Mixed practice 2

1. Solve the equation $3 \times 9^x - 10 \times 3^x + 3 = 0$.

2. Find the exact solution of the equation $2^{3x+1} = 5^{5-x}$.

3. Solve the simultaneous equations

$\ln x^2 + \ln y = 15$

$\ln x + \ln y^3 = 10$

4. Given that $y = \ln x - \ln(x+2) + \ln(x^2 - 4)$, express x in terms of y.

5. The graph with equation $y = 4\ln(x - a)$ passes through the point $(5, \ln 16)$. Find the value of a.

6. (a) An economic model predicts that the demand, D, for a new product will grow according to the equation $D = A - Ce^{-0.2t}$, where t is the number of days since the product launch. After 10 days the demand is 15 000 and it is increasing at a rate of 325 per day.

   (i) Find the value of C.

   (ii) Find the initial demand for the product.

   (iii) Find the long-term demand predicted by this model.

   (b) An alternative model is proposed, in which the demand grows according to the formula $D = B\ln\left(\frac{t+10}{5}\right)$. The initial demand is the same as that for the first model.

   (i) Find the value of B.

   (ii) What is the long-term prediction of this model?

   (c) After how many days will the demand predicted by the second model become larger than the demand predicted by the first model?

Going for the top 2

1. Find the exact solution of the equation $2^{3x-4} \times 3^{2x-5} = 36^{x-2}$, giving your answer in the form $\frac{\ln p}{\ln q}$ where p and q are integers.

2. Given that $\log_a b^2 = c^2$ and $\log_b a = c + 1$, express a in terms of b.

3. In a physics experiment, Maya measured how the force, F, exerted by a spring depends on its extension, x. She then plotted the values of $a = \ln F$ and $b = \ln x$ on a graph, with b on the horizontal axis and a on the vertical axis. The graph was a straight line, passing through the points (2, 4.5) and (4, 7.2). Find an expression for F in terms of x.

Mixed practice 5

1. The fourth term of an arithmetic sequence is 17. The sum of the first twenty terms is 990. Find the first term, a, and the common difference, d, of the sequence.

2. The fourth, tenth and thirteenth terms of a geometric sequence form an arithmetic sequence. Given that the geometric sequence has a sum to infinity, find its common ratio correct to three significant figures.

3. Evaluate $\sum_{r=1}^{12} 4r + \left(\frac{1}{3}\right)^r$ correct to four significant figures.

4. Find an expression for the sum of the first 20 terms of the series

   $\ln x + \ln x^4 + \ln x^7 + \ln x^{10} + \ldots$

   giving your answer as a single logarithm.

5. A rope of length 300 m is cut into several pieces, whose lengths form an arithmetic sequence with common difference d. If the shortest piece is 1 m long and the longest piece is 19 m, find d.

6. Aaron and Blake each open a savings account. Aaron deposits \$100 in the first month and then increases his deposits by \$10 each month. Blake deposits \$50 in the first month and then increases his deposits by 5% each month. After how many months will Blake have more money in his account than Aaron?

Going for the top 5

1. (a) (i) Prove that the sum of the first n terms of a geometric sequence with first term a and common ratio r is given by:

   $$S_n = \frac{a(r^n - 1)}{r - 1}$$

   (ii) Hence establish the formula for the sum to infinity, clearly justifying any conditions imposed on the common ratio r.

   (b) Show that in a geometric sequence with common ratio r, the ratio of the sum of the first n terms to the sum of the next n terms is $1 : r^n$.

   (c) In a geometric sequence, the sum of the seventh term and four times the fifth term equals the eighth term.

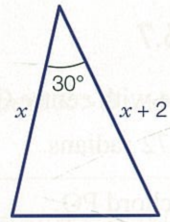
   (i) Find the ratio of the sum of the first 10 terms to the sum of the next 10 terms.

   (ii) Does the sequence have a sum to infinity? Explain your answer.

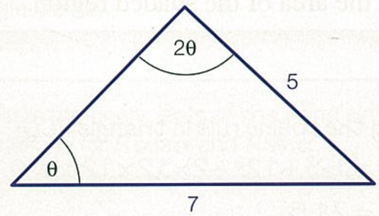
2. Find the sum of all integers between 1 and 1000 which are not divisible by 7.

Mixed practice 6

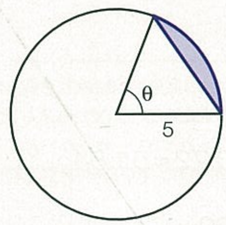
1. The area of the triangle shown in the diagram is $12\,\text{cm}^2$. Find the value of x.



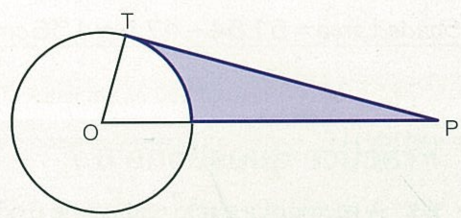
2. The triangle shown in the diagram has angles θ and 2θ. Find the value of θ in degrees.



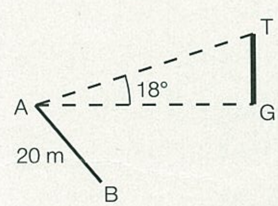
3. The area of the shaded region is $6.2\,\text{cm}^2$.
   (a) Show that $\theta - \sin\theta = 0.496$.
   (b) Find the value of θ in radians.



4. In the diagram, O is the centre of the circle and PT is the tangent to the circle at T. The radius of the circle is 7 cm and the distance PT is 12 cm.
   (a) Find the area of triangle OPT.
   (b) Find the size of $P\hat{O}T$.
   (c) Find the area of the shaded region.



5. Two observers, positioned on horizontal ground at A and B, are trying to measure the height of a vertical tree, GT. The distance AB is 20 m, $G\hat{A}B = 65°$ and $G\hat{B}A = 80°$. From A, the angle of elevation of the top of the tree is 18°.
   (a) Find the distance of A from the bottom of the tree.
   (b) Find the height of the tree.



6. (a) Given that $\cot x + \tan x = 4$ and $x \in \left[0, \frac{\pi}{4}\right]$, find the exact value of $\tan x$.
   (b) (i) Show that $\cot x + \tan x = 2\csc 2x$.
   (ii) Hence solve the equation $\cot x + \tan x = 4$ for $x \in [0, \pi]$.
   (c) Find the exact value of $\tan\left(\frac{\pi}{12}\right)$.

7. Solve the equation $3\sin 2\theta = \tan 2\theta$ for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

8. (a) The function f is defined by $f(x) = 3x^2 - 2x + 5$ for $-1 \le x \le 1$. Find the coordinates of the vertex of the graph of $y = f(x)$.

(b) The function g is defined by $g(\theta) = 3\cos 2\theta - 4\cos\theta + 13$ for $0 \le \theta \le 2\pi$.

(i) Show that $g(\theta) = 6\cos^2\theta - 4\cos\theta + 10$.

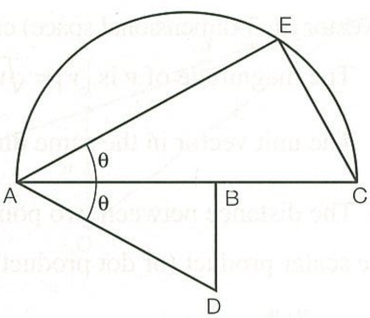
(ii) Hence find the minimum value of $g(\theta)$.

9. In the diagram, AC is the diameter of the semicircle, BD is perpendicular to AC, AB = 2 and BC = 1, and $B\hat{A}D = C\hat{A}E = \theta$. Let $R = AD - CE$.

(a) Find an expression for R in terms of θ.

(b) Show that R has a stationary value when $2\sin\theta = 3\cos^3\theta$.

(c) Assuming that this stationary value is a minimum, find the smallest possible value of AD – CE.



Going for the top 6

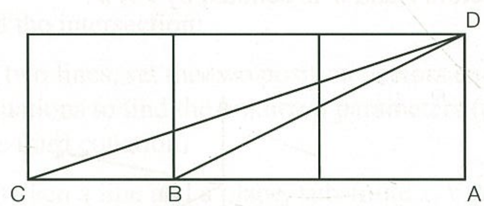
1. Let $y = \cos x$.

(a) Write $\cos 2x$ in terms of y.

(b) Show that $\arccos(2y^2 - 1) = 2\arccos y$.

(c) Evaluate $\arcsin y + \arccos y$.

2. The figure shown consists of three squares. Let $A\hat{D}B = \alpha$ and $A\hat{D}C = \beta$. Use a compound angle formula to find the sum $\alpha + \beta$.



3. (a) Show that $\cos[(A-B)x] - \cos[(A+B)x] = 2\sin Ax \sin Bx$.

(b) Hence show that $\sum_{r=1}^{n} \sin x \sin(2r-1)x = \frac{1-\cos 2nx}{2}$.

(c) Solve the equation $\sin^2 x + \sin x \sin 3x = \frac{1}{4}$ if $0 < x < \pi$.

(d) Evaluate $\sin\frac{\pi}{5} + \sin\frac{3\pi}{5} + \sin\frac{5\pi}{5} + \sin\frac{7\pi}{5} + \sin\frac{9\pi}{5}$.

2 EXPONENTS AND LOGARITHMS

Mixed practice 2

1. $$3\times 9^x - 10\times 3^x + 3 = 0$$
$$\Leftrightarrow 3\times (3^2)^x - 10\times 3^x + 3 = 0$$
$$\Leftrightarrow 3\times (3^x)^2 - 10\times 3^x + 3 = 0$$

Let $3^x = y$. Then

$$3y^2 - 10y + 3 = 0$$
$$\Leftrightarrow (3y-1)(y-3) = 0$$
$$\Leftrightarrow y = \frac{1}{3} \text{ or } y = 3$$

So $3^x = \frac{1}{3} \Rightarrow x = -1$ or $3^x = 3 \Rightarrow x = 1$

2. $$2^{3x+1} = 5^{5-x}$$
$$\Rightarrow \ln(2^{3x+1}) = \ln(5^{5-x})$$
$$\Rightarrow (3x+1)\ln 2 = (5-x)\ln 5$$
$$\Rightarrow (3\ln 2)x + \ln 2 = 5\ln 5 - x\ln 5$$
$$\Rightarrow (3\ln 2)x + x\ln 5 = 5\ln 5 - \ln 2$$
$$\Rightarrow x(3\ln 2 + \ln 5) = 5\ln 5 - \ln 2$$
$$\Rightarrow x = \frac{5\ln 5 - \ln 2}{3\ln 2 + \ln 5}$$

3. From the first equation:

$$\ln(x^2 y) = 15$$
$$\Rightarrow x^2 y = e^{15}$$
$$\Rightarrow y = \frac{e^{15}}{x^2}$$

Substituting this into the second equation gives

$$\ln x + \ln\left(\frac{e^{15}}{x^2}\right)^3 = 10$$
$$\Rightarrow \ln\left(x\times\frac{e^{45}}{x^6}\right) = 10$$
$$\Rightarrow \frac{e^{45}}{x^5} = e^{10}$$
$$\Rightarrow x^5 = e^{35}$$
$$\Rightarrow x = e^7$$

Then, substituting back to find y:

$$y = \frac{e^{15}}{x^2} = \frac{e^{15}}{e^{14}} = e$$

So the solution is $x = e^7$, $y = e$.

4. $$y = \ln x - \ln(x+2) + \ln(x^2-4)$$
$$= \ln\left(\frac{x}{x+2}\right) + \ln(x^2-4)$$
$$= \ln\left(\frac{x(x^2-4)}{x+2}\right)$$
$$= \ln\left(\frac{x(x-2)(x+2)}{x+2}\right)$$
$$= \ln(x(x-2))$$

$$\therefore\ x(x-2) = e^y$$
$$\Rightarrow x^2 - 2x - e^y = 0$$
$$\Rightarrow x = \frac{2\pm\sqrt{4+4e^y}}{2} = 1\pm\sqrt{1+e^y}$$

So $x = 1+\sqrt{1+e^y}$ (as $x > 0$)

5. Substituting $x = 5$ and $y = \ln 16$ into the equation:

$$\ln 16 = 4\ln(5-a)$$
$$\Rightarrow \ln 16 = \ln(5-a)^4$$
$$\Rightarrow 16 = (5-a)^4$$
$$\Rightarrow 5-a = \pm 2$$
$$\Rightarrow a = 3 \text{ or } 7$$

Putting each of these values into the original equation and checking that $(5, \ln 16)$ is a solution shows that $a = 7$ is not valid, because in that case we would get $\ln(5-7) = \ln(-2)$, which is not real.

So $a = 3$.

6. (a) (i) After 10 days the rate of increase is 325 per day, so $\frac{dD}{dt} = 325$ when $t = 10$:

$$\frac{dD}{dt} = 0.2Ce^{-0.2t}$$
$$325 = 0.2Ce^{-2}$$
$$\therefore C = 1625e^2$$

(ii) After 10 days the demand is 15 000, so we have $D = 15\,000$ when $t = 10$:

$$15\,000 = A - Ce^{-2}$$
$$15\,000 = A - (1625e^2)e^{-2}$$
$$\therefore A = 16\,625$$

The initial demand, D_0, is the value of D when $t = 0$:

$D_0 = A - C = 16\,625 - 1625e^2 = 4618$
(to the nearest integer)

(iii) As $t \to \infty$, $e^{-0.2t} \to 0$. Therefore $D \to A = 16\,625$

(b) (i) We are given that $D = 16625 - 1625e^2$ when $t = 0$, so

$$16625 - 1625e^2 = B\ln\left(\frac{0+10}{5}\right)$$

$$\therefore B = \frac{16625 - 1625e^2}{\ln 2} = 6662 \text{ (4 SF)}$$

(ii) As $t \to \infty$, $\ln\left(\frac{t+10}{5}\right) \to \infty$. Therefore $D \to \infty$.

(c) We need to find the first t for which

$B\ln\left(\frac{t+10}{5}\right) > A - Ce^{-0.2t}$; that is, we need to solve

$B\ln\left(\frac{t+10}{5}\right) = A - Ce^{-0.2t}$. From GDC, $t = 50.6$,

i.e. after 51 days.

Going for the top 2

1.

$$2^{3x-4} \times 3^{2x-5} = 36^{x-2}$$
$$\Rightarrow \ln\left(2^{3x-4} \times 3^{2x-5}\right) = \ln\left(36^{x-2}\right)$$
$$\Rightarrow \ln\left(2^{3x-4}\right) + \ln\left(3^{2x-5}\right) = \ln\left(36^{x-2}\right)$$
$$\Rightarrow (3x-4)\ln 2 + (2x-5)\ln 3 = (x-2)\ln 36$$
$$\Rightarrow (3\ln 2)x - 4\ln 2 + (2\ln 3)x - 5\ln 3 = x\ln 36 - 2\ln 36$$
$$\Rightarrow (3\ln 2 + 2\ln 3 - \ln 36)x = 4\ln 2 + 5\ln 3 - 2\ln 36$$
$$\Rightarrow (\ln 8 + \ln 9 - \ln 36)x = \ln 2^4 + \ln 3^5 - \ln 36^2$$
$$\Rightarrow x\ln\left(\frac{8\times 9}{36}\right) = \ln\left(\frac{2^4 \times 3^5}{36^2}\right)$$
$$\Rightarrow x\ln 2 = \ln\left(\frac{6^4 \times 3}{6^4}\right)$$
$$\Rightarrow x = \frac{\ln 3}{\ln 2}$$

2. Changing base a into base b, we have

$$\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}, \text{ so}$$

$$\log_a b^2 = c^2$$
$$\Leftrightarrow 2\log_a b = c^2$$
$$\Leftrightarrow 2\left(\frac{1}{\log_b a}\right) = c^2$$

Then, substituting $\log_b a = c + 1$ from the other equation gives

$$2\left(\frac{1}{c+1}\right) = c^2$$
$$\Leftrightarrow \frac{2}{c+1} = c^2$$
$$\Leftrightarrow 2 = c^3 + c^2$$
$$\Leftrightarrow c^3 + c^2 - 2 = 0$$
$$\Leftrightarrow (c-1)(c^2 + 2c + 2) = 0$$
$$\therefore c = 1$$

Therefore, $\log_b a = 1 + 1 = 2$ and hence $a = b^2$.

3. Gradient of the line through (2, 4.5) and (4, 7.2) is

$$m = \frac{7.2 - 4.5}{4 - 2} = 1.35$$

So

$$\ln F - 4.5 = 1.35(\ln x - 2)$$
$$\ln F = 1.35\ln x + 1.8$$
$$= \ln x^{1.35} + 1.8$$

and therefore $F = e^{\ln x^{1.35} + 1.8} = e^{1.8}x^{1.35}$.

3 POLYNOMIALS

Mixed practice 3

1. The function has a repeated root at $x = -2$, so it has a factor $(x + 2)^2$; it has another repeated root at $x = 3$, and hence also a factor $(x - 3)^2$.

This would give a y-intercept of $2^2 \times (-3)^2 = 36$, so a factor of 2 is also needed. Therefore

$$y = 2(x+2)^2(x-3)^2$$
$$= 2(x^2 + 4x + 4)(x^2 - 6x + 9)$$
$$= 2x^4 - 4x^3 - 22x^2 + 24x + 72$$

2. The general term of this binomial expansion is

$$\binom{8}{r}(x^3)^{8-r}\left(\frac{3}{x}\right)^r = \binom{8}{r}x^{24-3r}x^{-r}3^r = \binom{8}{r}x^{24-4r}3^r$$

The term independent of x will have power 0 for x; that is, $24 - 4r = 0$, so $r = 6$. The term is

$$\binom{8}{6}(x^3)^2\left(\frac{3}{x}\right)^6 = 28x^6\left(\frac{729}{x^6}\right) = 20412$$

3. Because it has a factor of $(x + 1)$:

$$(-1)^3 + 10(-1)^2 + c(-1) + d = 0$$
$$\Leftrightarrow -1 + 10 - c + d = 0$$
$$\Leftrightarrow c - d = 9$$

As it has a remainder of 5 when divided by $(x - 2)$:

$$2^3 + 10(2)^2 + c(2) + d = 5$$
$$\Leftrightarrow 8 + 40 + 2c + d = 5$$
$$\Leftrightarrow 2c + d = -43$$

Solving these equations simultaneously gives $c = -\frac{34}{3}$ and $d = -\frac{61}{3}$.

4. $y = x^2 + kx + 2$ never touches the x-axis and so has no real roots. Therefore

$b^2 - 4ac < 0$

$k^2 - 4 \times 1 \times 2 < 0$

$k^2 < 8$

$-2\sqrt{2} < k < 2\sqrt{2}$

5. Let the 3 roots be α, β and γ.

Then $\alpha + \beta + \gamma = \frac{-b}{1} = -b$.

Since α, β, γ form an arithmetic sequence:

$$\beta - \alpha = \gamma - \beta$$
$$\Leftrightarrow 2\beta = \alpha + \gamma$$
$$\Leftrightarrow 3\beta = \alpha + \beta + \gamma$$
$$\therefore \beta = \frac{\alpha + \beta + \gamma}{3} = \frac{-b}{3}$$

6. Using the binomial theorem:

$$(2+x)(3-2x)^5 = (2+x)\left(3^5 + \binom{5}{1}3^4(-2x) + \binom{5}{2}3^3(-2x)^2 + \ldots\right)$$
$$= (2+x)(243 - 810x + 1080x^2 + \ldots)$$

The quadratic term will be

$2 \times 1080x^2 + x \times (-810x) = 2160x^2 - 810x^2 = 1350x^2$

7. Solving simultaneously for the intersection point:

$$x^2 + (x+k)^2 = 9$$
$$\Leftrightarrow x^2 + x^2 + 2kx + k^2 = 9$$
$$\Leftrightarrow 2x^2 + 2kx + k^2 - 9 = 0$$

As $y = x + k$ is a tangent, there is only one solution to this quadratic, so the discriminant is 0:

$b^2 - 4ac = 0$

$(2k)^2 - 4 \times 2(k^2 - 9) = 0$

$4k^2 - 8k^2 + 72 = 0$

$4k^2 = 72$

$k^2 = 18$

$k = \pm 3\sqrt{2}$

8. (a) Using the binomial theorem:

$$(2-x)^5 = 2^5 + \binom{5}{1}2^4(-x) + \binom{5}{2}2^3(-x)^2 + \binom{5}{3}2^2(-x)^3 + \ldots$$
$$= 32 - 80x + 80x^2 - 40x^3 + \ldots$$

(b) Let $x = 0.01$ so that $(2-x)^5 = 1.99^5$. Then

$1.99^5 \approx 32 - 80 \times 0.01 + 80 \times 0.01^2 - 40 \times 0.01^3 = 31.20796$

9. From the original equation, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

The sum of the roots of the new equation is

$$(\alpha + 2\beta) + (2\alpha + \beta) = 3\alpha + 3\beta$$
$$= 3(\alpha + \beta)$$
$$= -\frac{3b}{a}$$

The product of the roots of the new equation is

$$(\alpha + 2\beta)(2\alpha + \beta) = 2\alpha^2 + 5\alpha\beta + 2\beta^2$$
$$= 2(\alpha + \beta)^2 + \alpha\beta$$
$$= 2\left(\frac{-b}{a}\right)^2 + \frac{c}{a}$$
$$= \frac{2b^2}{a^2} + \frac{c}{a}$$
$$= \frac{2b^2 + ac}{a^2}$$

Therefore, taking the coefficient of x^2 to be 1 in the new quadratic, a suitable equation is

$$x^2 + \frac{3b}{a}x + \frac{2b^2 + ac}{a^2} = 0$$

Or, multiplying through by a^2:

$$a^2x^2 + 3abx + 2b^2 + ac = 0$$

10. (a) Given that the polynomial has repeated roots α and β:

$$x^4 + bx^3 + cx^2 + dx + e = (x - \alpha)^2(x - \beta)^2$$
$$\therefore e = (-\alpha)^2(-\beta)^2 = (\alpha\beta)^2 \geq 0$$

(b) Sum of the roots: $\alpha + \alpha + \beta + \beta = \frac{-b}{1}$

$$\Rightarrow 2(\alpha + \beta) = -b$$

Product of the roots: $\alpha\alpha\beta\beta = \frac{e}{1} \Rightarrow (\alpha\beta)^2 = e$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$
$$= \left(\frac{-b}{2}\right)^2 - 2\sqrt{e}$$
$$= \frac{b^2}{4} - 2\sqrt{e}$$

(c) $$(\beta - \alpha)^2 = \beta^2 + \alpha^2 - 2\alpha\beta$$
$$= \frac{b^2}{4} - 2\sqrt{e} - 2\sqrt{ew}$$
$$= \frac{b^2}{4} - 4\sqrt{e}$$

$$\therefore \beta - \alpha = \sqrt{\frac{b^2}{4} - 4\sqrt{e}}$$

(take the positive square root as $\beta > \alpha$)

11. (a) Using the binomial expansion:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

$$= 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots + nx^{n-1} + x^n$$

Then

$$(1+x)^n = k$$

$$\Leftrightarrow (1+x)^n - k = 0$$

$$\Leftrightarrow 1 - k + nx + \frac{n(n-1)}{2}x^2 + \ldots + nx^{n-1} + x^n = 0$$

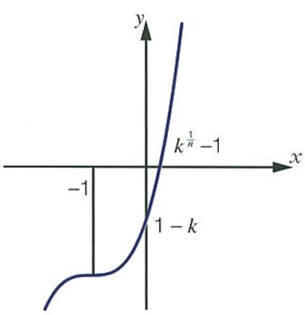
Sum of roots $= -\frac{a_{n-1}}{a_n} = -\frac{n}{1} = -n$, which is independent of k.

Product of roots $= (-1)^n \frac{a_0}{a_n} = (-1)^n \frac{1-k}{1}$

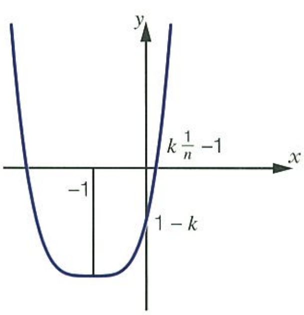
$= (-1)^n (1-k)$,

which has modulus $|1-k|$ independent of n.

(b) (i)



(ii)



(c) By the remainder theorem, $f(1) = 16$. That is,

$$(1+1)^n = 16$$

$$\Rightarrow 2^n = 16$$

$$\Rightarrow n = 4 \ \ (\text{as } n \in \mathbb{Z}^+)$$

(d) From the expansion above, the coefficient of x^2 is $\frac{n(n-1)}{2}$. So

$$\frac{n(n-1)}{2} = 136$$

$$\Leftrightarrow n^2 - n - 272 = 0$$

$$\Leftrightarrow (n-17)(n+16) = 0$$

$$\therefore\ n = 17 \ \ (\text{as } n \in \mathbb{Z}^+)$$

(e) Substitute $x = 1$ into the expansion of $f(x)$:

$$(1+1)^n = \binom{n}{0} + \binom{n}{1}1 + \binom{n}{2}1^2 + \ldots + \binom{n}{n-1}1^{n-1} + \binom{n}{n}1^n$$

$$\Rightarrow \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Going for the top 3

1. Since the remainder when divided by $x^2 + 7x + 12$ is $3x + 2$:

$$\frac{f(x)}{x^2+7x+12} = g(x) + \frac{3x+2}{x^2+7x+12}$$

$$\Rightarrow f(x) = (x^2+7x+12)g(x) + 3x + 2$$

for some polynomial $g(x)$.

Then, by the remainder theorem, the remainder, r, when divided by $x + 4$ is given by

$$r = f(-4)$$

$$= \left((-4)^2 + 7(-4) + 12\right)g(-4) + 3(-4) + 2$$

$$= 0 \times g(-4) - 10$$

$$= -10$$

2. $$(1+ax)^n = 1 + nax + \frac{n(n-1)}{2}(ax)^2 + \ldots$$

$$= 1 + nax + \frac{n(n-1)}{2}a^2x^2 + \ldots$$

So we have

$$\frac{n(n-1)}{2}a^2 = 54 \ \cdots\ (1)$$

$$na = 12 \ \cdots\ (2)$$

From (2), $a = \frac{12}{n}$. Substituting this into (1) gives

$$\frac{n(n-1)}{2}\left(\frac{12}{n}\right)^2 = 54$$

$$\Leftrightarrow \frac{72n^2 - 72n}{n^2} = 54$$

$$\Leftrightarrow 72n^2 - 72n = 54n^2$$

$$\Leftrightarrow 18n(n-4) = 0$$

$$\therefore\ n = 4$$

Then, substituting back into (2) gives $a = 3$.

3. The polynomial has a:

- single root at $x = -1$ and therefore a factor of $(x + 1)$
- triply repeated root at $x = 1$ and therefore a factor of $(x-1)^3$

$y = (x+1)(x-1)^3$ has constant term $1(-1)^3 = -1$, but the graph has y-intercept at -3.

So $y = 3(x+1)(x-1)^3$ is a possible polynomial that fits the graph.

It is also possible that the root at $x = 1$ is repeated five times, so another function that has these properties would be $y = 3(x+1)(x-1)^5$.

4. Let $x^2 = y$. Then the quartic becomes $y^2 + by + c = 0$.

For the quartic to have 4 roots, this quadratic needs to have 2 *positive* roots.

So, firstly,

$$b^2 - 4c > 0$$

$$\Rightarrow b^2 > 4c$$

The smaller root of the quadratic in y is $\frac{-b-\sqrt{b^2-4c}}{2}$, and this needs to be positive:

$$\frac{-b-\sqrt{b^2-4c}}{2} > 0$$

$$\Rightarrow -b - \sqrt{b^2-4c} > 0$$

$$\Rightarrow -b > \sqrt{b^2-4c}$$

(Note that this means b must be negative, so that $-b$ is positive.) Hence, squaring gives

$$b^2 > b^2 - 4c$$

$$\therefore\ 4c > 0$$

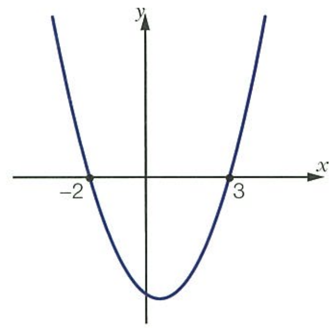
Therefore, $b^2 > 4c > 0$.

4 FUNCTIONS, GRAPHS AND EQUATIONS

Mixed practice 4

1. For an even function, $f(-x) = f(x)$, so for any point x in the domain there exists a second point $-x$ that maps to the same value in the range. Therefore the function is not one-to-one and so doesn't have an inverse.

2. (a)



(b) $x < -2$ or $x > 3$

3.
$$\begin{cases} x - 2y + z = 2 & \cdots\ (1) \\ x + y - 3z = k & \cdots\ (2) \\ 2x - y - 2z = k^2 & \cdots\ (3) \end{cases}$$

$$\begin{cases} x - 2y + z = 2 & \cdots\ (1) \\ x + y - 3z = k & \cdots\ (2) \\ (2)+(3)\quad 3x - 5z = k^2 + k & \cdots\ (4) \end{cases}$$

$$\begin{cases} (1)+2\times(2)\quad 3x - 5z = 2 + 2k & \cdots\ (5) \\ x + y - 3z = k & \cdots\ (2) \\ 3x - 5z = k^2 + k & \cdots\ (4) \end{cases}$$

$$\begin{cases} 3x - 5z = 2 + 2k & \cdots\ (5) \\ x + y - 3z = k & \cdots\ (2) \\ (4)-(5)\quad 0 = k^2 - k - 2 & \cdots\ (6) \end{cases}$$

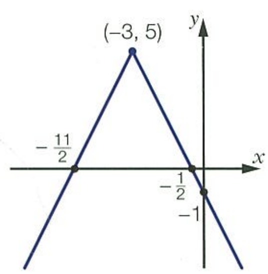
From (6), there will be infinitely many solutions if:

$$k^2 - k - 2 = 0$$

$$\Leftrightarrow (k-2)(k+1) = 0$$

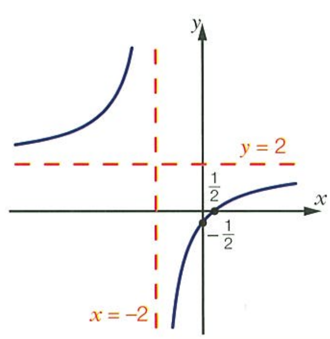
$$\Leftrightarrow k = 2 \text{ or } -1$$

4.



5. $x^2 - a^2 \geq 0 \Rightarrow x \geq a$ or $x \leq -a$

6. (a)



(b) $y = \dfrac{2x-1}{x+2}$

$\Rightarrow y(x+2) = 2x-1$

$\Rightarrow xy + 2y = 2x - 1$

$\Rightarrow 2x - xy = 2y + 1$

$\Rightarrow x(2-y) = 2y+1$

$\Rightarrow x = \dfrac{2y+1}{2-y}$

$\therefore\ f^{-1}(x) = \dfrac{2x+1}{2-x}$

Range is $y \in \mathbb{R}, y \neq -2$.

(c) $g(x) = f^{-1}(2-x)$

$= \dfrac{2(2-x)+1}{2-(2-x)}$

$= \dfrac{5-2x}{x}$

Therefore, the domain is $x \in \mathbb{R}, x \neq 0$.

(d) $f(x) = g(x)$

$\dfrac{2x-1}{x+2} = \dfrac{5-2x}{x}$

$\Leftrightarrow x(2x-1) = (5-2x)(x+2)$

$\Leftrightarrow 2x^2 - x = 10 + x - 2x^2$

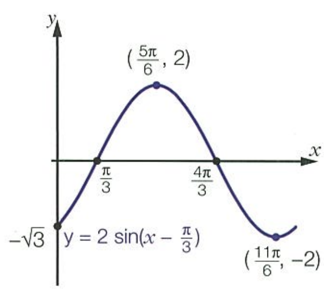
$\Leftrightarrow 4x^2 - 2x - 10 = 0$

$\Leftrightarrow 2x^2 - x - 5 = 0$

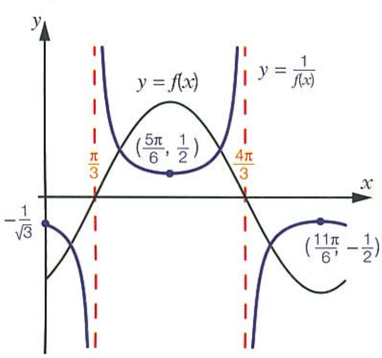
$\Leftrightarrow x = \dfrac{1 \pm \sqrt{1 - 4\times 2 \times(-5)}}{2\times 2}$

$= \dfrac{1\pm\sqrt{41}}{4}$

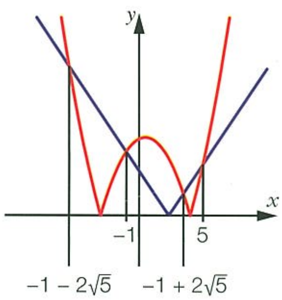
7. (a)



(b)



8. (a)



(b) The 4 points of intersection (from right to left on the graph) are given by the solutions to:

$x^2 - x - 12 = 3x - 7 \quad \cdots (1)$

$-(x^2 - x - 12) = 3x - 7 \quad \cdots (2)$

$-(x^2 - x - 12) = -(3x-7) \quad \cdots (3)$

$x^2 - x - 12 = -(3x-7) \quad \cdots (4)$

However, (1) and (3) are the same equation and (2) and (4) are the same, so just solve (1) and (2) to get 2 points of intersection from each:

$x^2 - x - 12 = 3x - 7$

$\Leftrightarrow x^2 - 4x - 5 = 0$

$\Leftrightarrow (x-5)(x+1) = 0$

$\Leftrightarrow x = 5,\ -1$

And

$-(x^2 - x - 12) = 3x - 7$

$\Leftrightarrow x^2 + 2x - 19 = 0$

$\Leftrightarrow x = \dfrac{-2 \pm \sqrt{2^2 - 4\times 1\times(-19)}}{2}$

$= \dfrac{-2\pm 4\sqrt{5}}{2}$

$= -1 \pm 2\sqrt{5}$

From the graph the solution is therefore

$-1 - 2\sqrt{5} < x < -1$ or $-1 + 2\sqrt{5} < x < 5$

9. (a) $f(x) = \dfrac{ax - a^2 + 1}{x-a}$

$= \dfrac{a(x-a)+1}{x-a}$

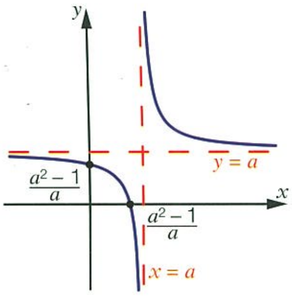
$= \dfrac{a(x-a)}{x-a} + \dfrac{1}{x-a}$

$= a + \dfrac{1}{x-a}$

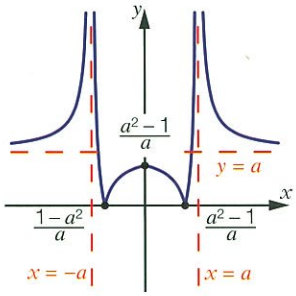
So $p = a$ and $q = 1$.

(b) The graph $y = \dfrac{1}{x}$ has been translated up by a and to the right by a.

(c)



(d)



(e) $f\left(f(x)\right) = f\left(a + \dfrac{1}{x-a}\right)$

$$= a + \frac{1}{\left(a + \frac{1}{x-a}\right) - a}$$

$$= a + \frac{1}{\frac{1}{x-a}}$$

$$= a + x - a$$

$$= x$$

(f) Since $f \circ f(x) = x$, f is self-inverse and so

$$f^{-1}(x) = a + \frac{1}{x-a}.$$

(g) It is symmetric in the line $y = x$.

Going for the top 4

1. (a) $y = \dfrac{x^2 - 3x + 3}{x - 2}$

$\Rightarrow xy - 2y = x^2 - 3x + 3$

$\Rightarrow x^2 - 3x - xy + 2y + 3 = 0$

$\Rightarrow x^2 - (3+y)x + (2y+3) = 0$

$$\Rightarrow x = \frac{3 + y \pm \sqrt{(3+y)^2 - 4(2y+3)}}{2}$$

$$= \frac{3 + y \pm \sqrt{y^2 + 6y + 9 - 8y - 12}}{2}$$

$$= \frac{3 + y \pm \sqrt{y^2 - 2y - 3}}{2}$$

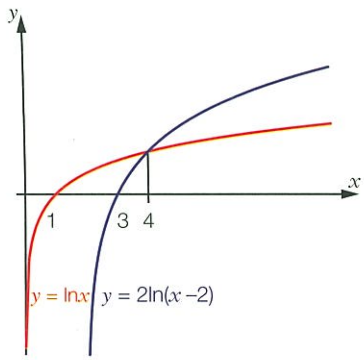
(b) For the square root to give a real number:

$y^2 - 2y - 3 \geq 0$

$\Leftrightarrow (y-3)(y+1) \geq 0$

$\therefore y \leq -1$ or $y \geq 3$

2. (a)



(b) Intersection when $\ln x = 2\ln(x-2)$

$\Leftrightarrow \ln x = \ln(x-2)^2$

$\Leftrightarrow x = (x-2)^2$

$\Leftrightarrow x^2 - 5x + 4 = 0$

$\Leftrightarrow (x-1)(x-4) = 0$

$\therefore x - 4$ (Reject $x = 1$)

$2\ln(x-2)$ has no real value for $x \leq 2$

From the graph, $\ln x \geq 2\ln(x-2)$ for $2 < x \leq 4$

3. Eliminating x and y from the last equation:

$$\begin{cases} 3x + 2y - z = 4 & \cdots \ (1) \\ 2x - 4y + 3z = 1 & \cdots \ (2) \\ x + y + az = k & \cdots \ (3) \end{cases}$$

$$\begin{array}{r} \\ (2) - 2\times(3) \\ 3\times(3) - (1) \end{array} \begin{cases} 3x + 2y - z = 4 & \cdots \ (1) \\ -6y + (3-2a)z = 1 - 2k & \cdots \ (4) \\ y + (3a+1)z = 3k - 4 & \cdots \ (5) \end{cases}$$

$$\begin{array}{r} \\ \\ (4) + 6\times(5) \end{array} \begin{cases} 3x + 2y - z = 4 & \cdots \ (1) \\ -6y + (3-2a)z = 1 - 2k & \cdots \ (4) \\ (16a+9)z = 16k - 23 & \cdots \ (6) \end{cases}$$

(a) The solution will be unique when the coefficient of z in the last equation is not 0:

$16a + 9 \neq 0$

$\therefore a \neq -\dfrac{9}{16}$

(b) There will be infinitely many solutions when the last equation is $0 = 0$, so:

$16k - 23 = 0$

$\Rightarrow k = \dfrac{23}{16}$

The conditions for infinitely many solutions are $k = \dfrac{23}{16}$ and $a = -\dfrac{9}{16}$.

(c) There will be no solutions when the LHS of the last equation is 0 but the RHS isn't, i.e. when $a = -\dfrac{9}{16}$ and $k \neq \dfrac{23}{16}$.

5 SEQUENCES AND SERIES

Mixed practice 5

1. $u_4 = 17 \Rightarrow u_1 + 3d = 17 \quad \cdots (1)$

$S_{20} = 990 \Rightarrow \frac{20}{2}[2u_1 + 19d] = 990$

$\Rightarrow 2u_1 + 19d = 99 \quad \cdots (2)$

From (1), $u_1 = 17 - 3d$; substituting this into (2) gives

$2(17-3d) + 19d = 99$

$\Leftrightarrow 34 + 13d = 99$

$\Leftrightarrow d = 5$

So $u_1 = 17 - 3d = 17 - 3 \times 5 = 2$.

2. The fourth, tenth and thirteenth terms of the geometric sequence are:

$u_4 = u_1 r^3$

$u_{10} = u_1 r^9$

$u_{13} = u_1 r^{12}$

As these form an arithmetic sequence:

$u_{10} - u_4 = u_{13} - u_{10}$

$\Rightarrow u_1 r^9 - u_1 r^3 = u_1 r^{12} - u_1 r^9$

$\Rightarrow r^{12} - 2r^9 + r^3 = 0 \quad (\text{as } u_1 \neq 0)$

$\Rightarrow r^9 - 2r^6 + 1 = 0 \quad (\text{as } r \neq 0)$

$\Rightarrow r = 1,\ 1.17,\ -0.852 \quad$ (from GDC to 3 SF)

$\therefore\ r = -0.852 \quad$ (for sum to infinity to exist, $|r| < 1$)

3. $\sum_{r=1}^{12} 4r + \left(\frac{1}{3}\right)^r = 4\sum_{r=1}^{12} r + \sum_{r=1}^{12}\left(\frac{1}{3}\right)^r$

The first sum is an arithmetic series, and the second sum is a geometric series. So, using the formulae:

$$4\left[\frac{12}{2}(2\times 1 + (12-1)\times 1)\right] + \frac{\frac{1}{3}\left[1-\left(\frac{1}{3}\right)^{12}\right]}{1-\frac{1}{3}} = 312.5 \text{ (4 SF)}$$

4. For 20 terms of this series, i.e. with $n = 20$:

$\ln x + \ln x^4 + \ln x^7 + \ln x^{10} + \ldots = \ln x + 4\ln x + 7\ln x + 10\ln x + \ldots$

$= \ln x(1+4+7+10+\ldots)$

$= \ln x\left(\frac{20}{2}(2+19\times 3)\right)$

$= 590\ln x$

$= \ln x^{590}$

5. We know that the total length of the pieces is 300, i.e. $S_n = 300$:

$S_n = \frac{n}{2}(u_1 + u_n)$

$300 = \frac{n}{2}(1+19)$

$\Leftrightarrow 300 = 10n$

$\Leftrightarrow n = 30$

So $u_{30} = 19$:

$u_n = u_1 + (n-1)d$

$19 = 1 + 29d$

$\Leftrightarrow d = \frac{18}{29}$ metres

6. The amount Aaron has in his account for the first few months is:

1st: 100
2nd: 100 + 110
3rd: 100 + 110 + 120

His monthly balance forms an arithmetic series with $a = 100$ and $d = 10$. So after n months he will have:

$S_n = \frac{n}{2}[2\times 100 + (n-1)10]$

$= \frac{n}{2}[200 + 10n - 10]$

$= \frac{n}{2}[190 + 10n]$

$= 5n^2 + 95n$

The amount Blake has in his account for the first few months is:

1st: 50
2nd: 50×1.05
3rd: 50×1.05^2

His monthly balance forms a geometric series with $a = 50$ and $r = 1.05$. So after n months he will have:

$S_n = \frac{50(1.05^n - 1)}{1.05 - 1}$

$= 1000(1.05^n - 1)$

Therefore, Blake will have more in his account than Aaron does when:

$1000(1.05^n - 1) > 5n^2 + 95n$

$\therefore\ n = 73$ months (from GDC)

Going for the top 5

1. (a) (i) Writing out the sum from the first term, a, to the nth term, ar^{n-1}:

$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \quad (1)$

Multiplying through by r:

$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \quad (2)$

We can see that (1) and (2) have many terms in common:

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} && (1)\\ rS_n &= \quad\ \ ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} + ar^n && (2)\end{aligned}$$

So (2) − (1) gives:

$rS_n - S_n = ar^n - a$

$\Rightarrow S_n(r-1) = a(r^n - 1)$

$\Rightarrow S_n = \dfrac{a(r^n-1)}{r-1}$

(ii) If $-1 < r < 1$ (or equivalently $|r| < 1$), then as $n \to \infty$, $r^n \to 0$. So

$S_n \to \dfrac{a(0-1)}{r-1} = \dfrac{-a}{r-1} = \dfrac{a}{1-r}$

If $|r| \geq 1$, then r^n has no limit as $n \to \infty$, and so there is no finite limit for S_n.

Thus, $S_\infty = \dfrac{a}{1-r}$ only for $|r| < 1$.

(b) The sum of the first n terms is $S_n = \dfrac{u_1(1-r^n)}{1-r}$.

The sum of the next n terms is given by

$$\begin{aligned} S_{2n} - S_n &= \frac{u_1(1-r^{2n})}{1-r} - \frac{u_1(1-r^n)}{1-r} \\ &= \frac{u_1(1-r^{2n}-1+r^n)}{1-r} \\ &= \frac{u_1(r^n - r^{2n})}{1-r} \\ &= \frac{u_1 r^n(1-r^n)}{1-r} \end{aligned}$$

So the ratio of the sum of the first n terms to the sum of the next n terms is

$\dfrac{u_1(1-r^n)}{1-r} : \dfrac{u_1 r^n(1-r^n)}{1-r} = 1 : r^n$

(c) (i) $u_7 + 4u_5 = u_8$

i.e. $u_1 r^6 + 4u_1 r^4 = u_1 r^7$

$\Rightarrow r^2 + 4 = r^3$ (since $u_1, r \neq 0$)

$\Rightarrow r^3 - r^2 - 4 = 0$

To factorise this, substitute small positive and negative integers into $f(r) = r^3 - r^2 - 4$ until you find an r such that $f(r) = 0$:

$f(1) = 1^3 - 1^2 - 4 = -4$

$f(-1) = (-1)^3 - (-1)^2 - 4 = -6$

$f(2) = 2^3 - 2^2 - 4 = 0$

Therefore, by the factor theorem, $(x-2)$ is a factor and so by long division or equating coefficients we get:

$r^3 - r^2 - 4 = (r-2)(r^2 + r + 2)$

For the quadratic $r^2 + r + 2$,

$\Delta = 1^2 - 4 \times 1 \times 2 = -7 < 0$

Hence there are no real roots for this quadratic factor, and so the only root of $f(r)$ is $r = 2$.

Then, from part (b), the ratio is

$1 : r^{10} = 1 : 2^{10} = 1 : 1024$

(ii) No, because $r > 1$ in this case, and as shown in part (a)(ii), the condition for a sum to infinity to exist is that $|r| < 1$.

2. The integers from 1 to 1000 form an arithmetic sequence with $u_1 = 1$, $u_n = 1000$ and $n = 1000$. So

$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \frac{1000}{2}(1+1000) \\ &= 500\,500 \end{aligned}$$

The multiples of 7 between 1 and 1000 form an arithmetic sequence with $u_1 = 7$, $u_n = 994$ and $n = \dfrac{994}{7} = 142$. So

$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \frac{142}{2}(7+994) \\ &= 71\,071 \end{aligned}$$

Therefore the sum of the integers between 1 and 1000 that are not divisible by 7 is

$500\,500 - 71\,071 = 429\,429$.

6 TRIGONOMETRY

Mixed practice 6

1. $A = \dfrac{1}{2}ab\sin C$

$12 = \dfrac{1}{2}x(x+2)\sin 30°$

$\Rightarrow 12 = \dfrac{1}{2}x(x+2)\dfrac{1}{2}$

$\Rightarrow 48 = x^2 + 2x$

$\Rightarrow x^2 + 2x - 48 = 0$

$\Rightarrow (x+8)(x-6) = 0$

$\therefore x = 6 \quad (\text{as } x > 0)$

2. By the sine rule:

$\dfrac{\sin\theta}{5} = \dfrac{\sin 2\theta}{7}$

$\Rightarrow 7\sin\theta = 5\sin 2\theta$

$\Rightarrow 7\sin\theta = 5(2\sin\theta\cos\theta)$

$\Rightarrow 7\sin\theta - 10\sin\theta\cos\theta = 0$

$\Rightarrow \sin\theta(7 - 10\cos\theta) = 0$

$\Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{7}{10}$

$\Rightarrow \theta = 0°, 180°$ or $\theta = 45.6°$

$\therefore\ \theta = 45.6°$

3. (a) Shaded area = area of sector − area of triangle

$$6.2 = \frac{1}{2}r^2\theta - \frac{1}{2}ab\sin C$$

$$6.2 = \frac{1}{2}\times 5^2\theta - \frac{1}{2}\times 5\times 5\sin\theta$$

$$6.2 = \frac{25}{2}(\theta - \sin\theta)$$

$$\therefore\ \theta - \sin\theta = \frac{12.4}{25} = 0.496$$

(b) From GDC, $\theta = 1.49$ (3 SF)

4. (a) Since $\mathrm{O\hat{T}P}$ is a right angle,

$$\text{Area} = \frac{1}{2}bh$$

$$= \frac{1}{2}\times 12\times 7$$

$$= 42$$

(b) Let $\mathrm{P\hat{O}T} = \theta$. Then

$$\tan\theta = \frac{12}{7}$$

$$\therefore\ \theta = \tan^{-1}\left(\frac{12}{7}\right) = 1.04\ (3\text{ SF})$$

(c) Shaded area = area of triangle OPT − area of sector

$$= 42 - \frac{1}{2}\times 7^2\times \mathrm{P\hat{O}T}$$

$$= 16.5\ \ (3\,\text{SF})$$

5. (a) In triangle ABG, we know $\mathrm{G\hat{A}B} = 65°$ and $\mathrm{G\hat{B}A} = 80°$, so

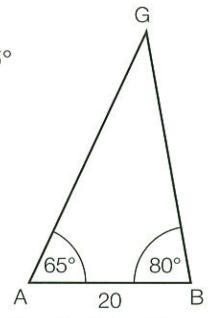
$\mathrm{A\hat{G}B} = 180° - 65° - 80° = 35°$

By the sine rule:

$$\frac{\text{AG}}{\sin 80°} = \frac{20}{\sin 35°}$$

$$\Rightarrow \text{AG} = \frac{20}{\sin 35°}\times\sin 80°$$

$$= 34.3\text{ m}$$

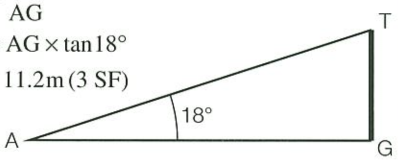


(b) In triangle AGT, we know the length AG and the angle $\mathrm{G\hat{A}T}$:

$$\tan 18° = \frac{\text{GT}}{\text{AG}}$$

$$\therefore\ \text{GT} = \text{AG}\times\tan 18°$$

$$= 11.2\text{m}\ (3\text{ SF})$$



6. (a) $\cot x + \tan x = 4$

$$\Leftrightarrow \frac{1}{\tan x} + \tan x = 4$$

$$\Leftrightarrow 1 + \tan^2 x = 4\tan x$$

$$\Leftrightarrow \tan^2 x - 4\tan x + 1 = 0$$

$$\Leftrightarrow \tan x = \frac{4 \pm \sqrt{(-4)^2 - 4\times 1\times 1}}{2\times 1}$$

$$= \frac{4\pm\sqrt{12}}{2}$$

$$= \frac{4\pm 2\sqrt{3}}{2}$$

$$= 2\pm\sqrt{3}$$

For $x \in \left[0, \frac{\pi}{4}\right[$, $0 \le \tan x < 1$, so only the smaller value is possible. Hence $\tan x = 2-\sqrt{3}$.

(b) (i) $\cot x + \tan x = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$

$$= \frac{\cos^2 x}{\sin x\cos x} + \frac{\sin^2 x}{\sin x\cos x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\sin x\cos x}$$

$$= \frac{1}{\sin x\cos x}$$

$$= \frac{1}{\frac{1}{2}\sin 2x}$$

$$= \frac{2}{\sin 2x}$$

$$= 2\csc 2x$$

(ii) $\cot x + \tan x = 4$

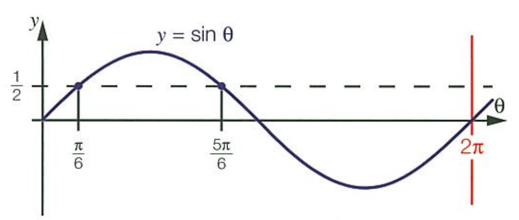
$$\Leftrightarrow 2\csc 2x = 4$$

$$\Leftrightarrow \csc 2x = 2$$

$$\Leftrightarrow \sin 2x = \frac{1}{2}$$

$x \in [0, \pi] \Rightarrow 2x \in [0, 2\pi]$

The graph shows that there are two solutions to $\sin\theta = \frac{1}{2}$ in $[0, 2\pi]$:



One solution is $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

And another is $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\therefore\ x = \frac{\pi}{12}, \frac{5\pi}{12}$$

(c) From (b)(ii), $x = \dfrac{\pi}{12}$ is a solution of the equation $\cot x + \tan x = 4$, and it satisfies $x \in \left[0, \dfrac{\pi}{4}\right[$.

Therefore, from part (a), $\tan\left(\dfrac{\pi}{12}\right) = 2 - \sqrt{3}$.

7. $3\sin 2\theta = \tan 2\theta$

$\Leftrightarrow 3\sin 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta}$

$\Leftrightarrow 3\sin 2\theta \cos 2\theta = \sin 2\theta$

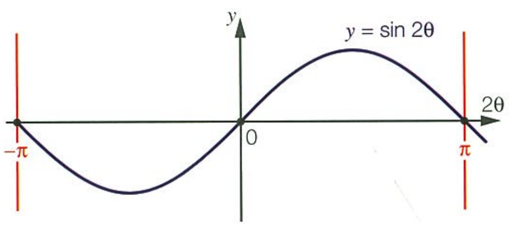
$\Leftrightarrow 3\sin 2\theta \cos 2\theta - \sin 2\theta = 0$

$\Leftrightarrow \sin 2\theta(3\cos 2\theta - 1) = 0$

$\therefore\ \sin 2\theta = 0 \quad \text{or} \quad \cos 2\theta = \dfrac{1}{3}$

$\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \Rightarrow 2\theta \in [-\pi, \pi]$

For $\sin 2\theta = 0$, the graph shows there to be three solutions in $[-\pi, \pi]$:



One solution is $\sin^{-1} 0 = 0$

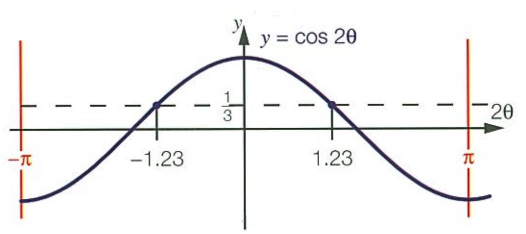
Another is $\pi - 0 = \pi$

Then, adding/subtracting 2π gives one further value in the interval: $\pi - 2\pi = -\pi$

$\therefore 2\theta = 0, \pm\pi$

$\Rightarrow \theta = 0, \pm\dfrac{\pi}{2}$

For $\cos 2\theta = \dfrac{1}{3}$, the graph shows there to be two solutions:



One solution is $\cos^{-1}\left(\dfrac{1}{3}\right) = 1.23$

$2\pi - 1.23 = 5.05$ is not in the interval $[-\pi, \pi]$, but $5.05 - 2\pi = -1.23$ is in the interval.

$\therefore\ 2\theta = \pm 1.23$

$\Rightarrow \theta = \pm 0.615$

So the solutions are $\theta = 0, \pm 0.615, \pm\dfrac{\pi}{2}$

8. (a)
$$\begin{aligned} 3x^2 - 2x + 5 &= 3\left[x^2 - \frac{2}{3}x\right] + 5 \\ &= 3\left[\left(x - \frac{1}{3}\right)^2 - \frac{1}{9}\right] + 5 \\ &= 3\left(x - \frac{1}{3}\right)^2 - \frac{1}{3} + 5 \\ &= 3\left(x - \frac{1}{3}\right)^2 + \frac{14}{3} \end{aligned}$$

So the coordinates of the vertex are $\left(\dfrac{1}{3}, \dfrac{14}{3}\right)$.

(b) (i)
$$\begin{aligned} g(\theta) &= 3\cos 2\theta - 4\cos\theta + 13 \\ &= 3(2\cos^2\theta - 1) - 4\cos\theta + 13 \\ &= 6\cos^2\theta - 4\cos\theta + 10 \end{aligned}$$

(ii) Let $\cos\theta = x$. Then
$$\begin{aligned} 6\cos^2\theta - 4\cos\theta + 10 &= 6x^2 - 4x + 10 \\ &= 2(3x^2 - 2x + 5) \\ &= 2f(x) \end{aligned}$$

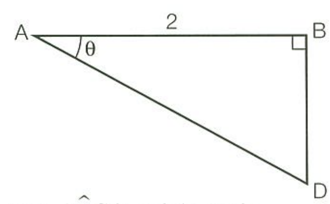
and $\theta \le \theta \le 2\pi \Rightarrow -1 \le x \le 1$

Since the minimum value of $f(x)$ is $\dfrac{14}{3}$,

the minimum value of $2f(x)$ is $\dfrac{28}{3}$.

That is, the minimum value of $g(\theta)$ is $\dfrac{28}{3}$.

9. (a) In triangle ABD:

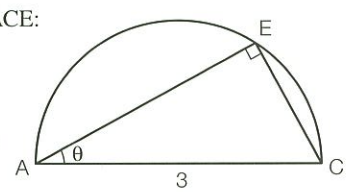


$\cos\theta = \dfrac{2}{AD}$

$\Rightarrow AD = \dfrac{2}{\cos\theta}$

Since AC is a diameter, $A\hat{E}C$ is a right angle.

So in triangle ACE:



$\sin\theta = \dfrac{CE}{3}$

$\Rightarrow CE = 3\sin\theta$

Therefore

$$R = \text{AD} - \text{CE}$$
$$= \frac{2}{\cos\theta} - 3\sin\theta$$
$$= 2\sec\theta - 3\sin\theta$$

(b) For stationary values, $\frac{dR}{d\theta} = 0.$

$$\frac{dR}{d\theta} = 2\sec\theta\tan\theta - 3\cos\theta$$

$$0 = 2\sec\theta\tan\theta - 3\cos\theta$$
$$\Rightarrow 2\sec\theta\tan\theta = 3\cos\theta$$
$$\Rightarrow 2\times\frac{1}{\cos\theta}\times\frac{\sin\theta}{\cos\theta} = 3\cos\theta$$
$$\Rightarrow 2\sin\theta = 3\cos^3\theta$$

(c) From GDC, the minimum occurs at $\theta = 0.711$ and $R = 0.682$ (3 SF). So the smallest value of AD – CE is 0.682.

Going for the top 6

1. (a) By the double angle formula:

$$\cos 2x = 2\cos^2 x - 1$$
$$= 2y^2 - 1$$

(b) $\cos 2x = 2y^2 - 1$

$$\Rightarrow 2x = \arccos(2y^2 - 1)$$

But $\cos x = y \Rightarrow x = \arccos y$

$$\therefore 2\arccos y = 2x = \arccos(2y^2 - 1)$$

(c) $y = \cos x \Rightarrow \arccos y = x$

Also, $y = \cos x = \sin\left(\frac{\pi}{2} - x\right) \Rightarrow \arcsin y = \frac{\pi}{2} - x$

$$\therefore \arcsin y + \arccos y = \left(\frac{\pi}{2} - x\right) + x = \frac{\pi}{2}$$

2. $\tan\alpha = \frac{\text{AB}}{\text{AD}} = \frac{2\text{AD}}{\text{AD}} = 2$

$$\tan\beta = \frac{\text{AC}}{\text{AD}} = \frac{3\text{AD}}{\text{AD}} = 3$$

By the compound angle formula:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$= \frac{2+3}{1-2\times 3} = -1$$

$$\tan^{-1}(-1) = -\frac{\pi}{4}$$

$$\therefore \alpha+\beta = -\frac{\pi}{4} + \pi \quad (\text{as } \alpha+\beta > 0)$$
$$= \frac{3\pi}{4}$$

3. (a) $\cos[(A-B)x] - \cos[(A+B)x] = \cos(Ax - Bx) - \cos(Ax + Bx)$

$$= \cos Ax\cos Bx + \sin Ax\sin Bx - (\cos Ax\cos Bx - \sin Ax\sin Bx)$$
$$= 2\sin Ax\sin Bx$$

(b) Let $A = 1$ and $B = 2r - 1$.

Then, by (a), $2\sin x\sin[(2r-1)x] = \cos[(1-(2r-1))x] - \cos[(1+(2r-1))x]$

$$\sum_{r=1}^{n} 2\sin x\sin(2r-1)x = \sum_{r=1}^{n}\cos[(1-(2r-1))x] - \cos[(1+(2r-1))x]$$
$$= \sum_{r=1}^{n}\cos[2(1-r)x] - \cos 2rx$$
$$= (\cos 0 - \cos 2x) + (\cos(-2x) + \cos 4x) + (\cos(-4x) + \cos 6x) + \ldots + (\cos[2(1-n)x] + \cos 2nx)$$
$$= (\cos 0 - \cos 2x) + (\cos 2x - \cos 4x) + (\cos 4x - \cos 6x) + \ldots + (\cos[2(n-1)x] - \cos 2nx)$$
$$= 1 - \cos 2nx \qquad \text{(because all the terms in the middle cancel out)}$$
$$\therefore \sum_{r=1}^{n}\sin x\sin(2r-1)x = \frac{1-\cos 2nx}{2}$$

(c) Let $n = 2$ in part (b). Then

$$\sum_{r=1}^{2} \sin x \sin(2r-1)x = \frac{1-\cos 4x}{2}$$

$$\Leftrightarrow \sin x \sin x + \sin x \sin 3x = \frac{1-\cos 4x}{2}$$

$$\Leftrightarrow \sin^2 x + \sin x \sin 3x = \frac{1-\cos 4x}{2}$$

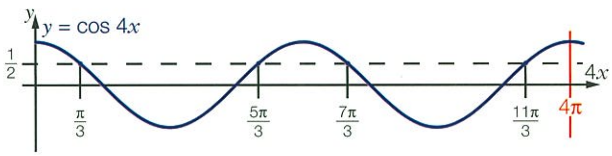
Therefore, solve

$$\frac{1-\cos 4x}{2} = \frac{1}{4}$$

$$\Leftrightarrow \cos 4x = \frac{1}{2}$$

$0 < x < \pi \Rightarrow 0 < 4x < 4\pi$

The graph shows there to be four solutions:



One solution is $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Another is $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

Then, adding 2π to these gives two more solutions in $[0, 4\pi]$:

$$\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$$

$$\frac{5\pi}{3} + 2\pi = \frac{11\pi}{3}$$

$$\therefore 4x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

$$\Rightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$$

(d) Let $n = 5$ and $x = \frac{\pi}{5}$ in part (b). Then

$$\sum_{r=1}^{5} \sin\left(\frac{\pi}{5}\right) \sin\left[(2r-1)\frac{\pi}{5}\right] = \frac{1-\cos\left[2\times 5\left(\frac{\pi}{5}\right)\right]}{2}$$

$$\Rightarrow \sin\frac{\pi}{5}\sin\frac{\pi}{5} + \sin\frac{\pi}{5}\sin\frac{3\pi}{5} + \sin\frac{\pi}{5}\sin\frac{5\pi}{5} + \sin\frac{\pi}{5}\sin\frac{7\pi}{5} + \sin\frac{\pi}{5}\sin\frac{9\pi}{5} = \frac{1-\cos 2\pi}{2}$$

$$\Rightarrow \sin\frac{\pi}{5}\left(\sin\frac{\pi}{5} + \sin\frac{3\pi}{5} + \sin\frac{5\pi}{5} + \sin\frac{7\pi}{5} + \sin\frac{9\pi}{5}\right) = 0$$

$$\Rightarrow \sin\frac{\pi}{5} + \sin\frac{3\pi}{5} + \sin\frac{5\pi}{5} + \sin\frac{7\pi}{5} + \sin\frac{9\pi}{5} = 0$$

7 VECTORS

Mixed practice 7

1. (a) $\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} (-1)\times p - 2\times 1 \\ 2\times 1 - 3\times p \\ 3\times 1 - (-1)\times 1 \end{pmatrix}$

$$= \begin{pmatrix} -p-2 \\ 2-3p \\ 4 \end{pmatrix}$$

(or $-(p+2)\boldsymbol{i} + (2-3p)\boldsymbol{j} + 4\boldsymbol{k}$)

(b) If $\boldsymbol{a} \times \boldsymbol{b}$ is parallel to $\boldsymbol{c}$, then $\boldsymbol{a} \times \boldsymbol{b} = \lambda \boldsymbol{c}$ for some constant λ.

So $\begin{pmatrix} -p-2 \\ 2-3p \\ 4 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ 22 \\ 8 \end{pmatrix}$

The third component gives $4 = 8\lambda \Rightarrow \lambda = \frac{1}{2}$

And the first component gives

$-p - 2 = 2\lambda$

$\therefore p = -3$

Check these values in the second equation: $2 - 3p = 11 = 22\lambda$, as stated.