MAA HL

Test on Exponents and Logarithms (2)

(without GDC)

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Marks: /40

Name of student: SOLUTIONS

1. [Maximum mark: 6]

Let $\log_2 3 = a$ and $\log_2 7 = b$. Express the following in terms of a and b.

(a) $\log_2 63$ [2]

(b) $\log_2 42$ [2]

(c) $\log_7 \sqrt{3}$ [2]

(a) $\log_2 63 = \log_2 7\cdot 3^2 = \log_2 7 + 2\log_2 3$

$= \boxed{b+2a}$

(b) $\log_2 42 = \log_2 2\cdot 3\cdot 7 = \log_2 2 + \log_2 3 + \log_2 7$

$= \boxed{a+b+1}$

(c) $\log_7 \sqrt{3} = \dfrac{\log_2 \sqrt{3}}{\log_2 7} = \dfrac{\frac{1}{2}\log_2 3}{\log_2 7} = \boxed{\dfrac{a}{2b}}$

Turn over

2. [Maximum mark: 6]

Solve the logarithmic equation

$$2\log_3 x + \log_{\frac{1}{3}}(x-2) = 2$$

$$2\log_3 x + \frac{\log_3(x-2)}{\log_3 1/3} = 2$$

$$\Leftrightarrow \quad 2\log_3 x - \log_3(x-2) = 2$$

$$\Leftrightarrow \quad \log_3 \frac{x^2}{x-2} = 2$$

$$\Leftrightarrow \quad \frac{x^2}{x-2} = 3^2$$

$$\Leftrightarrow \quad x^2 = 9x - 18$$

$$\Leftrightarrow \quad x^2 - 9x + 18 = 0$$

$$\Leftrightarrow \quad (x-3)(x-6) = 0$$

$$\Leftrightarrow \quad \boxed{x=3} \text{ or } \boxed{x=6}$$

3. [Maximum mark: 5]

Solve the system of equations

$$\ln(x-5+e^2)=2$$

$$2^{2y+1}+1=x$$

$\ln(x-5+e^2)=2 \Leftrightarrow x-5+e^2=e^2$

$\Leftrightarrow \boxed{x=5}$

$2^{2y+1}+1=5 \Leftrightarrow 2^{2y+1}=4$

$\Leftrightarrow 2^{2y+1}=2^2$

$\Leftrightarrow 2y+1=2$

$\Leftrightarrow \boxed{y=\frac{1}{2}}$

Solution $(x,y)=(5,\frac{1}{2})$

Turn over

4. [Maximum mark: 4]

Show that

$$\log_a b \cdot \log_b c \cdot \log_c a = 1$$

$\log_a b \cdot \log_b c \cdot \log_c a$

$= \frac{\ln b}{\ln a} \cdot \frac{\ln c}{\ln b} \cdot \frac{\ln a}{\ln c} = 1$

5. [Maximum mark: 6]

Solve the equation

$$\log_2(\log_2 x) = \log_4(\log_4 x)$$

$$\log_2(\log_2 x) = \frac{\log_2(\log_4 x)}{\log_2 4}$$

$$\Leftrightarrow \log_2(\log_2 x) = \frac{\log_2(\log_4 x)}{2}$$

$$\Leftrightarrow \log_2(\log_2 x)^2 = \log_2(\log_4 x)$$

$$\Leftrightarrow (\log_2 x)^2 = \log_4 x$$

$$\Leftrightarrow (\log_2 x)^2 = \frac{\log_2 x}{\log_2 4}$$

$$\Leftrightarrow (\log_2 x)^2 = \frac{\log_2 x}{2}$$

$$\Leftrightarrow 2(\log_2 x)^2 - \log_2 x = 0$$

$$\Leftrightarrow \log_2 x\,(2\log_2 x - 1) = 0$$

$$\Leftrightarrow \log_2 x = 0 \quad \text{or} \quad \log_2 x = \frac{1}{2}$$

$$\Leftrightarrow x = 2^0 \quad \text{or} \quad x = 2^{1/2}$$

$$\Leftrightarrow x = 1 \quad \text{or} \quad \boxed{x = \sqrt{2}}$$

rejected

Turn over

6. [Maximum mark: 6]

(a) Show that $6xy+2y-15x-5 \equiv (3x+1)(2y-5)$ [1]

(b) **Hence** solve the equation $6xe^{x}+2e^{x}-15x=5$ [5]

(a) $(3x+1)\cdot(2y-5) = 3x\cdot 2y - 3x\cdot 5 + 2y - 5$

$= 6xy - 15x + 2y - 5$

$= 6xy + 2y - 15x - 5$

(b) Let $e^x = y$. Then

$6xy + 2y - 15x = 5$

$\Leftrightarrow (3x+1)(2y-5) = 0$

$\Leftrightarrow \boxed{x = -\frac{1}{3}}$ or $y = \frac{5}{2}$

$e^x = \frac{5}{2}$

$\boxed{x = \ln\frac{5}{2}}$

7. [Maximum mark: 7]

The cubic function $f(t) = t^3 - 10t^2 + 7t + 18$ is divisible by $(t^2 - t - 2)$.

(a) Find the three real roots of $f(t)$. [4]

(b) Solve the logarithmic equation

$$(\ln x)^3 + 7\ln x + 18 = 10(\ln x)^2$$ [3]

(a) $t^2 - t - 2 = 0$ $\Delta = 1 + 8 = 9$

$t = \frac{1 \pm 3}{2} \Rightarrow t = 2$ or $t = -1$

Sum of roots $S = \frac{10}{1} \Leftrightarrow S = 10$

Hence $2 + (-1) + x_3 = 10 \Rightarrow x_3 = 9$

The three roots are $\boxed{-1, 2, 9}$

(b) Let $t = \ln x$ Then

$t^3 + 7t + 18 = 10t^2 \Leftrightarrow t^3 - 10t^2 + 7t + 18 = 0$

From (a) $t = -1$ or 2 or 9

$\ln x = -1 \Rightarrow x = e^{-1} \Rightarrow \boxed{x = \frac{1}{e}}$

$\ln x = 2 \Rightarrow \boxed{x = e^2}$

$\ln x = 9 \Rightarrow \boxed{x = e^9}$