Sequences [119 marks]

1. [Maximum mark: 5] 24M.1.AHL.TZ1.5

Consider a geometric sequence with first term 1 and common ratio 10.

S_n is the sum of the first n terms of the sequence.

(a) Find an expression for S_n in the form $\frac{a^n-1}{b}$, where $a,\ b \in \mathbb{Z}^+$. [1]

Markscheme

$S_n = \frac{10^n-1}{9}$ ***A1***

$(a = 10,\ b = 9)$

[1 mark]

(b) Hence, show that $S_1 + S_2 + S_3 + \ldots + S_n = \frac{10(10^n-1)-9n}{81}$. [4]

Markscheme

METHOD 1

$S_1 + S_2 + S_3 + \ldots + S_n$

$= \frac{10-1}{9} + \frac{10^2-1}{9} + \ldots + \frac{10^n-1}{9}$ ***(A1)***

$= \frac{10-1+10^2-1+10^3-1+\ldots+10^n-1}{9}$ OR $\frac{9\left(10-1+10^2-1+10^3-1+\ldots+10^n-1\right)}{81}$

attempt to use geometric series formula on powers of 10, and collect -1's together ***M1***

$10 + 10^2 + 10^3 + \ldots + 10^n = \frac{10(10^n-1)}{10-1}$ and
$-1 - 1 - 1 \ldots = -n$ ***A1***

$$= \frac{\frac{10(10^n-1)}{10-1} - n}{9} \text{ OR } \frac{9\left(\frac{10(10^n-1)}{10-1}\right) - 9n}{81}$$ ***A1***

Note: Award ***A1*** for any correct intermediate expression.

$$= \frac{10(10^n-1) - 9n}{81}$$ ***AG***

METHOD 2

attempt to create sum using sigma notation with S_n ***M1***

$$\sum_{i=1}^{n} \frac{10^i - 1}{9} \quad \left(= \frac{1}{9}\left(\sum_{i=1}^{n} 10^i - \sum_{i=1}^{n} 1\right)\right)$$

$$\sum_{i=1}^{n} 10^i = \frac{10(10^n-1)}{9}$$ ***A1***

$$\sum_{i=1}^{n} 1 = n$$ ***A1***

$$= \frac{1}{9}\left(\frac{10(10^n-1)}{9} - n\right) \text{ OR } \frac{1}{9}\left(\frac{10(10^n-1) - 9n}{9}\right)$$ ***A1***

$$= \frac{10(10^n-1) - 9n}{81}$$ ***AG***

METHOD 3

let $\mathrm{P}(n)$ be the proposition that
$S_1 + S_2 + S_3 + \ldots + S_n = \frac{10(10^n-1) - 9n}{81}$

considering $\mathrm{P}(1)$:

$\mathrm{LHS} = S_1 = \frac{10^1-1}{9} = 1$ and $\mathrm{RHS} = \frac{10(10^1-1)-9(1)}{81} = 1$ and so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true *i.e.* $S_1 + S_2 + S_3 + \ldots + S_k = \frac{10(10^k-1)-9k}{81}$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

considering $\mathrm{P}(k+1)$:

$S_1 + S_2 + S_3 + \ldots + S_{k+1} = \frac{10(10^k-1)-9k}{81} + \frac{10^{k+1}-1}{9}$

$= \frac{10^{k+1}-10-9k+9(10^{k+1})-9}{81}$ ***A1***

$= \frac{10(10^{k+1}-1)-9(k+1)}{81}$

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true ***R1***

(for all integers $n \geq 1$)

Note: To obtain the final ***R1***, the first ***R1*** and ***A1*** must have been awarded.

[4 marks]

2. [Maximum mark: 16] 24M.1.AHL.TZ2.10

Consider the arithmetic sequence $a,\ p,\ q\ldots,$ where $a,\ p,\ q \neq 0$.

(a) Show that $2p - q = a$. [2]

Markscheme

attempt to find a difference *(M1)*

$d = p - a, 2d = q - a, d = q - p$ OR
$p = a + d, q = a + 2d, q = p + d$

correct equation ***A1***

$p - a = q - p$ OR $q - a = 2(p - a)$ OR $p = \frac{a+q}{2}$ (or equivalent)

$2p - q = a$ ***AG***

[2 marks]

Consider the geometric sequence $a,\ s,\ t\ldots,$ where $a,\ s,\ t \neq 0$.

(b) Show that $s^2 = at$. [2]

Markscheme

attempt to find a ratio *(M1)*

$r = \frac{s}{a},\ r^2 = \frac{t}{a},\ r = \frac{t}{s}$ OR $s = ar, t = ar^2, t = sr$

correct equation ***A1***

$\left(\frac{s}{a}\right)^2 = \frac{t}{a}$ OR $\frac{s}{a} = \frac{t}{s}$ (or equivalent)

$s^2 = at$ ***AG***

[2 marks]

The first term of both sequences is a.

It is given that $q = t = 1$.

(c) Show that $p > \frac{1}{2}$. [2]

Markscheme

EITHER

$2p - 1 = s^2$ (or equivalent) ***A1***

$\left(s^2 > 0\right) \Rightarrow 2p - 1 > 0$ OR $s = \sqrt{2p-1} \Rightarrow 2p - 1 > 0$ OR $p = \frac{s^2+1}{2}$ (and $s^2 > 0$) ***R1***

OR

$2p - 1 = a$ and $s^2 = a$ ***A1***

($s^2 > 0,$ so) $a > 0 \Rightarrow 2p - 1 > 0$ OR $p\frac{a+1}{2}$ and $a > 0$ ***R1***

$\Rightarrow p > \frac{1}{2}$ ***AG***

Note: Do not award ***A0R1***.

[2 marks]

Consider the case where $a = 9,\ s > 0$ and $q = t = 1$.

(d) Write down the first four terms of the (d.i) arithmetic sequence; [2]

Markscheme

$9, 5, 1, -3$ ***A1A1***

Note: Award ***A1*** for each of 2nd term and 4th term

[2 marks]

(d.ii) geometric sequence. [2]

Markscheme

$9, 3, 1, \frac{1}{3}$ ***A1A1***

Note: Award ***A1*** for each of 2nd term and 4th term

[2 marks]

The arithmetic and the geometric sequence are used to form a new arithmetic sequence u_n.

The first three terms of u_n are $u_1 = 9 + \ln 9,\ u_2 = 5 + \ln 3,$ and $u_3 = 1 + \ln 1$.

(e.i) Find the common difference of the new sequence in terms of $\ln 3$. [3]

Markscheme

attempt to find the difference between two consecutive terms ***(M1)***

$d = u_2 - u_1 = 5 + \ln 3 - 9 - \ln 9$ OR
$d = u_3 - u_2 = 1 + \ln 1 - 5 - \ln 3$

$\ln 9 = 2 \ln 3$ OR $\ln 1 = 0$ OR
$\ln 3 - \ln 9 = \ln \frac{1}{3} \left(= \ln 3^{-1} = -\ln 3\right)$ (seen anywhere) ***(A1)***

$d = -4 - \ln 3$ ***A1***

[3 marks]

(e.ii) Show that $\sum_{i=1}^{10} = -90 - 25 \ln 3.$ [3]

Markscheme

METHOD 1

attempt to substitute first term and their common difference into S_{10} ***(M1)***

$\frac{10}{2}(2(9 + \ln 9) + 9(-4 - \ln 3))$ OR
$\frac{10}{2}(2(9 + 2 \ln 3) + 9(-4 - \ln 3))$ (or equivalent) ***A1***

$= 5(-18 - 5 \ln 3)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i = -90 - 25 \ln 3$ ***AG***

METHOD 2

$u_{10} = 9 + \ln 9 + 9(-4 - \ln 3)(= -27 + \ln 9 - 9 \ln 3)$

attempt to substitute first term and their u_{10} into S_{10} ***(M1)***

$\frac{10}{2}(2(9 + \ln 9) + 9(-4 - \ln 3))$ OR
$\frac{10}{2}(9 + \ln 9 - 27 + \ln 9 - 9 \ln 3)$ OR

$\frac{10}{2}(2(9+2\ln 3)+9(-4-\ln 3))$ OR
$\frac{10}{2}(9+\ln 9-27-7\ln 3)$ (or equivalent) ***A1***

$=5(-18-5\ln 3)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i = -90-25\ln 3$ ***AG***

[3 marks]

3. [Maximum mark: 7] 23N.1.SL.TZ1.4

The sum of the first n terms of an arithmetic sequence is given by $S_n = pn^2 - qn$, where p and q are positive constants.

It is given that $S_5 = 65$ and $S_6 = 96$.

(a) Find the value of p and the value of q. [5]

Markscheme

METHOD 1

attempt to form at least one equation, using either S_5 or S_6 ***(M1)***

$65 = 25p - 5q$ $(13 = 5p - q)$ **and** $96 = 36p - 6q$ $(16 = 6p - q)$ ***(A1)***

valid attempt to solve simultaneous linear equations in p and q and by substituting or eliminating one of the variables. ***(M1)***

$p = 3$, $q = 2$ ***A1A1***

Note: If candidate does not explicitly state their values of p and q, but gives $S_n = 3n^2 - 2n$, award final two marks as ***A1A0***.

METHOD 2

attempt to form at least one equation, using either S_5 or S_6 ***(M1)***

$$65 = \frac{5}{2}(2u_1 + 4d)\ (26 = 2u_1 + 4d)$$ **and**
$$96 = 3(2u_1 + 5d)\ (32 = 2u_1 + 5d)$$ ***(A1)***

valid attempt to solve simultaneous linear equations in u_1 and d by substituting or eliminating one of the variables. ***(M1)***

$u_1 = 1,\ d = 6$ ***A1***

$S_n = \frac{n}{2}(2 + 6(n - 1)) = 3n^2 - 2n$

$p = 3$ and $q = 2$ ***A1***

Note: If candidate does not explicitly state their values of *p* and *q*, do not award the final mark.

[5 marks]

(b) Find the value of u_6. [2]

Markscheme

$u_6 = S_6 - S_5$ OR substituting their values of u_1 and d into
$u_6 = u_1\ +\ 5d$

OR substituting their value of u_1 into $96 = \frac{6}{2}(u_1 + u_6)$ ***(M1)***

$(u_6 =)96 - 65$ OR $(u_6 =)1 + 5 \times 6$ OR $96 = 3(1 + u_6)$

$= 31$ ***A1***

[2 marks]

4. [Maximum mark: 14] 23M.1.AHL.TZ1.10

Consider the arithmetic sequence $u_1, u_2, u_3, \ldots$.

The sum of the first n terms of this sequence is given by $S_n = n^2 + 4n$.

(a.i) Find the sum of the first five terms. [2]

Markscheme

recognition that $n = 5$ ***(M1)***

$S_5 = 45$ ***A1***

[2 marks]

(a.ii) Given that $S_6 = 60$, find u_6. [2]

Markscheme

METHOD 1

recognition that $S_5 + u_6 = S_6$ ***(M1)***

$u_6 = 15$ ***A1***

METHOD 2

recognition that $60 = \frac{6}{2}\left(S_1 + u_6\right)$ ***(M1)***

$60 = 3(5 + u_6)$

$u_6 = 15$ ***A1***

METHOD 3

substituting their u_1 and d values into $u_1 + (n-1)d$ ***(M1)***

$u_6 = 15$ ***A1***

[2 marks]

(b) Find u_1. [2]

Markscheme

recognition that $u_1 = S_1$ (may be seen in (a)) OR substituting their u_6 into S_6 ***(M1)***

OR equations for S_5 and S_6 in terms of u_1 and d

$1 + 4$ OR $60 = \frac{6}{2}\left(U_1 + 15\right)$

$u_1 = 5$ ***A1***

[2 marks]

(c) Hence or otherwise, write an expression for u_n in terms of n. [3]

Markscheme

EITHER

valid attempt to find d (may be seen in (a) or (b)) ***(M1)***

$d = 2$ ***(A1)***

OR

valid attempt to find $S_n - S_{n-1}$ ***(M1)***

$n^2 + 4n - \left(n^2 - 2n + 1 + 4n - 4\right)$ ***(A1)***

OR

equating $n^2 + 4n = \frac{n}{2}\left(5 + u_n\right)$ ***(M1)***

$2n + 8 = 5 + u_n$ (or equivalent) ***(A1)***

THEN

$u_n = 5 + 2(n - 1)$ OR $u_n = 2n + 3$ ***A1***

[3 marks]

Consider a geometric sequence, v_n, where $v_2 = u_1$ and $v_4 = u_6$.

(d) Find the possible values of the common ratio, r. [3]

Markscheme

recognition that $v_2 r^2 = v_4$ OR $\left(v_3\right)^2 = v_2 \times v_4$ ***(M1)***

$r^2 = 3$ OR $v_3 = \left(\pm\right)5\sqrt{3}$ ***(A1)***

$r = \pm\sqrt{3}$ ***A1***

Note: If no working shown, award ***M1A1A0*** for $\sqrt{3}$.

[3 marks]

(e) Given that $v_{99} < 0$, find v_5. [2]

Markscheme

recognition that r is negative ***(M1)***

$v_5 = -15\sqrt{3}\ \left(= -\frac{45}{\sqrt{3}}\right)$ ***A1***

[2 marks]

5. [Maximum mark: 18] 22M.1.AHL.TZ1.10

Consider the series $\ln x + p \ln x + \frac{1}{3}\ln x + \ldots$, where $x \in \mathbb{R},\ x > 1$ and $p \in \mathbb{R},\ p \neq 0$.

Consider the case where the series is geometric.

(a.i) Show that $p = \pm\frac{1}{\sqrt{3}}$. [2]

Markscheme

EITHER

attempt to use a ratio from consecutive terms ***M1***

$\frac{p \ln x}{\ln x} = \frac{\frac{1}{3}\ln x}{p \ln x}$ OR $\frac{1}{3}\ln x = (\ln x)r^2$ OR $p \ln x = \ln x\left(\frac{1}{3p}\right)$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} + \ldots$ and consider the powers of x in geometric sequence

Award ***M1*** for $\frac{p}{1} = \frac{\frac{1}{3}}{p}$.

OR

$r = p$ and $r^2 = \frac{1}{3}$ ***M1***

THEN

$p^2 = \frac{1}{3}$ OR $r = \pm\frac{1}{\sqrt{3}}$ ***A1***

$p = \pm\frac{1}{\sqrt{3}}$ ***AG***

Note: Award ***M0A0*** for $r^2 = \frac{1}{3}$ or $p^2 = \frac{1}{3}$ with no other working seen.

[2 marks]

(a.ii) Hence or otherwise, show that the series is convergent. [1]

Markscheme

EITHER

since, $|p| = \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}} < 1$ ***R1***

OR

since, $|p| = \frac{1}{\sqrt{3}}$ and $-1 < p < 1$ ***R1***

THEN

$\Rightarrow$ the geometric series converges. ***AG***

Note: Accept r instead of p.
Award ***R0*** if both values of p not considered.

[1 mark]

(a.iii) Given that $p > 0$ and $S_\infty = 3 + \sqrt{3}$, find the value of x. [3]

Markscheme

$\frac{\ln x}{1-\frac{1}{\sqrt{3}}} \ \left(= 3 + \sqrt{3}\right)$ ***(A1)***

$\ln x = 3 - \frac{3}{\sqrt{3}} + \sqrt{3} - \frac{\sqrt{3}}{\sqrt{3}}$ OR
$\ln x = 3 - \sqrt{3} + \sqrt{3} - 1 \ \ (\Rightarrow \ln x = 2)$ ***A1***

$x = \mathrm{e}^2$ ***A1***

[3 marks]

Now consider the case where the series is arithmetic with common difference d.

(b.i) Show that $p = \frac{2}{3}$. [3]

Markscheme

METHOD 1

attempt to find a difference from consecutive terms or from u_2 ***M1***

correct equation ***A1***

$p \ln x - \ln x = \frac{1}{3}\ln x - p \ln x$ OR
$\frac{1}{3}\ln x = \ln x + 2(p \ln x - \ln x)$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} + \ldots$ and consider the powers of x in arithmetic sequence.

Award ***M1A1*** for $p - 1 = \frac{1}{3} - p$

$2p \ln x = \frac{4}{3}\ln x \ \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

METHOD 2

attempt to use arithmetic mean $u_2 = \frac{u_1+u_3}{2}$ ***M1***

$p \ln x = \frac{\ln x + \frac{1}{3}\ln x}{2}$ ***A1***

$2p \ln x = \frac{4}{3}\ln x \ \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

METHOD 3

attempt to find difference using u_3 ***M1***

$\frac{1}{3}\ln x = \ln x + 2d \ \left(\Rightarrow d = -\frac{1}{3}\ln x\right)$

$u_2 = \ln x + \frac{1}{2}\left(\frac{1}{3}\ln x - \ln x\right)$ OR $p \ln x - \ln x = -\frac{1}{3}\ln x$
A1

$p \ln x = \frac{2}{3}\ln x$ ***A1***

$p = \frac{2}{3}$ ***AG***

[3 marks]

(b.ii) Write down d in the form $k \ln x$, where $k \in \mathbb{Q}$. [1]

Markscheme

$d = -\frac{1}{3}\ln x$ ***A1***

[1 mark]

(b.iii) The sum of the first n terms of the series is $\ln\left(\frac{1}{x^3}\right)$.

Find the value of n. [8]

Markscheme

METHOD 1

$S_n = \frac{n}{2}\left\lfloor 2 \ln x + (n-1) \times \left(-\frac{1}{3}\ln x\right)\right\rfloor$

attempt to substitute into S_n and equate to $\ln\left(\frac{1}{x^3}\right)$ ***(M1)***

$\frac{n}{2}\left\lfloor 2 \ln x + (n-1) \times \left(-\frac{1}{3}\ln x\right)\right\rfloor = \ln\left(\frac{1}{x^3}\right)$

$\ln\left(\frac{1}{x^3}\right) = -\ln x^3 \left(= \ln x^{-3}\right)$ ***(A1)***

$= -3 \ln x$ *(A1)*

correct working with S_n (seen anywhere) *(A1)*

$\frac{n}{2}\left[2 \ln x - \frac{n}{3}\ln x + \frac{1}{3}\ln x\right]$ OR $n \ln x - \frac{n(n-1)}{6}\ln x$ OR $\frac{n}{2}\left(\ln x + \left(\frac{4-n}{3}\right) \ln x\right)$

correct equation without $\ln x$ ***A1***

$\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$ OR $n - \frac{n(n-1)}{6} = -3$ or equivalent

Note: Award as above if the series $1 + p + \frac{1}{3} + \ldots$ is considered leading to $\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$.

attempt to form a quadratic $= 0$ *(M1)*

$n^2 - 7n - 18 = 0$

attempt to solve their quadratic *(M1)*

$(n - 9)(n + 2) = 0$

$n = 9$ ***A1***

METHOD 2

$\ln\left(\frac{1}{x^3}\right) = -\ln x^3 (= \ln x^{-3})$ *(A1)*

$= -3 \ln x$ *(A1)*

listing the first 7 terms of the sequence *(A1)*

$\ln x + \frac{2}{3}\ln x + \frac{1}{3}\ln x + 0 - \frac{1}{3}\ln x - \frac{2}{3}\ln x - \ln x + \ldots$

recognizing first 7 terms sum to 0 ***M1***

8^{th} term is $-\frac{4}{3}\ln x$ ***(A1)***

9^{th} term is $-\frac{5}{3}\ln x$ ***(A1)***

sum of 8^{th} and 9^{th} term $= -3\ \ln x$ ***(A1)***

$n = 9$ ***A1***

[8 marks]

6. [Maximum mark: 15] 21N.1.SL.TZ0.8

Consider the function $f(x) = a^x$ where $x,\ a \in \mathbb{R}$ and $x > 0,\ a > 1$.

The graph of f contains the point $\left(\frac{2}{3}, 4\right)$.

(a) Show that $a = 8$. [2]

Markscheme

$f\left(\frac{2}{3}\right) = 4$ OR $a^{\frac{2}{3}} = 4$ ***(M1)***

$a = 4^{\frac{3}{2}}$ OR $a = \left(2^2\right)^{\frac{3}{2}}$ OR $a^2 = 64$ OR $\sqrt[3]{a} = 2$ ***A1***

$a = 8$ ***AG***

[2 marks]

(b) Write down an expression for $f^{-1}(x)$. [1]

Markscheme

$f^{-1}(x) = \log_8 x$ ***A1***

Note: Accept $f^{-1}(x) = \log_a x$.
Accept any equivalent expression for f^{-1} e.g. $f^{-1}(x) = \frac{\ln x}{\ln 8}$.

[1 mark]

(c) Find the value of $f^{-1}\left(\sqrt{32}\right)$. [3]

Markscheme

correct substitution ***(A1)***

$\log_8 \sqrt{32}$ OR $8^x = 32^{\frac{1}{2}}$

correct working involving log/index law ***(A1)***

$\frac{1}{2}\log_8 32$ OR $\frac{5}{2}\log_8 2$ OR $\log_8 2 = \frac{1}{3}$ OR $\log_2 2^{\frac{5}{2}}$ OR $\log_2 8 = 3$ OR $\frac{\ln 2^{\frac{5}{2}}}{\ln 2^3}$ OR $2^{3x} = 2^{\frac{5}{2}}$

$f^{-1}\left(\sqrt{32}\right) = \frac{5}{6}$ ***A1***

[3 marks]

Consider the arithmetic sequence $\log_8 27\ ,\ \log_8 p\ ,\ \log_8 q\ ,\ \log_8 125$, where $p > 1$ and $q > 1$.

(d.i) Show that $27,\ p,\ q$ and 125 are four consecutive terms in a geometric sequence. [4]

Markscheme

METHOD 1

equating a pair of differences ***(M1)***

$u_2 - u_1 = u_4 - u_3 (= u_3 - u_2)$

$\log_8 p - \log_8 27 = \log_8 125 - \log_8 q$

$\log_8 125 - \log_8 q = \log_8 q - \log_8 p$

$\log_8 \left(\frac{p}{27}\right) = \log_8 \left(\frac{125}{q}\right), \ \log_8 \left(\frac{125}{q}\right) = \log_8 \left(\frac{q}{p}\right)$ ***A1A1***

$\frac{p}{27} = \frac{125}{q}$ and $\frac{125}{q} = \frac{q}{p}$ ***A1***

$27, \ p, \ q$ and 125 are in geometric sequence ***AG***

Note: If candidate assumes the sequence is geometric, award no marks for part (i). If $r = \frac{5}{3}$ has been found, this will be awarded marks in part (ii).

METHOD 2

expressing a pair of consecutive terms, in terms of d ***(M1)***

$p = 8^d \times 27$ and $q = 8^{2d} \times 27$ OR $q = 8^{2d} \times 27$ and $125 = 8^{3d} \times 27$

two correct pairs of consecutive terms, in terms of d ***A1***

$\frac{8^d \times 27}{27} = \frac{8^{2d} \times 27}{8^d \times 27} = \frac{8^{3d} \times 27}{8^{2d} \times 27}$ (must include 3 ratios) ***A1***

all simplify to 8^d ***A1***

$27, \ p, \ q$ and 125 are in geometric sequence ***AG***

[4 marks]

(d.ii) Find the value of p and the value of q. [5]

Markscheme

METHOD 1 (geometric, finding r)

$u_4 = u_1 r^3$ OR $125 = 27(r)^3$ ***(M1)***

$r = \frac{5}{3}$ (seen anywhere) ***A1***

$p = 27r$ OR $\frac{125}{q} = \frac{5}{3}$ ***(M1)***

$p = 45,\ q = 75$ ***A1A1***

METHOD 2 (arithmetic)

$u_4 = u_1 + 3d$ OR $\log_8 125 = \log_8 27 + 3d$ ***(M1)***

$d = \log_8\left(\frac{5}{3}\right)$ (seen anywhere) ***A1***

$\log_8 p = \log_8 27 + \log_8\left(\frac{5}{3}\right)$ OR
$\log_8 q = \log_8 27 + 2\log_8\left(\frac{5}{3}\right)$ ***(M1)***

$p = 45,\ q = 75$ ***A1A1***

METHOD 3 (geometric using proportion)

recognizing proportion ***(M1)***

$pq = 125 \times 27$ OR $q^2 = 125p$ OR $p^2 = 27q$

two correct proportion equations *A1*

attempt to eliminate either p or q *(M1)*

$q^2 = 125 \times \frac{125 \times 27}{q}$ OR $p^2 = 27 \times \frac{125 \times 27}{p}$

$p = 45,\ q = 75$ *A1A1*

[5 marks]

7. [Maximum mark: 9] 21N.2.SL.TZ0.6

The sum of the first n terms of a geometric sequence is given by

$S_n = \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r.$

(a) Find the first term of the sequence, u_1. [2]

Markscheme

$u_1 = S_1 = \frac{2}{3} \times \frac{7}{8}$ *(M1)*

$= \frac{14}{24}\left(= \frac{7}{12} = 0.583333\ldots\right)$ *A1*

[2 marks]

(b) Find S_∞. [3]

Markscheme

$r = \frac{7}{8}(= 0.875)$ *(A1)*

substituting their values for u_1 and r into $S_\infty = \frac{u_1}{1-r}$ *(M1)*

$= \frac{14}{3} (= 4.66666\ldots)$ ***A1***

[3 marks]

(c) Find the least value of n such that $S_\infty - S_n < 0.001$. [4]

Markscheme

attempt to substitute their values into the inequality or formula for S_n ***(M1)***

$\frac{14}{3} - \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r < 0.001$ OR $S_n = \frac{\frac{7}{12}\left(1-\left(\frac{7}{8}\right)^n\right)}{\left(1-\frac{7}{8}\right)}$

attempt to solve their inequality using a table, graph or logarithms

(must be exponential) ***(M1)***

Note: Award ***(M0)*** if the candidate attempts to solve $S_\infty - u_n < 0.001$.

correct critical value or at least one correct crossover value ***(A1)***

$63.2675\ldots$ OR $S_\infty - S_{63} = 0.001036\ldots$ OR $S_\infty - S_{64} = 0.000906\ldots$

OR $S_\infty - S_{63} - 0.001 = 0.0000363683\ldots$ OR $S_\infty - S_{64} - 0.001 = 0.0000931777\ldots$

least value is $n = 64$ ***A1***

[4 marks]

8. [Maximum mark: 5] 21M.1.SL.TZ1.3

Consider an arithmetic sequence where $u_8 = S_8 = 8$. Find the value of the first term, u_1, and the value of the common difference, d. [5]

Markscheme

METHOD 1 (finding u_1 first, from S_8)

$4(u_1 + 8) = 8$ ***(A1)***

$u_1 = -6$ ***A1***

$u_1 + 7d = 8$ OR $4(2u_1 + 7d) = 8$ (may be seen with their value of u_1) ***(A1)***

attempt to substitute their u_1 ***(M1)***

$d = 2$ ***A1***

METHOD 2 (solving simultaneously)

$u_1 + 7d = 8$ ***(A1)***

$4(u_1 + 8) = 8$ OR $4(2u_1 + 7d) = 8$ OR $u_1 = -3d$ ***(A1)***

attempt to solve linear or simultaneous equations ***(M1)***

$u_1 = -6,\ d = 2$ ***A1A1***

[5 marks]

9. [Maximum mark: 5] 20N.1.AHL.TZ0.H_5

The first term in an arithmetic sequence is 4 and the fifth term is $\log_2 625$.

Find the common difference of the sequence, expressing your answer in the form $\log_2 p$, where $p \in \mathbb{Q}$. [5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$u_5 = 4 + 4d = \log_2 625$ ***(A1)***

$4d = \log_2 625 - 4$

attempt to write an integer (*eg* 4 or 1) in terms of $\log_2$ ***M1***

$4d = \log_2 625 - \log_2 16$

attempt to combine two logs into one ***M1***

$4d = \log_2 \left(\frac{625}{16}\right)$

$d = \frac{1}{4}\log_2 \left(\frac{625}{16}\right)$

attempt to use power rule for logs ***M1***

$d = \log_2 \left(\frac{625}{16}\right)^{\frac{1}{4}}$

$d = \log_2 \left(\frac{5}{2}\right)$ ***A1***

[5 marks]

Note: Award method marks in any order.

10. [Maximum mark: 8] 20N.2.SL.TZ0.S_6

An infinite geometric series has first term $u_1 = a$ and second term $u_2 = \frac{1}{4}a^2 - 3a$, where $a > 0$.

(a) Find the common ratio in terms of a. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

evidence of dividing terms (in any order) ***(M1)***

eg $\frac{u_1}{u_2}$, $\frac{\frac{1}{4}a^2 - 3a}{a}$

$r = \frac{1}{4}a - 3$ ***A1 N2***

[2 marks]

(b) Find the values of a for which the sum to infinity of the series exists. [3]

Markscheme

recognizing $|r| < 1$ (must be in terms of a) ***(M1)***

eg
$\left|\frac{1}{4}a - 3\right| < 1, \quad -1 \le \frac{1}{4}a - 3 \le 1, \quad -4 < a - 12 < 4$

$8 < a < 16$ ***A2 N3***

[3 marks]

(c) Find the value of a when $S_\infty = 76$. [3]

Markscheme

correct equation ***(A1)***

eg $\frac{a}{1-\left(\frac{1}{4}a-3\right)} = 76,\ a = 76\left(4 - \frac{1}{4}a\right)$

$a = \frac{76}{5}\ (= 15.2)$ (exact) ***A2 N3***

[3 marks]

11. [Maximum mark: 5] 19N.2.AHL.TZ0.H_1

A geometric sequence has $u_4 = -70$ and $u_7 = 8.75$. Find the second term of the sequence. [5]

Markscheme

$u_1 r^3 = -70$, $u_1 r^6 = 8.75$ ***(M1)***

$r^3 = \frac{8.75}{-70} = -0.125$ ***(A1)***

$\Rightarrow r = -0.5$ ***(A1)***

valid attempt to find u_2 ***(M1)***

for example: $u_1 = \frac{-70}{-0.125} = 560$

$u_2 = 560 \times -0.5$

$= -280$ ***A1***

[5 marks]

12. [Maximum mark: 7] 19M.2.SL.TZ1.S_7

The first terms of an infinite geometric sequence, u_n, are 2, 6, 18, 54, …

The first terms of a second infinite geometric sequence, v_n, are 2, –6, 18, –54, …

The terms of a third sequence, w_n, are defined as $w_n = u_n + v_n$.

The finite series, $\sum_{k=1}^{225} w_k$, can also be written in the form $\sum_{k=0}^{m} 4r^k$.

(a) Write down the first three **non-zero** terms of w_n. [3]

Markscheme

attempt to add corresponding terms ***(M1)***

eg $2+2,\ 6+(-6),\ 2(3)^{n-1} + 2(-3)^{n-1}$

correct value for w_5 ***(A1)***

eg 324

4, 36, 324 (accept 4 + 36 + 324) ***A1 N3***

[3 marks]

(b.i) Find the value of r. [2]

Markscheme

valid approach ***(M1)***

eg $4 \times r^1 = 36$, $4 \times 9^{n-1}$

$r = 9$ (accept $\sum_{k=0}^{m} 4 \times 9^k$; m may be incorrect) ***A1 N2***

[2 marks]

(b.ii) Find the value of m. [2]

Markscheme

recognition that 225 terms of w_n consists of 113 non-zero terms ***(M1)***

eg $\sum_1^{113}, \sum_0^{112}$, 113

$m = 112$ (accept $\sum_{k=0}^{1} 124 \times r^k$; r may be incorrect) ***A1 N2***

[2 marks]

13. [Maximum mark: 5] 18N.2.AHL.TZ0.H_1

Consider a geometric sequence with a first term of 4 and a fourth term of −2.916.

(a) Find the common ratio of this sequence. [3]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$u_4 = u_1 r^3 \Rightarrow -2.916 = 4r^3$ ***(A1)***

solving, $r = -0.9$ ***(M1)A1***

[3 marks]

(b) Find the sum to infinity of this sequence. [2]

Markscheme

$S_\infty = \frac{4}{1-(-9)}$ ***(M1)***

$= \frac{40}{19} \, (= 2.11)$ ***A1***

[2 marks]