**MATH HL**

**TEST**

**EXPONENTS – LOGARITHMS**
without GDC
***by Christos Nikolaidis***

**Name:** SOLUTIONS

**Date:**________________

**Marks:**____**/100**

**Grade:**______

## Questions

**1.** *[Maximum mark: 9]*

Let $\ln x = a$, $\ln y = b$ and $\ln 5 = c$. Express the following in terms of $a, b$ and $c$:

(a) $\ln \frac{25\sqrt{x}}{y^3}$ (b) $\log_5 xy$ (c) $\log_y 5e$ *[3+3+3 marks]*

(a) $\ln 25 + \ln\sqrt{x} - \ln y^3 = 2c + \frac{1}{2}a - 3b$

(b) $\frac{\ln xy}{\ln 5} = \frac{a+b}{c}$

(c) $\frac{\ln 5e}{\ln y} = \frac{c+1}{b}$

**2.** *[Maximum mark: 11]*

Solve the exponential equations

(a) $8^{x+3} = \left(\frac{1}{16}\right)^{-x-5}$ (b) $7e^{2x+3} = 1$ (c) $2^{x-1}3^{x+3} = 7^{x+2}$ *[3+3+5 marks]*

(a) $2^{3x+9} = 2^{4x+20} \Leftrightarrow 3x+9 = 4x+20 \Leftrightarrow \boxed{x=-11}$

(b) $e^{2x+3} = \frac{1}{7} \Leftrightarrow 2x+3 = \ln\frac{1}{7} \Leftrightarrow x = \frac{\ln\frac{1}{7} - 3}{2} = \frac{-\ln 7 - 3}{2}$

(c) $\frac{2^x}{2} \cdot 3^x \cdot 27 = 7^x \cdot 49 \Leftrightarrow \left(\frac{2\cdot 3}{7}\right)^x = \frac{98}{27}$

$\Leftrightarrow \left(\frac{6}{7}\right)^x = \frac{49}{27} \Leftrightarrow \boxed{x = \log_{\frac{6}{7}} \frac{98}{27}}$

OR $\boxed{x = \frac{\ln\frac{98}{27}}{\ln\frac{6}{7}}}$

**3.** *[Maximum mark: 6]*

Solve the exponential equation

$$2(4^{x+1}) = 2 + \frac{3}{4^x}$$

Give your answer in the following forms

(i) $x = a - \log_4 b$, where $a$ and $b$ are integers.

(ii) $x = \dfrac{\ln c}{\ln d}$, where $c$ and $d$ are real numbers.

Set $y = 4^x$

$2(4^x \cdot 4) = 2 + \frac{3}{4^x} \Leftrightarrow 8y = 2 + \frac{3}{y}$

$\Leftrightarrow 8y^2 - 2y - 3 = 0$

$\Delta = 4 + 96 = 100 \qquad y = \frac{2 \pm 10}{16} = \begin{cases} -\frac{1}{2} \text{ (rejected)} \\ \frac{12}{16} = \frac{3}{4} \end{cases}$

Hence $4^x = \frac{3}{4}$

(i) $x = \log_4 \frac{3}{4} = \log_4 3 - \log_4 4 = \boxed{-1 + \log_4 3}$

(ii) $\ln 4^x = \ln \frac{3}{4} \Leftrightarrow \boxed{x = \frac{\ln \frac{3}{4}}{\ln 4}}$

**4.** *[Maximum mark: 7]*
Solve the logarithmic equations

(a) $\dfrac{\log_5(4-x)}{2} - \log_5 x = \dfrac{1}{2}$ (b) $\log_{\sqrt{5}}(4-x) - 4\log_5 x = 2$ *[4+3 marks]*

(a) $\log_5(4-x) - 2\log_5 x = 1$

$\Leftrightarrow \log_5 \dfrac{4-x}{x^2} = 1 \Leftrightarrow \dfrac{4-x}{x^2} = 5 \Leftrightarrow 4-x = 5x^2$

$\Leftrightarrow 5x^2 + x - 4 = 0$

$\Delta = 1 + 80 = 81 \qquad x = \dfrac{-1 \pm 9}{10} = \begin{cases} -1 \text{ (rejected)} \\ \dfrac{8}{10} = \dfrac{4}{5} \end{cases}$

Hence $\boxed{x = \dfrac{4}{5}}$

(b) $\dfrac{\log_5(4-x)}{\log_5 \sqrt{5}} - 4\log_5 x = 2$

$\Leftrightarrow \dfrac{\log_5(4-x)}{\frac{1}{2}} - 4\log_5 x = 2$

$\Leftrightarrow \log_5(4-x) - 2\log_5 x = 1$

It is exactly the same equation

Hence $\boxed{x = \dfrac{4}{5}}$

**5.** *[Maximum mark: 8]*

Solve the logarithmic equations

(a) $2\log_3(x-3) = 2 - \log_{1/3}(x+1)$ (b) $\log_x 2 - 3\log_2 x = 2$ *[4+4 marks]*

(a) $2\log_3(x-3) = 2 - \dfrac{\log_3(x+1)}{\log_3 \frac{1}{3}}$

$\Leftrightarrow 2\log_3(x-3) = 2 + \log_3(x+1)$

$\Leftrightarrow \log_3 \dfrac{(x-3)^2}{x+1} = 2 \Leftrightarrow \dfrac{(x-3)^2}{x+1} = 3^2 \Leftrightarrow (x-3)^2 = 9x+9$

$\Leftrightarrow x^2 - 6x + 9 = 9x + 9 \Leftrightarrow x^2 - 15x = 0 \Leftrightarrow x = 0$ (rejected) or $x = 15$

Hence $\boxed{x = 15}$

(b) $\dfrac{\log_2 2}{\log_2 x} - 3\log_2 x = 2 \Leftrightarrow \dfrac{1}{\log_2 x} - 3\log_2 x = 2$

Set $y = \log_2 x$ $\quad \dfrac{1}{y} - 3y = 2 \Leftrightarrow 1 - 3y^2 = 2y$

$\Leftrightarrow 3y^2 + 2y - 1 = 0 \quad \Delta = 16 \quad y = \dfrac{-2 \pm 4}{6} = \begin{cases} -1 \\ \frac{2}{6} = \frac{1}{3} \end{cases}$

For $y = -1$ $\quad \log_2 x = -1 \Leftrightarrow x = 2^{-1} \Leftrightarrow \boxed{x = \frac{1}{2}}$

For $y = \frac{1}{3}$ $\quad \log_2 x = \frac{1}{3} \Leftrightarrow x = 2^{1/3} \Leftrightarrow \boxed{x = \sqrt[3]{2}}$

**6.** *[Maximum mark: 9]*
Solve the equations

(a) $x^{\ln x} = e^4$ (b) $\dfrac{x^{\log x}}{x^2} = \dfrac{\sqrt{x}}{10}$ *[4+5 marks]*

(a) $\ln\left(x^{\ln x}\right) = \ln e^4 \Leftrightarrow \ln x \cdot \ln x = 4$

$\Leftrightarrow (\ln x)^2 = 4 \Leftrightarrow \ln x = \pm 2 \Leftrightarrow \boxed{x = e^{\pm 2}}$

(b) $10x^{\log x} = x^2\sqrt{x} \Leftrightarrow 10x^{\log x} = x^{5/2}$

$\Leftrightarrow \log\left(10x^{\log x}\right) = \log x^{5/2}$

$\Leftrightarrow \log 10 + \log x \cdot \log x = \frac{5}{2}\log x$

$\Leftrightarrow 1 + (\log x)^2 = \frac{5}{2}\log x \Leftrightarrow 2(\log x)^2 - 5\log x + 2 = 0$

Set $y = \log x$ $2y^2 - 5y + 2 = 0$

$\Delta = 25 - 16 = 9$ $y = \dfrac{5 \pm 3}{4} = \begin{cases} 2 \\ \frac{1}{2} \end{cases}$

$\log x = 2 \Leftrightarrow x = 10^2 \Leftrightarrow \boxed{x = 100}$

$\log x = \frac{1}{2} \Leftrightarrow x = 10^{1/2} \Leftrightarrow \boxed{x = \sqrt{10}}$

**7.** *[Maximum mark: 6]*

Solve the equation

$$2^{\log x} + 3(4^{\log x}) = 52$$

Set $y = 2^{\log x}$ then $4^{\log x} = 2^{2\log x} = y^2$

$y + 3y^2 = 52 \Leftrightarrow 3y^2 + y - 52 = 0$

$\Delta = 1 + 624 = 625 \qquad y = \frac{-1 \pm 25}{6} = \begin{cases} -\frac{26}{6} = -\frac{13}{3} \text{ (rej)} \\ \frac{24}{6} = 4 \end{cases}$

Hence $2^{\log x} = 4 \Leftrightarrow \log x = 2 \Leftrightarrow x = 10^2$

$\Leftrightarrow \boxed{x = 100}$

**8.** *[Maximum mark: 6]*

Solve the simultaneous equations

$$27^{y} = 9^{2x+3} \qquad \text{and} \qquad \log_4 y = 2\log_{16} x + 2$$

- $3^{3y} = 3^{4x+6} \Leftrightarrow \boxed{3y = 4x + 6}$ (1)

- $\log_4 y = 2\log_{16} x + 2 \Leftrightarrow \log_4 y = 2\dfrac{\log_4 x}{\log_4 16} + 2$

$\Leftrightarrow \log_4 y = \log_4 x + 2 \Leftrightarrow \log_4 \dfrac{y}{x} = 2 \Leftrightarrow \dfrac{y}{x} = 16$

$\Leftrightarrow \boxed{y = 16x}$ (2)

- We solve the system (1) (2)

$3(16x) = 4x + 6 \Leftrightarrow 48x = 4x + 6 \Leftrightarrow 44x = 6 \Leftrightarrow \boxed{x = \frac{3}{22}}$

Then $y = 16\frac{3}{22} \Leftrightarrow \boxed{y = \frac{24}{11}}$

**9.** *[Maximum mark: 8]*

It is given that $\log_a(x^2y) = p$ and $\log_a(\frac{x}{y^2}) = q$

a) Find $\log_a x$ and $\log_a y$ in terms of $p$ and $q$. *[6 marks]*

b) Express $\log_a(xy)$ in terms of $p$ and $q$. *[2 marks]*

(a) $2\log_a x + \log_a y = p$ (a)

$\log_a x - 2\log_a y = q$ (b)

(a) $- 2\times$ (b) : $5\log_a y = p - 2q \Leftrightarrow \boxed{\log_a y = \frac{p-2q}{5}}$

$2\times$(a) + (b) : $5\log_a x = 2p + q \Leftrightarrow \boxed{\log_a x = \frac{2p+q}{5}}$

(b) $\log_a(xy) = \log_a x + \log_a y = \frac{p-2q}{5} + \frac{2p+q}{5}$

$= \boxed{\frac{3p-q}{5}}$

**10.** *[Maximum mark: 8]*

The mass m kg of a radio-active substance at time t hours is given by $m = m_0 e^{-kt}$.

a) If the half live time of the material is 3 hours show that $k = \frac{\ln 2}{3}$

b) If the mass of the substance after 6 hours is 1kg, find its mass after 12 hours..

(a) $\frac{m_0}{2} = m_0 e^{-k3} \Leftrightarrow \frac{1}{2} = e^{-3k} \Leftrightarrow -3k = \ln\frac{1}{2}$

$\Leftrightarrow 3k = \ln 2 \Leftrightarrow \boxed{k = \frac{\ln 2}{3}}$

(b) $1 = m_0 e^{-\frac{\ln 2}{3}\cdot 6} \Leftrightarrow m_0 e^{-2\ln 2} = 1$

$\Leftrightarrow m_0 = \frac{1}{e^{-2\ln 2}} \Leftrightarrow m_0 = e^{2\ln 2} \Leftrightarrow m_0 = e^{\ln 4}$

$\Leftrightarrow \boxed{m_0 = 4}$

Hence, for $t=12$ $m = 4e^{-\frac{\ln 2}{3}\cdot 12} = 4e^{-4\ln 2}$

$\Rightarrow m = 4e^{\ln 2^{-4}} \Rightarrow m = 4\cdot 2^{-4} \Rightarrow \boxed{m = \frac{4}{16} = \frac{1}{4}}$

(Otherwise, since every 3 hours it halves,

after 9 h it is $\frac{1}{2}$

after 12 h it is $\frac{1}{4}$ )

**11.** *[Maximum mark: 10]*

Let $f(x) = e^{3-2x}$ and $g(x) = \dfrac{3 - \ln x}{2}$

a) Show that $f$ and $g$ are inverse to each other.

b) Find $(fog)(x)$ in the simplest form.

c) Sketch the graphs of $f$ and $g$; Indicate intercepts and asymptotes.

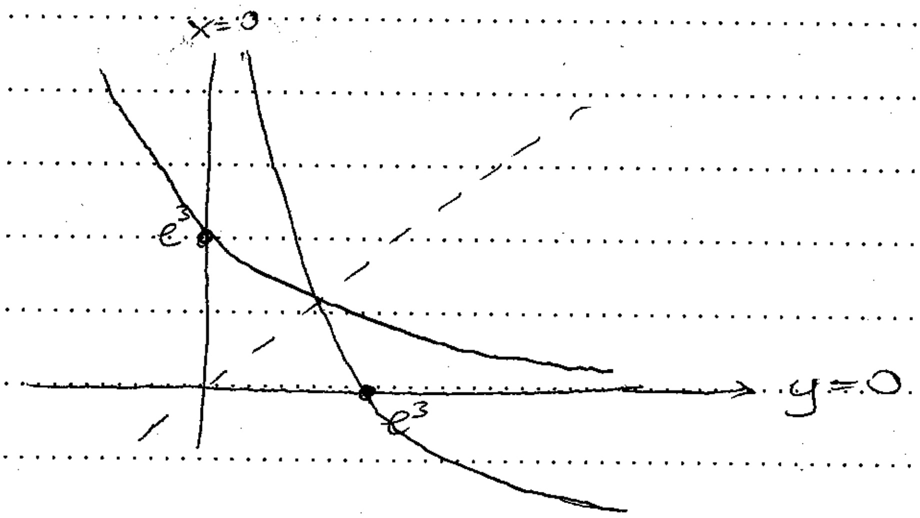
d) Write down the number of solutions of the equation $f(x) = g(x)$.

(a) $y = e^{3-2x} \Leftrightarrow \ln y = 3 - 2x \Leftrightarrow 2x = 3 - \ln y$

$\Leftrightarrow x = \dfrac{3 - \ln y}{2}$ hence $f^{-1}(x) = \dfrac{3 - \ln x}{2}$

(b) $(f \circ g)(x) = (f \circ f^{-1})(x) = x$

(c)




(d) one solution only.

**12.** *[Maximum mark: 12]*

Consider the function $f(x) = e^{x+2}$.

a) Sketch the graph of $f(x)$ by indicating clearly the intercepts and any asymptotes.

b) Write down the domain and the range of $f$

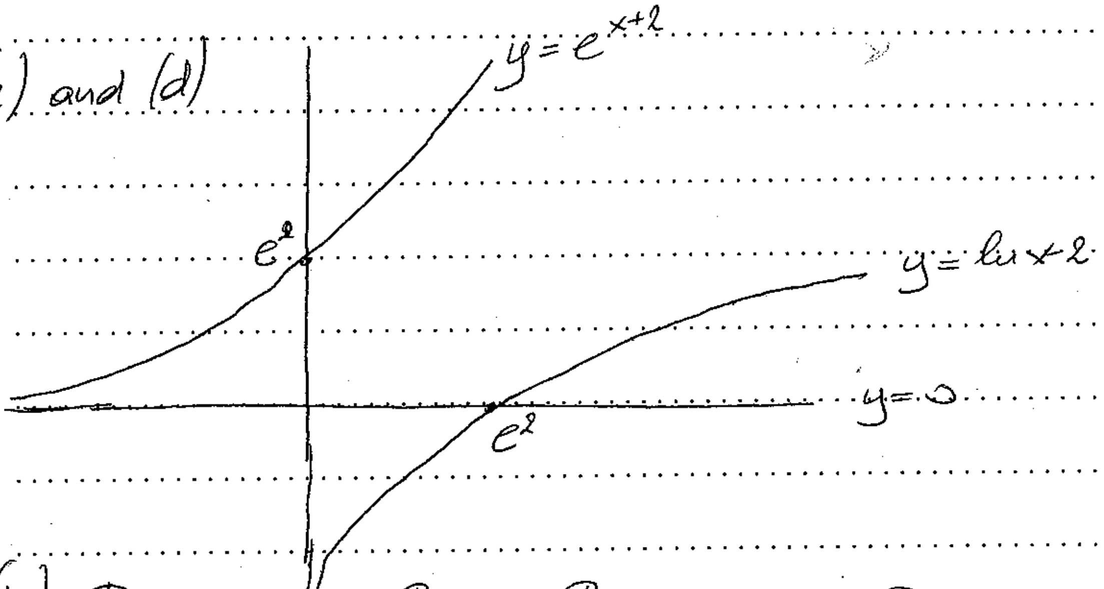
c) Find $f^{-1}$

d) On the same axes, sketch the graph of $f^{-1}$ by indicating clearly the intercepts and any asymptotes.

e) Write down the domain and the range of $f^{-1}$.

*[12 marks]*

(a) and (d)




(b) Domain $x \in \mathbb{R}$ Range: $y > 0$

(c) $y = e^{x+2} \Leftrightarrow \ln y = x + 2 \Leftrightarrow x = \ln y - 2$

Hence $f^{-1}(x) = \ln x - 2$

(e) Domain $x > 0$, Range $y \in \mathbb{R}$