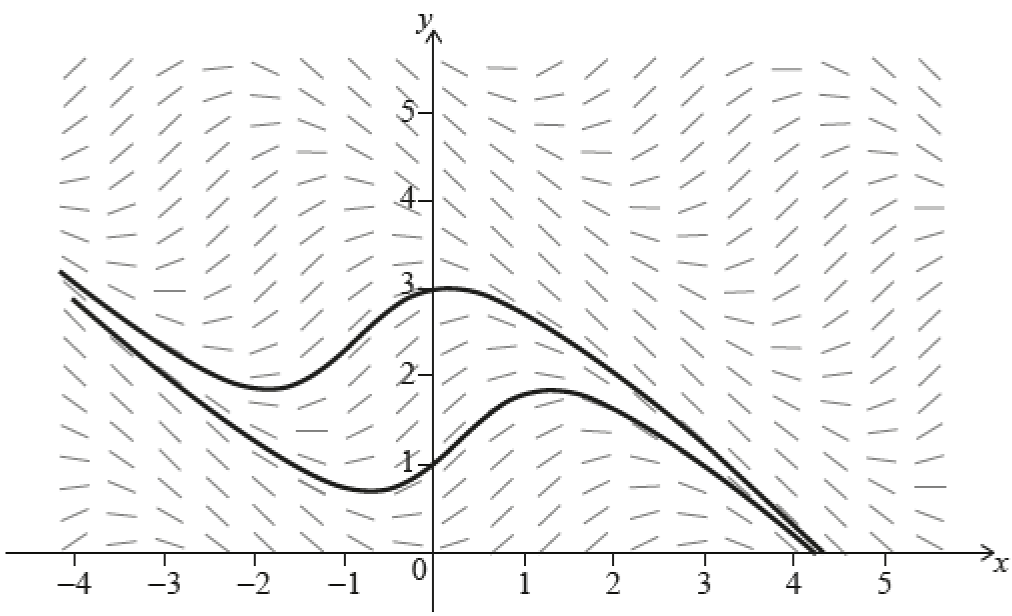
(b) Find the time taken for the tank to empty. [2]

9. [Maximum mark: 5] 21M.1.AHL.TZ1.15

The diagram shows the slope field for the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(x + y), \ -4 \le x \le 5, \ 0 \le y \le 5.$$

The graphs of the two solutions to the differential equation that pass through points $(0,\ 1)$ and $(0,\ 3)$ are shown.



For the two solutions given, the local minimum points lie on the straight line L_1.

(a) Find the equation of L_1, giving your answer in the form $y = mx + c$. [3]

(b) For the two solutions given, the local maximum points lie on the straight line L_2.

Find the equation of L_2. [2]

10. [Maximum mark: 20] SPM.2.AHL.TZ0.7

An object is placed into the top of a long vertical tube, filled with a thick viscous fluid, at time $t = 0$ seconds.

Initially it is thought that the resistance of the fluid would be proportional to the velocity of the object. The following model was proposed, where the object's displacement, x, from the top of the tube, measured in metres, is given by the differential equation

$\frac{d^2x}{dt^2} = 9.81 - 0.9\left(\frac{dx}{dt}\right)$.

(a) By substituting $v = \frac{dx}{dt}$ into the equation, find an expression for the velocity of the particle at time t. Give your answer in the form $v = f(t)$. [7]

The maximum velocity approached by the object as it falls is known as the terminal velocity.

(b) From your solution to part (a), or otherwise, find the terminal velocity of the object predicted by this model. [2]

An experiment is performed in which the object is placed in the fluid on a number of occasions and its terminal velocity recorded. It is found that the terminal velocity was consistently smaller than that predicted by the model used. It was suggested that the resistance to motion is actually proportional to the velocity squared and so the following model was set up.

$$\frac{d^2x}{dt^2} = 9.81 - 0.9\left(\frac{dx}{dt}\right)^2$$

(c) Write down the differential equation as a system of first order differential equations. [2]

(d) Use Euler's method, with a step length of 0.2, to find the displacement and velocity of the object when $t = 0.6$. [4]

(e) By repeated application of Euler's method, find an approximation for the terminal velocity, to five significant figures. [1]

At terminal velocity the acceleration of an object is equal to zero.

(f) Use the differential equation to find the terminal velocity for the object. [2]

(g) Use your answers to parts (d), (e) and (f) to comment on the accuracy of the Euler approximation to this model. [2]

11. [Maximum mark: 28] 23N.3.AHL.TZ0.1

This question uses differential equations to model the maximum velocity of a skydiver in free fall.

In 2012, Felix Baumgartner jumped from a height of $40\,000\ \mathrm{m}$. He was attempting to travel at the speed of sound, $330\ \mathrm{m\,s^{-1}}$, whilst free-falling to the Earth.

Before making his attempt, Felix used mathematical models to check how realistic his attempt would be. The simplest model he used suggests that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g$$

where $v\,\mathrm{m\,s^{-1}}$ is Felix's velocity and $g\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity. The time since he began to free-fall is t seconds and the displacement from his initial position is s metres.

Throughout this question, the direction towards the centre of the Earth is taken to be positive and v is a positive quantity.

When $s = 0$, it is given that Felix jumps with an initial velocity $v = 10$.

(a.i) Use the chain rule to show that $\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s}$. [1]

(a.ii) Assuming that g is a constant, solve the differential equation $v\frac{\mathrm{d}v}{\mathrm{d}s} = g$ to find v as a function of s. [4]

(a.iii) Using $g = 9.8$, determine whether the model predicts that Felix will succeed in travelling at the speed of sound at some point before $s = 40\,000$. Justify your answer. [3]

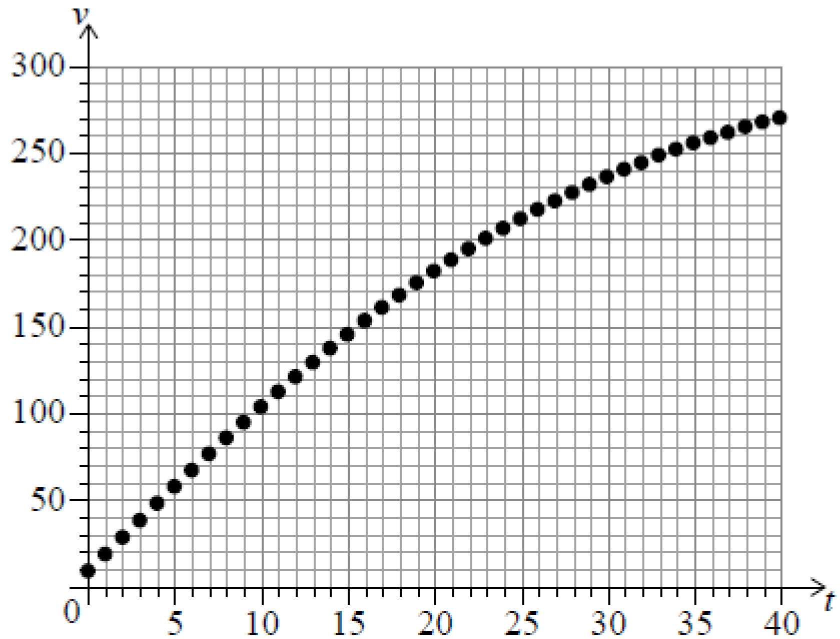
(b) To test the model

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g,$$

Felix conducted a trial jump from a lower height, and data for v against t was found.

(b.i) If the model is correct, describe the shape of the graph of v against t. [2]

Felix's data are plotted on the following graph.



(b.ii) Use the plot to comment on the validity of the model in part (a). [1]

(c) An improved model considers air resistance, using

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - kv^2$$

where k is a positive constant. You are reminded that initially $s = 0$ and $v = 10$.

(c.i) By using $\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s}$, solve the differential equation to find v in terms of s, g and k. You may assume that $g - kv^2 > 0$. [5]

Felix uses the graph of v against t shown in part (b) to estimate the value of k.

(c.ii) The gradient is estimated to be 9.672 when $v = 40$. Taking g to be 9.8, use this information to show that Felix found that $k = 8 \times 10^{-5}$. [2]

(c.iii) Hence, find the value of v predicted by this model, as s tends to infinity. [2]

(c.iv) Find the upper bound for the velocity according to this model, given that $0 < s \leq 40\,000$. Give your answer to four significant figures. [2]

The assumption that the value of g is constant is not correct. It can be shown that

$$g = \frac{3.98\times10^{14}}{\left(6.41\times10^{6}-s\right)^{2}}.$$

Hence, the new model is given by

$$v\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{3.98\times10^{14}}{\left(6.41\times10^{6}-s\right)^{2}} - \left(8\times10^{-5}\right)v^{2}.$$

When $s = 0$, it is known that $v = 10$.

(d) Use Euler's method with a step length of 4000 to estimate the value of v when $s = 40\,000$. [4]

(e) After Felix completed his record-breaking jump, he found that the answer from part (d) was not supported by data collected during the jump.

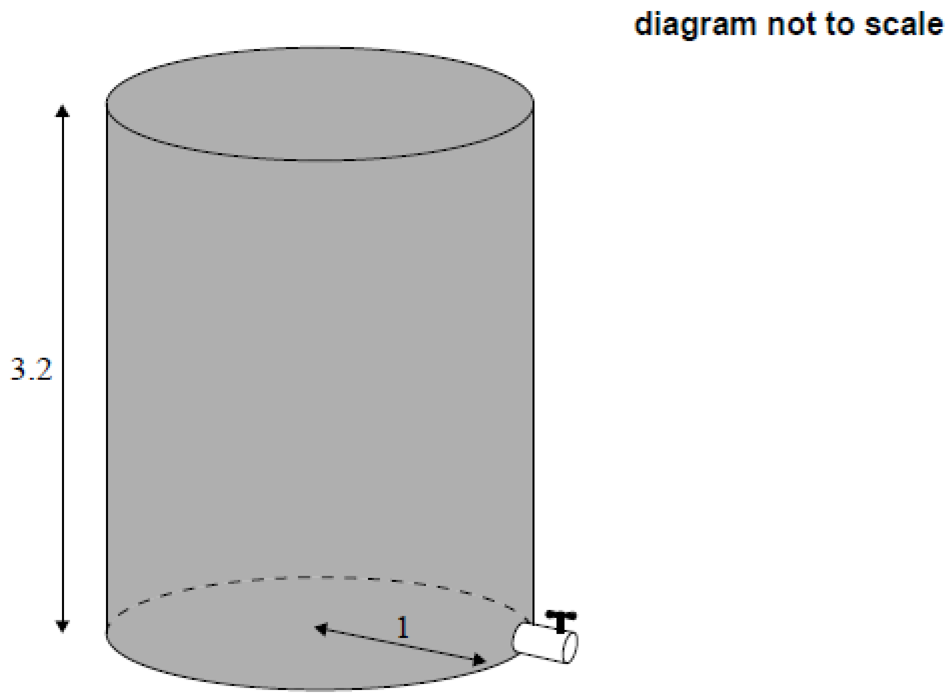
(e.i) Suggest **one** improvement to the use of Euler's method which might increase the accuracy of the prediction of the model. [1]

(e.ii) Suggest **one** factor **not** explicitly considered by the model in part (d) which might lead to a difference between the model's prediction and the data collected. [1]

12. [Maximum mark: 30] 21N.3.AHL.TZ0.2

This question explores models for the height of water in a cylindrical container as water drains out.

The diagram shows a cylindrical water container of height 3.2 metres and base radius 1 metre. At the base of the container is a small circular valve, which enables water to drain out.



Eva closes the valve and fills the container with water.

At time $t = 0$, Eva opens the valve. She records the height, h metres, of water remaining in the container every 5 minutes.

Time, t (minutes)	Height, h (metres)
0	3.2
5	2.4
10	1.6
15	1.1
20	0.5

Eva first tries to model the height using a linear function, $h(t) = at + b$, where $a,\ b \in \mathbb{R}$.

(a.i) Find the equation of the regression line of h on t. [2]

(a.ii) Interpret the meaning of parameter a in the context of the model. [1]

Eva uses the equation of the regression line of h on t, to predict the time it will take for all the water to drain out of the container.

(a.iii) Suggest why Eva's use of the linear regression equation in this way could be unreliable. [1]

Eva thinks she can improve her model by using a quadratic function, $h(t) = pt^2 + qt + r$, where $p,\ q,\ r \in \mathbb{R}$.

(b.i) Find the equation of the least squares quadratic regression curve. [1]

Eva uses this equation to predict the time it will take for all the water to drain out of the container and obtains an answer of k minutes.

(b.ii) Find the value of k. [2]

(b.iii) Hence, write down a suitable domain for Eva's function $h(t) = pt^2 + qt + r$. [1]

Let V be the volume, in cubic metres, of water in the container at time t minutes. Let R be the radius, in metres, of the circular valve.

Eva does some research and discovers a formula for the rate of change of V.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\pi R^2\sqrt{70\,560h}$$

(c) Show that $\frac{\mathrm{d}h}{\mathrm{d}t} = -R^2\sqrt{70\,560h}$. [3]

(d) By solving the differential equation $\frac{\mathrm{d}h}{\mathrm{d}t} = -R^2\sqrt{70\,560h}$, show that the general solution is given by $h = 17\,640\left(c - R^2t\right)^2$, where $c \in \mathbb{R}$. [5]

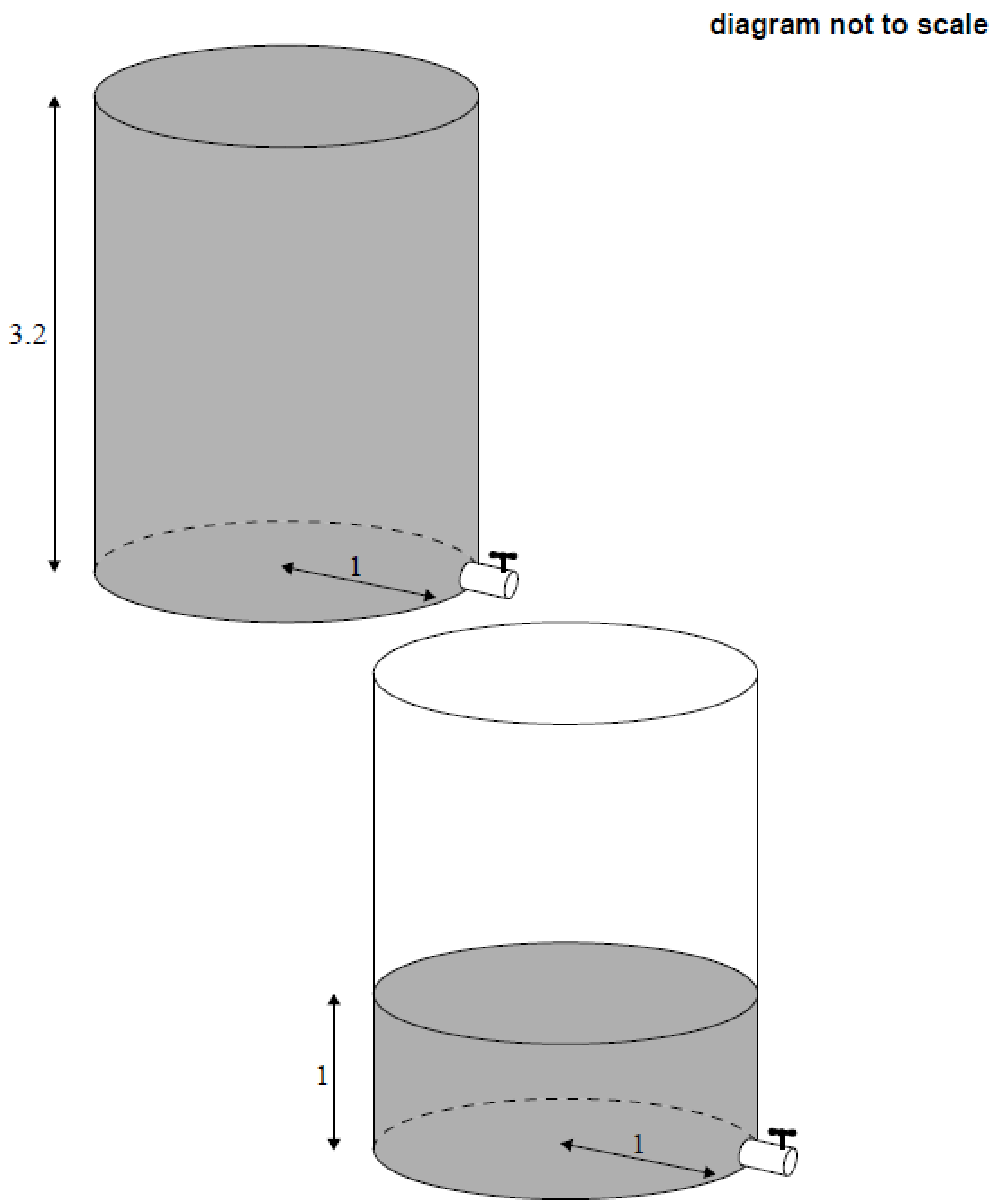
Eva measures the radius of the valve to be 0.023 metres. Let T be the time, in minutes, it takes for all the water to drain out of the container.

(e) Use the general solution from part (d) and the initial condition $h(0) = 3.2$ to predict the value of T. [4]

Eva wants to use the container as a timer. She adjusts the initial height of water in the container so that all the water will drain out of the container in 15 minutes.

(f) Find this new height. [3]

Eva has another water container that is identical to the first one. She places one water container above the other one, so that all the water from the highest container will drain into the lowest container. Eva completely fills the highest container, but only fills the lowest container to a height of 1 metre, as shown in the diagram.



At time $t = 0$ Eva opens both valves. Let H be the height of water, in metres, in the lowest container at time t.

(g.i) Show that $\frac{\mathrm{d}H}{\mathrm{d}t} \approx 0.2514 - 0.009873t - 0.1405\sqrt{H}$, where $0 \leq t \leq T$. [4]

(g.ii) Use Euler's method with a step length of 0.5 minutes to estimate the maximum value of H. [3]