MATH HL

TEST

EXPONENTS – LOGARITHMS

(without GDC)

by Christos Nikolaidis

Date: 12 – 12 – 2017

SOLUTIONS

1. *[Maximum mark: 7]*

Let $\log_2 3 = a$, $\log_2 5 = b$. Express the following in terms of a and b :2

(a) $\log_2 60$ (b) $\log_4 0.2$ (c) $\log_3 25$ *[2+3+2 marks]*

Solution

(a) $log_2 60 = log_2(2^2 \cdot 3 \cdot 5) = 2log_2 2 + log_2 3 + log_2 5 = a + b + 2$

(b) $log_4 0.2 = \frac{log_2 0.2}{log_2 4} = \frac{log_2 0.2}{2} = \frac{1}{2}log_2\frac{1}{5} = -\frac{1}{2}log_2 5 = -\frac{b}{2}$

(c) $log_3 25 = \frac{log_2 25}{log_2 3} = \frac{2b}{a}$

2. *[Maximum mark: 5]*

Solve the exponential equation $2^{x+5} = \frac{3\cdot(5^x)}{2^{x-2}}$.

Give your answer in the form $x = \frac{\ln a}{\ln b}$, where $a, b \in Q$.

Solution

$$2^{x+5} = \frac{3\cdot(5^x)}{2^{x-2}} \Leftrightarrow 2^x \cdot 32 = \frac{3\cdot(5^x)\cdot 4}{2^x} \Leftrightarrow 2^x \cdot 8 = \frac{3\cdot(5^x)}{2^x}$$

$$\Leftrightarrow \frac{2^x\cdot 2^x}{5^x} = \frac{3}{8} \Leftrightarrow \left(\frac{4}{5}\right)^x = \frac{3}{8}$$

$$\Leftrightarrow ln\left(\frac{4}{5}\right)^x = ln\frac{3}{8} \Leftrightarrow x = \frac{ln\frac{3}{8}}{ln\frac{4}{5}}$$

3. *[Maximum mark: 6]*

Solve the exponential equation

$$2^{x+5} = 8 + \frac{3}{2^{x-2}}.$$

Give your answer in the form $x = a + \log_2 b$, where $a, b \in Z$.

Solution

$$2^x 2^5 = 8 + \frac{3 \cdot 4}{2^x} \Leftrightarrow 32(2^x) = 8 + \frac{12}{2^x}$$

Let $y = 2^x$. The equation becomes

$$32y = 8 + \frac{12}{y} \quad \Leftrightarrow 32y^2 = 8y + 12$$

$$\Leftrightarrow 32y^2 - 8y - 12 = 0 \quad \Leftrightarrow 8y^2 - 2y - 3 = 0$$

$\Delta = 4 + 96 = 100$ $\qquad y = \dfrac{2 \pm 10}{16} = \dfrac{3}{4}$ or $-\dfrac{1}{2}$ (rejected)

Therefore,

$$2^x = \frac{3}{4} \Leftrightarrow x = \log_2 \frac{3}{4} \Leftrightarrow x = \log_2 3 - \log_2 4 \Leftrightarrow x = -2 + \log_2 3$$

4. *[Maximum mark: 8]*

Find the values of

(a) $A = (\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{15} 16)$, as an integer *[4 marks]*

(b) $B = \log_2 3 + \log_2 3^2 + \log_2 3^3 + \cdots + \log_2 3^{20}$ in the form $a \log_2 b$ *[4 marks]*

Solution

(a) $A = (\log_2 3)\left(\dfrac{\log_2 4}{\log_2 3}\right)\left(\dfrac{\log_2 5}{\log_2 4}\right)\cdots\left(\dfrac{\log_2 16}{\log_2 15}\right) = \log_2 16 = 4$

(b) $B = \log_2 3 + 2\log_2 3 + 3\log_2 3 + \cdots + 20\log_2 3$

Arithmetic series with $u_1 = \log_2 3$ and $d = \log_2 3$

$$S_{20} = 10(\log_2 3 + 20\log_2 3) = 210\log_2 3$$

5. *[Maximum mark: 7]*
Solve the logarithmic equation

$$\log_{\sqrt{5}}(4-\frac{x}{5})-4\log_5(\frac{x}{5})=2$$

Solution

$$\frac{\log_5(4-\frac{x}{5})}{\log_5\sqrt{5}}-4\log_5(\frac{x}{5})=2 \Leftrightarrow \frac{\log_5\left(\frac{20-x}{5}\right)}{1/2}-4\log_5(\frac{x}{5})=2$$

$$\Leftrightarrow \log_5\left(\frac{20-x}{5}\right)-2\log_5\left(\frac{x}{5}\right)=1$$

$$\Leftrightarrow \log_5(20-x)-\log_5 5-2\log_5 x+2\log_5 5=1$$

$$\Leftrightarrow \log_5(20-x)-2\log_5 x=0$$

$$\Leftrightarrow \log_5(20-x)=\log_5 x^2$$

$$\Leftrightarrow 20-x=x^2$$

$$\Leftrightarrow x^2+x-20=0$$

$$\Leftrightarrow x=4 \quad or \quad x=-5\,(rejected)$$

6. *[Maximum mark: 6]*
Solve the logarithmic equation

$$\log_x 2-3\log_2 x=2$$

Solution

$$\frac{\log_2 2}{\log_2 x}-3\log_2 x=2 \Leftrightarrow \frac{1}{\log_2 x}-3\log_2 x=2 \qquad Let \quad y=\log_2 x$$

$$\frac{1}{y}-3y=2 \Leftrightarrow 1-3y^2=2y \Leftrightarrow 3y^2+2y-1=0$$

$$\Delta=4+12=16 \qquad y=\frac{-2\pm4}{6} \Leftrightarrow y=-1 \text{ or } y=1/3$$

Therefore, $\log_2 x=-1 \Leftrightarrow x=2^{-1}=\frac{1}{2}$

$\log_2 x=1/3 \Leftrightarrow x=2^{1/3}=\sqrt[3]{2}$

7. *[Maximum mark: 6]*
Solve the equation

$$\ln(e^{4x} - 8) = 2x + \ln 7$$

Solution

$$\ln(e^{4x} - 8) = 2x + \ln 7 \iff e^{4x} - 8 = e^{2x + \ln 7}$$

$$\iff e^{4x} - 8 = e^{2x} e^{\ln 7} \qquad \iff e^{4x} - 8 = 7e^{2x}$$

Let $y = e^{2x}$. $\quad y^2 - 7y - 8 = 0 \iff y = 8$ or $y = -1$ (rejected)

Therefore, $e^{2x} = 8 \iff 2x = \ln 8 \iff x = \dfrac{\ln 8}{2}$

8. *[Maximum mark: 7]*

It is given that $\log_a(x^2 y) = p$ and $\log_a(\frac{x}{y^2}) = q$

(a) Find $\log_a x$ and $\log_a y$ in terms of p and q. *[5 marks]*

(b) **Hence** find $\log_x y$ in terms of p and q. *[2 marks]*

Solution

(a) $\log_a(x^2 y) = p \iff 2\log_a x + \log_a y = p \quad (1)$

$\log_a(\frac{x}{y^2}) = q \iff \log_a x - 2\log_a y = q \quad (2)$

We solve the system:

$2\times(1) + (2)$: $\quad 5\log_a x = 2p + q \iff \log_a x = \dfrac{2p+q}{5}$

$(1) - 2\times(2)$: $\quad 5\log_a y = p - 2q \iff \log_a y = \dfrac{p-2q}{5}$

(b) $\log_x y = \dfrac{\log_a y}{\log_a x} = \dfrac{2p+q}{p-2q}$

9. *[Maximum mark: 6]*

The mass m kg of a radio-active substance at time t hours is given by $m = 8e^{-kt}$.
The half life time of the substance is 3 hours.

(a) Find the initial mass of the substance. *[1 mark]*

(b) Show that $k = \dfrac{\ln 2}{3}$ *[3 marks]*

(c) Given that the mass of the substance after t hours is 1kg, find the value of t. *[2 marks]*

Solution

(a) $m = 8$

(b) $4 = 8e^{-3k} \Leftrightarrow e^{-3k} = \dfrac{1}{2} \Leftrightarrow e^{3k} = 2 \Leftrightarrow 3k = \ln 2 \Leftrightarrow k = \dfrac{\ln 2}{3}$

(c) $1 = 8e^{-\frac{\ln 2}{3}t} \Leftrightarrow e^{\frac{\ln 2}{3}t} = 8 \Leftrightarrow \dfrac{\ln 2}{3}t = \ln 8 \Leftrightarrow \dfrac{\ln 2}{3}t = 3\ln 2 \Leftrightarrow t = 9$

(or directly, since the half life time is 3 hours,

the initial mass of 8kg becomes 1kg in 9 hours)

10. *[Maximum mark: 6]*

Solve the equation

$$8(3^x) + 5(2^x) - 6^x = 40$$

Solution

$8(3^x) + 5(2^x) - 6^x = 40 \Leftrightarrow 8(3^x) + 5(2^x) - (3^x)(2^x) - 40 = 0$

$\Leftrightarrow 3^x(8 - 2^x) - 5(8 - 2^x) = 0 \Leftrightarrow -3^x(2^x - 8) + 5(2^x - 8) = 0$

$\Leftrightarrow (2^x - 8)(5 - 3^x) = 0 \quad \Leftrightarrow 2^x = 8 \quad or \quad 3^x = 5$

$\Leftrightarrow x = 3 \quad or \quad x = \log_3 5$

11. *[Maximum mark: 16]*

Let $f(x) = 2e^{5x} - 3$

(a) Write down the domain and the range of *f*. *[2 marks]*

(b) Find f^{-1} *[3 marks]*

(c) Sketch the graph of *f*. Indicate intercepts and asymptotes. *[3 marks]*

(d) Sketch the graph of f^{-1}. Indicate intercepts and asymptotes. *[2 marks]*

(e) Sketch the graph of $\frac{1}{f}$. Indicate intercepts and asymptotes. *[3 marks]*

(f) Sketch the graph of $\frac{1}{f^{-1}}$. Indicate intercepts and asymptotes. *[3 marks]*

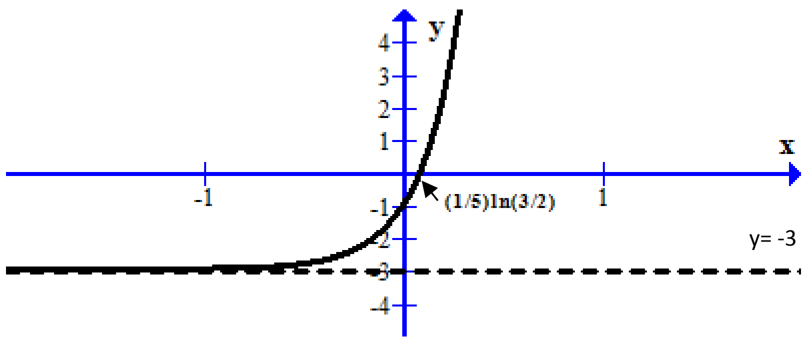
Solution

(a) Domain: $x \in R$ Range: $y > -3$

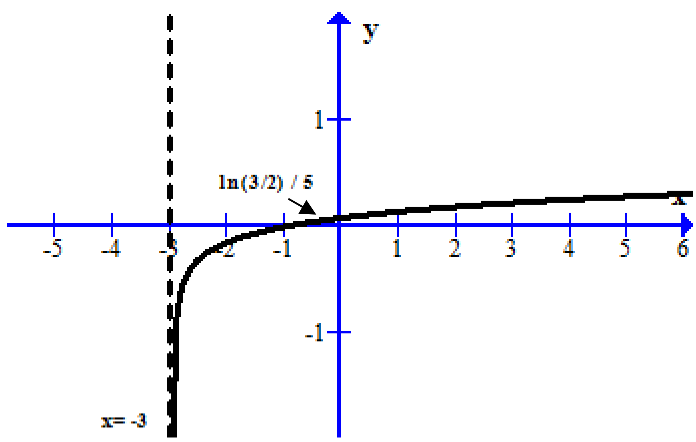
(b) $2e^{5x} - 3 = y \Leftrightarrow 2e^{5x} = y + 3 \Leftrightarrow e^{5x} = \frac{y+3}{2} \Leftrightarrow 5x = \ln\left(\frac{y+3}{5}\right)$

$\Leftrightarrow x = \frac{1}{5}\ln\left(\frac{y+3}{2}\right)$ Hence $f^{-1}(x) = \frac{1}{5}\ln\left(\frac{x+3}{2}\right)$

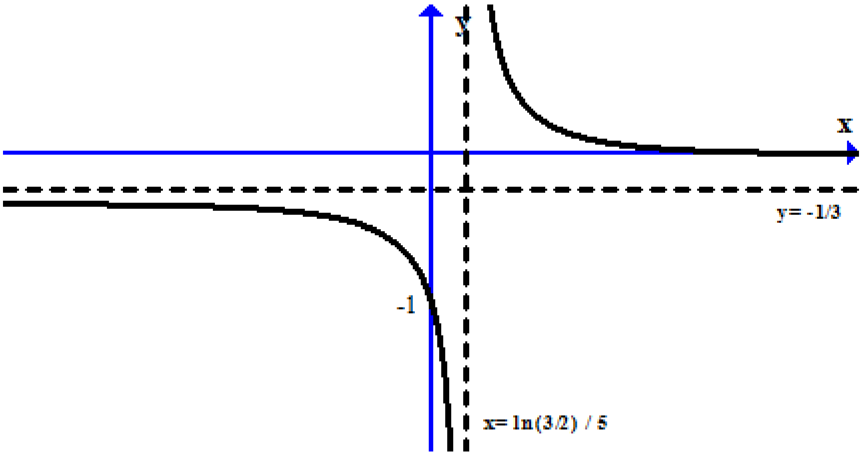
(c)



(d)



(e)



(f)